SOME PROPERTIES OF SOLUTIONS OF A FUNCTIONAL-DIFFERENTIAL EQUATION OF SECOND ORDER WITH DELAY

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ABSTRACT. Existence, uniqueness, data dependence (monotony, continuity, differentiability with respect to parameter) and Ulam-Hyers stability results for the solutions of a system of functional-differential equations with delays are proved. The techniques used are Perov's fixed point theorem and weakly Picard operator theory.

1. INTRODUCTION

Functional-differential equations with delay arise when modeling biological, physical, engineering and other processes whose rate of change of state at any moment of time t is determined not only by the present state, but also by past state.

The description of certain phenomena in physics has to take into account that the rate of propagation is finite. For example, oscillation in a vacuum tube can be described by the following equation in dimensionless variables [4], [9]

$$x''(t) + 2rx'(t) + \omega^2 x(t) + 2qx'(t-1) = \epsilon x'^3(t-1).$$

In this equation, time delay is due to the fact that the time necessary for electrons to pass from the cathode to the anode in the tube is finite. The same equation has been used in the theory of stabilization of ships [9]. The dynamics of an autogenerator with delay and second-order

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filter was described in [1] by the equation

$$x''(t) + 2\delta x'(t) + x(t) = f(x(t-h)).$$

The model of ship course stabilization under conditions of uncertainty may be described by the following equation [3],

$$Ix''(t) + Hx'(t) = -K\Psi(t) + b_0\xi_0'(t),\ x(t_0) = x_0,\ x'(t_0) = 0,$$

with $x(t)$ being the angle of the deviation from course, $\Psi(t)$ the turning angle of the rudder and $\xi_0(t)$ the stochastic disturbance. In the process of mathematical modeling often small delays are neglected, that is why false conclusions appear. As an example we can give the following equation [4],

$$x''(t) + x'(t) + x(t) = a[x''(t-h) + x'(t-h) + x(t-h)],$$

which is asymptotically stable for $h = 0$, but unstable for arbitrary $h > 0$. Here $a > 1$. If $h = 0$ the above system is asymptotically stable. The characteristic equation is $\Delta(z) = (z^2 + z + 1)(1 - ae^{-hz}) = 0$ and has the following zeros with positive real part if $h > 0:\ z_k = \frac{1}{h}(lna + 2k\pi i),\ i^2 = -1,\ k = 0, \pm 1, \ldots$. So the trivial solution is unstable for any $h > 0$.

In this paper we continue the research in this field and develop the study of the following general functional differential equation with delay

$$\left\{ \begin{array}{l} x''(t) = f(t, x(t), x'(t), x(t-h), x'(t-h)),\ t \in [a,b] \\ x(t) = \varphi(t),\ t \in [a-h, a]. \end{array} \right. \tag{1.1}$$

Existence, uniqueness, data dependence (monotony, continuity, differentiability with respect to parameter) and Ulam-Hyers stability results of solution for the Cauchy problem are obtained. Our results are essentially based on Perov's fixed point theorem and weakly Picard operator technique, which will be presented in Section 2. More results about functional and integral differential equations using these techniques can be found in [2], [5]-[7]. The problem (1.1) is equivalent to the following system

$$\left\{ \begin{array}{l} x'(t) = z(t),\ t \in [a,b] \\ z'(t) = f(t, x(t), z(t), x(t-h), z(t-h)),\ t \in [a,b], \end{array} \right. \tag{1.2}$$

with the initial conditions

$$\left\{ \begin{array}{l} x(t) = \varphi(t),\ t \in [a-h, a] \\ z(t) = \varphi'(t),\ t \in [a-h, a]. \end{array} \right. \tag{1.3}$$

By a solution of the system (1.2) we understand a function $\binom{x}{z} \in C([a-h,b], \mathbb{R}^2) \cap C^1([a,b], \mathbb{R}^2)$ that verifies the system.

We suppose that

(C_1) $a < b,\ h > 0$;
(C_2) $f \in C([a,b] \times \mathbb{R}^4, \mathbb{R}), \varphi \in C^1([a-h,a], \mathbb{R})$;
(C_3) there exists $L_1, L_2 > 0$ such that $\forall t \in [a,b], u_i, v_i, \widetilde{u}_i, \widetilde{v}_i \in \mathbb{R}$, i=1, 2, we have

$$|f(t,u_1,v_1,u_2,v_2) - f(t,\widetilde{u}_1,\widetilde{v}_1,\widetilde{u}_2,\widetilde{v}_2)| \leq$$
$$\leq L_1 \max\left(|u_1 - \widetilde{u}_1|, |u_2 - \widetilde{u}_2|\right) + L_2 \max\left(|v_1 - \widetilde{v}_1|, |v_2 - \widetilde{v}_2|\right).$$

If $\binom{x}{z} \in C([a-h,b], \mathbb{R}^2) \cap C^1([a,b], \mathbb{R}^2)$ is a solution of the problem (1.2)-(1.3) then $\binom{x}{z}$ is a solution of the following integral system

$$(1.4) \qquad \binom{x}{z}(t) = \begin{cases} \binom{\varphi}{\varphi'}(t), \text{ for } t \in [a-h,a] \\ \binom{\varphi(a)+\int_a^t z(s)ds}{\varphi'(a)+\int_a^t f(s,x(s),z(s),x(s-h),z(s-h))ds}, \text{ for } t \in [a,b]. \end{cases}$$

If $\binom{x}{z} \in C([a-h,b], \mathbb{R}^2)$ is a solution of (1.4) then $\binom{x}{z} \in C([a-h,b], \mathbb{R}^2) \cap C^1([a,b], \mathbb{R}^2)$ and $\binom{x}{z}$ is a solution of (1.2)-(1.3).

Moreover, the system (1.2) is equivalent to the functional integral system

$$(1.5) \qquad \binom{x}{z}(t) = \begin{cases} \binom{x}{z}(t), \text{ for } t \in [a-h,a] \\ \binom{x(a)+\int_a^t z(s)ds}{z(a)+\int_a^t f(s,x(s),z(s),x(s-h),z(s-h))ds}, \text{ for } t \in [a,b]. \end{cases}$$

We consider the operators $B, E : C([a-h,b], \mathbb{R}^2) \to C([a-h,b], \mathbb{R}^2)$, $B = \binom{B_1\binom{x}{z}}{B_2\binom{x}{z}}$, $E = \binom{E_1\binom{x}{z}}{E_2\binom{x}{z}}$ defined by $B\binom{x}{z}(t)$:=the right hand side of (1.4), for $t \in [a-h,b]$ and $E\binom{x}{z}(t)$:=the right hand side of (1.5), for $t \in [a-h,b]$.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary results which are used throughout this paper, see [10]–[18]. Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$.

Definition 2.1. *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;
(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. *It is clear that $A^\infty(X) = F_A$.*

Definition 2.5. *Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \le cd(x, A(x)),\ \forall x \in X.$$

The following concept is important for our further considerations.

Definition 2.6. *Let (X, d) be a metric space and $f : X \to X$ be an operator. The fixed point equation*

$$x = f(x) \tag{2.1}$$

is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \le \varepsilon$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*, x^*) \le c_f\varepsilon.$$

Now we have

Theorem 2.7. [18] *If $f : X \to X$ is c-WPO, then the equation*

$$x = f(x)$$

is Ulam-Hyers stable.

Another result from the WPO theory is the following (see, e.g., [12]).

Theorem 2.8. (Fibre contraction principle). *Let (X, d) and (Y, ρ) be two metric spaces and $A : X \times Y \to X \times Y$, $A = (B, C)$, ($B : X \to X$, $C : X \times Y \to Y$) a triangular operator. We suppose that*

(i) *(Y, ρ) is a complete metric space;*
(ii) *the operator B is Picard operator;*
(iii) *there exists $l \in [0, 1)$ such that $C(x, \cdot) : Y \to Y$ is a l-contraction, for all $x \in X$;*
(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is Picard operator.

Throughout this paper we denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. A square matrix Q with nonnegative elements is said to be convergent to zero if $Q^k \to 0$ as $k \to \infty$. It is known that the property of being convergent to zero is equivalent to each of the following three conditions (see [10], [11]):

(a) $I - Q$ is nonsingular and $(I-Q)^{-1} = I + Q + Q^2 + \cdots$ (where I stands for the unit matrix of the same order as Q);
(b) the eigenvalues of Q are located inside the open unit disc of the complex plane;
(c) $I - Q$ is nonsingular and $(I-Q)^{-1}$ has nonnegative elements.

We finish this section by recalling the following fundamental result (see [8], [10]).

Theorem 2.9. (Perov's fixed point theorem). *Let (X,d) with $d(x,y) \in \mathbb{R}^m$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, such that*

(i) *$d(A(x), A(y)) \le Q d(x,y)$, for all $x, y \in X$;*
(ii) *$Q^n \to 0$ as $n \to \infty$.*

Then

(a) *$F_A = \{x^*\}$;*
(b) *$A^n(x) = x^*$ as $n \to \infty$, $\forall x \in X$;*
(c) *$d(A^n(x), x^*) \le (I-Q)^{-1} Q^n d(x_0, A(x_0))$.*

3. Main results

In this section we present existence, uniqueness and data dependence (monotony, continuity, differentiability with respect to parameter) results of solution for the Cauchy problem (1.2)-(1.3).

3.1. Existence and uniqueness.

Using Perov's fixed point theorem we obtain existence and uniqueness theorem for the solution of the problem (1.2)-(1.3).

Theorem 3.1. *We suppose that:*

(i) *the conditions (C_1)-(C_3) are satisfied;*
(ii) *$Q^n \to 0$ as $n \to \infty$, where $Q = (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$.*

Then:

(a) *the problem (1.2)-(1.3) has a unique solution $\binom{x^*}{z^*} \in C^1([a,b], \mathbb{R}^2)$;*

(b) *for all* $\binom{x^0}{z^0} \in C^1([a,b],\mathbb{R}^2)$, *the sequence* $\binom{x^n}{z^n}_{n\in\mathbb{N}}$ *defined by* $\binom{x^{n+1}}{z^{n+1}} = B\binom{x^n}{z^n}$, *converges uniformly to* $\binom{x^*}{z^*}$, *for all* $t \in [a,b]$, *and*

$$\left\| \begin{pmatrix} x^n \\ z^n \end{pmatrix} - \begin{pmatrix} x^* \\ z^* \end{pmatrix} \right\| \leq (I-Q)^{-1} Q^n \left\| \begin{pmatrix} x^0 \\ z^0 \end{pmatrix} - \begin{pmatrix} x^1 \\ z^1 \end{pmatrix} \right\|;$$

(c) *the operator* B *is Picard operator in* $(C([a-h,b],\mathbb{R}^2), \xrightarrow{unif})$;

(d) *the operator* E *is weakly Picard operator in* $(C([a-h,b],\mathbb{R}^2), \xrightarrow{unif})$.

Proof. Consider on the space $X := C([a-h,b],\mathbb{R}^2)$ the norm

$$\left\| \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\| := \max \begin{pmatrix} |u_1 - u_2| \\ |v_1 - v_2| \end{pmatrix},$$

which endows X with the uniform convergence.

Let $X_{\binom{\chi}{\chi'}} = \left\{ \binom{x}{z} \in X \mid \binom{x}{z}\Big|_{[a-h,a]} = \binom{\chi}{\chi'}, \text{ for } \binom{\chi}{\chi'} \in C([a-h,a],\mathbb{R}^2) \right\}$. Then $X = \bigcup_{\binom{\chi}{\chi'} \in C([a-h,a],\mathbb{R}^2)} X_{\binom{\chi}{\chi'}}$ is a partition of X and from [13] we have

(1) $B(X) \subset X_{\binom{\chi}{\chi'}}, B(X_{\binom{\chi}{\chi'}}) \subset X_{\binom{\chi}{\chi'}}$;

(2) $B|_{X_{\binom{\chi}{\chi'}}} = E|_{X_{\binom{\chi}{\chi'}}}$.

On the other hand, for $t \in [a-h,a] \cup [a,b]$

$$\left\| \begin{pmatrix} B_1\binom{x_1}{z_1} \\ B_2\binom{x_1}{z_1} \end{pmatrix} - \begin{pmatrix} B_1\binom{x_2}{z_2} \\ B_2\binom{x_2}{z_2} \end{pmatrix} \right\| \leq (b-a) \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \left\| \begin{pmatrix} x_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ z_2 \end{pmatrix} \right\|,$$

whence B is a contraction in $(X, \|\cdot\|)$ with $Q = (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$. Applying Perov's theorem we obtain (a), (b) and (c). Moreover the operator B is c-PO and E is c-WPO with $c = \left[1 - (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \right]^{-1}$. □

3.2. **Inequalities of Čaplygin type.** Now we establish the Čaplygin type inequalities.

Theorem 3.2. *We suppose that*

(i) *the conditions (a), (b) and (c) in Theorem 3.1 are satisfied;*

(ii) $u_i, v_i, \widetilde{u}_i, \widetilde{v}_i \in \mathbb{R}$, $\binom{u_i}{v_i} \leq \binom{\widetilde{u}_i}{\widetilde{v}_i}$, $i = 1,2$ *imply that*

$$f(t, u_1, v_1, u_2, v_2) \leq f(t, \widetilde{u}_1, \widetilde{v}_1, \widetilde{u}_2, \widetilde{v}_2), \ \forall t \in [a,b].$$

Let $\binom{x^*}{z^*}$ *be a solution of (1.2) and* $\binom{y^*}{w^*}$ *be a solution of the system*

$$\begin{cases} y'(t) \le w(t), \ t \in [a,b] \\ w'(t) \le f(t, y(t), w(t), y(t-h), w(t-h)), \ t \in [a,b]. \end{cases}$$

Then

$$\begin{pmatrix} y^* \\ w^* \end{pmatrix}\Bigg|_{[a-h,a]} \le \begin{pmatrix} x^* \\ z^* \end{pmatrix}\Bigg|_{[a-h,a]} \quad \textit{implies that} \begin{pmatrix} y^* \\ w^* \end{pmatrix} \le \begin{pmatrix} x^* \\ z^* \end{pmatrix}.$$

Proof. We have that

$$\begin{pmatrix} x^* \\ z^* \end{pmatrix} = E\begin{pmatrix} x^* \\ z^* \end{pmatrix}, \begin{pmatrix} y^* \\ w^* \end{pmatrix} \le E\begin{pmatrix} y^* \\ w^* \end{pmatrix}.$$

From Theorem 3.1, (c), E is weakly Picard operator. From condition (ii), we obtain that E^∞ is increasing ([12]). So

$$\begin{pmatrix} y^* \\ w^* \end{pmatrix} \le E^\infty\begin{pmatrix} y^* \\ w^* \end{pmatrix} = E^\infty\begin{pmatrix} \widetilde{y}^* \\ \widetilde{w}^* \end{pmatrix} \le E^\infty\begin{pmatrix} \widetilde{x}^* \\ \widetilde{z}^* \end{pmatrix} = \begin{pmatrix} x^* \\ z^* \end{pmatrix}$$

where $\binom{\widetilde{x}^*}{\widetilde{z}^*} \in X_{\binom{x}{z}|_{[a-h,a]}}$. □

3.3. **Data dependence: monotony.** In this subsection we study the monotony of the solution of the problem (1.2)-(1.3) with respect to φ and f.

Theorem 3.3. *(Comparison theorem) Let* $f_i \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$, $i = 1,2,3$, *be as in Theorem 3.1. We suppose that*

(i) $f_1 \le f_2 \le f_3$;

(ii) $f_2(t,\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^4 \to \mathbb{R}$ *is increasing,* $\forall t \in [a,b]$.

Let $\binom{x_i^*}{z_i^*}$ *be a solution of the system*

$$\begin{cases} x'(t) = z(t), \ t \in [a,b] \\ z'(t) = f_i(t, x(t), z(t), x(t-h), z(t-h)), \ t \in [a,b]. \end{cases} \tag{3.1}$$

Then

$$\begin{pmatrix} x_1^* \\ z_1^* \end{pmatrix}\Bigg|_{[a-h,a]} \le \begin{pmatrix} x_2^* \\ z_2^* \end{pmatrix}\Bigg|_{[a-h,a]} \le \begin{pmatrix} x_3^* \\ z_3^* \end{pmatrix}\Bigg|_{[a-h,a]}$$

imply that $\binom{x_1^*}{z_1^*}\Big|_{[a-h,b]} \le \binom{x_2^*}{z_2^*}\Big|_{[a-h,b]} \le \binom{x_3^*}{z_3^*}\Big|_{[a-h,b]}$.

Proof. We consider the operators E_i corresponding to each system (3.1). The operators $E_i, i = 1,2,3$ are weakly Picard operators. Taking into consideration the condition (ii), E_2 is increasing. From (i) we have $E_1 \le E_2 \le E_3$. On the other hand we have that

$$\begin{pmatrix} x_i^* \\ z_i^* \end{pmatrix} = E_i^\infty\begin{pmatrix} \widetilde{x}_i^* \\ \widetilde{z}_i^* \end{pmatrix}, i = 1,2,3.$$

where $\binom{\widetilde{x}^*}{\widetilde{z}^*} \in X_{\binom{x}{z}|_{[a-h,a]}}$. The proof follows from the abstract comparison Lemma (see [12]). $\square$

3.4. **Data dependence: continuity.** Consider the problem (1.2)-(1.3) with the dates $f_i \in C([a,b]\times\mathbb{R}^4,\mathbb{R}), i=1,2$ and suppose that f_i satisfy the conditions from Theorem 3.1 with the same Lipshitz constants. We obtain the data dependence result.

Theorem 3.4. *Let $f_i \in C([a,b]\times\mathbb{R}^4,\mathbb{R}), \varphi_i \in C([a-h,a],\mathbb{R})$, $i=1,2$, be as in Theorem 3.1. We suppose that*

(i) *there exists $\eta_1,\eta_2>0$ such that*

$$\left|\begin{pmatrix}\varphi_1\\ \varphi_1'\end{pmatrix}(t)-\begin{pmatrix}\varphi_2\\ \varphi_2'\end{pmatrix}(t)\right|\le\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},\ \forall t\in[a-h,a];$$

(ii) *there exists $\eta_3>0$ such that*

$$|f_1(t,u_1,u_2,u_3,u_4)-f_2(t,u_1,u_2,u_3,u_4)|\le\eta_3, \forall t\in[a,b], u_i\in\mathbb{R}, i=\overline{1,4}.$$

Then

$$\left\|\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_1,\varphi_1',f_1)-\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right\|\le(I-Q)^{-1}\begin{pmatrix}\eta_1\\ \eta_2+(b-a)\eta_3\end{pmatrix},$$

where $\binom{x^*}{z^*}(\cdot,\varphi,\varphi',f)$ denote the unique solution of (1.2)-(1.3).

Proof. Consider the operators $B_{\varphi_i,f_i}, i=1,2$. From Theorem 3.1 it follows that

$$\left\|B_{\varphi_1,f_1}\begin{pmatrix}x_1\\ z_1\end{pmatrix}-B_{\varphi_1,f_1}\begin{pmatrix}x_2\\ z_2\end{pmatrix}\right\|\le Q\left\|\begin{pmatrix}x_1\\ z_1\end{pmatrix}-\begin{pmatrix}x_2\\ z_2\end{pmatrix}\right\|,\forall\begin{pmatrix}x_i\\ z_i\end{pmatrix}\in X.$$

Additionally,

$$\left\|B_{\varphi_1,f_1}\begin{pmatrix}x\\ z\end{pmatrix}-B_{\varphi_2,f_2}\begin{pmatrix}x\\ z\end{pmatrix}\right\|\le\begin{pmatrix}\eta_1\\ \eta_2+(b-a)\eta_3\end{pmatrix}.$$

Thus

$$\begin{aligned}
&\left\|\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_1,\varphi_1',f_1)-\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right\|\\
&=\left\|B_{\varphi_1,f_1}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_1,\varphi_1',f_1)\right)-B_{\varphi_2,f_2}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right)\right\|\\
&\le\left\|B_{\varphi_1,f_1}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_1,\varphi_1',f_1)\right)-B_{\varphi_1,f_1}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right)\right\|\\
&\quad+\left\|B_{\varphi_1,f_1}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right)-B_{\varphi_2,f_2}\left(\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right)\right\|\\
&\le Q\left\|\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_1,\varphi_1',f_1)-\begin{pmatrix}x^*\\ z^*\end{pmatrix}(\cdot,\varphi_2,\varphi_2',f_2)\right\|+\begin{pmatrix}\eta_1\\ \eta_2+(b-a)\eta_3\end{pmatrix},
\end{aligned}$$

and since $Q^n \to \infty$, as $n \to \infty$ implies that $(I-Q)^{-1} \in M_{22}(\mathbb{R}_+)$, we finally obtain

$$\left\| \begin{pmatrix} x^* \\ z^* \end{pmatrix}(\cdot, \varphi_1, \varphi_1', f_1) - \begin{pmatrix} x^* \\ z^* \end{pmatrix}(\cdot, \varphi_2, \varphi_2', f_2) \right\| \leq (I-Q)^{-1} \begin{pmatrix} \eta_1 \\ \eta_2 + (b-a)\eta_3 \end{pmatrix}.$$

□

3.5. **Data dependence: differentiability.** Consider the following differential system with parameter

$$\begin{cases} x'(t) = z(t),\ t \in [a,b] \\ z'(t) = f(t, x(t), z(t), x(t-h), z(t-h); \lambda),\ t \in [a,b], \lambda \in J, \end{cases} \tag{3.2}$$

with the initial conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a-h, a] \\ z(t) = \varphi'(t),\ t \in [a-h, a], \end{cases} \tag{3.3}$$

where $J \subset \mathbb{R}$ is a compact interval.

Suppose that the following conditions are satisfied:

(C_1) $a < b, h > 0,\ J \subset \mathbb{R}$ a compact interval;
(C_2) $f \in C([a,b] \times \mathbb{R}^4 \times J, \mathbb{R})$;
(C_3) $\varphi \in C^1([a-h, a], \mathbb{R})$;
(C_4) there exists $L_f > 0$, such that

$$\left| \frac{\partial f(t, u_1, u_2, u_3, u_4; \lambda)}{\partial u_i} \right| \leq L_1, u_1, u_3 \in \mathbb{R},\ i = 1, 3, \lambda \in J;$$

$$\left| \frac{\partial f(t, u_1, u_2, u_3, u_4; \lambda)}{\partial u_i} \right| \leq L_2, u_2, u_4 \in \mathbb{R},\ i = 2, 4, \lambda \in J;$$

(C_5) for $Q = (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$ we have $Q^n \to 0$ as $n \to \infty$.

Then, from Theorem 3.1, we have that the problem (1.2)-(1.3) has a unique solution, $\binom{x^*}{z^*} \in C^1([a,b], \mathbb{R}^2)$. We prove that $\binom{x^*}{z^*} \in C^1(J, \mathbb{R}^2),\ \forall\, t \in [a-h, b]$.

For this we consider the system

(3.4)

$$\begin{cases} x'(t;\lambda) = z(t;\lambda),\ t \in [a,b], \lambda \in J \\ z'(t;\lambda) = f(t, x(t;\lambda), z(t;\lambda), x(t-h;\lambda), z(t-h;\lambda); \lambda),\ t \in [a,b], \lambda \in J \end{cases}$$

with $\binom{x^*}{z^*} \in C([a-h, b] \times J, \mathbb{R}^2) \cap C^1([a,b] \times J, \mathbb{R}^2)$.

Theorem 3.5. *Consider the problem (3.4)-(3.3) and suppose the conditions* (C_1)–(C_5) *hold. Then,*

(i) *(3.4)-(3.3) has a unique solution* $\binom{x^*}{z^*}(\cdot, \lambda)$*, in* $C([a-h, b] \times J, \mathbb{R}^2)$;
(ii) $\binom{x^*}{z^*}(\cdot, \lambda) \in C^1(J, \mathbb{R}^2), \forall t \in [a-h, b]$.

Proof. The problem (3.4)-(3.3) is equivalent with the following functional-integral system

(3.5)
$$\begin{pmatrix} x \\ z \end{pmatrix}(t;\lambda) = \begin{pmatrix} \varphi(a) + \int_a^t z(s;\lambda)ds \\ \varphi'(a) + \int_a^t f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)ds \end{pmatrix},$$

for $t \in [a,b]$ and

$$\begin{pmatrix} x \\ z \end{pmatrix}(t;\lambda) = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}(t), \text{ for } t \in [a-h,a]. \tag{3.6}$$

Now let us take the operator $A : C([a-h,b] \times J, \mathbb{R}^2) \to C([a-h,b] \times J, \mathbb{R}^2)$, defined by

$A\begin{pmatrix} x \\ z \end{pmatrix}(t;\lambda) := \begin{pmatrix} A_1\binom{x}{z} \\ A_2\binom{x}{z} \end{pmatrix}$:= the right hand side of (3.5), for $t \in [a,b]$ and

$A\begin{pmatrix} x \\ z \end{pmatrix}(t;\lambda) := \begin{pmatrix} A_1\binom{x}{z} \\ A_2\binom{x}{z} \end{pmatrix}$:= the right hand side of (3.6), for $t \in [a-h,a]$.

Let $X = C([a-h,b] \times J, \mathbb{R}^2)$.

It is clear, from the proof of the Theorem 3.1, that in the condition (C_1)–(C_5), the operator$A : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is Picard operator.

Let $\binom{x^*}{z^*}$ be the unique fixed point of A.

Supposing that there exists $\begin{pmatrix} \frac{\partial x^*}{\partial\lambda} \\ \frac{\partial z^*}{\partial\lambda} \end{pmatrix}$, from (3.5)-(3.6), we obtain that

$$\tfrac{\partial x^*}{\partial\lambda} = \int_a^t \frac{\partial z(s;\lambda)}{\partial\lambda} ds$$

and

$$\begin{aligned}
\tfrac{\partial z^*}{\partial\lambda} = & \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_1} \frac{\partial x(s;\lambda)}{\partial\lambda} ds \\
& + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_2} \frac{\partial z(s;\lambda)}{\partial\lambda} ds \\
& + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_3} \frac{\partial x(s-h;\lambda)}{\partial\lambda} ds \\
& + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_4} \frac{\partial z(s-h;\lambda)}{\partial\lambda} ds \\
& + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial\lambda} ds,
\end{aligned}$$

for all $t \in [a,b], \lambda \in J$.

This relation suggest us to consider the following operator

$$C : X \times X \to X,$$
$$\left(\begin{pmatrix} x_{12} \\ z_{12} \end{pmatrix}, \begin{pmatrix} u_{13} \\ u_{24} \end{pmatrix}\right) \to C\left(\begin{pmatrix} x_{12} \\ z_{12} \end{pmatrix}, \begin{pmatrix} u_{13} \\ u_{24} \end{pmatrix}\right),$$

where $C\left(\binom{x_{12}}{z_{12}}, \binom{u_{13}}{u_{24}}\right)(t;\lambda) = 0$ for $t \in [a-h,a], \lambda \in J$ and

$$C\left(\begin{pmatrix} x_{12} \\ z_{12} \end{pmatrix}, \begin{pmatrix} u_{13} \\ u_{24} \end{pmatrix}\right)(t;\lambda) := \begin{pmatrix} C_1\left(\binom{x_{12}}{z_{12}}, \binom{u_{13}}{u_{24}}\right) \\ C_2\left(\binom{x_{12}}{z_{12}}, \binom{u_{13}}{u_{24}}\right) \end{pmatrix}$$

where

$$C_1\left(\begin{pmatrix} x_{12} \\ z_{12} \end{pmatrix}, \begin{pmatrix} u_{13} \\ u_{24} \end{pmatrix}\right)(t;\lambda) := \int_a^t \frac{\partial z(s;\lambda)}{\partial \lambda} ds$$

and

$$\begin{aligned}
&C_2\left(\begin{pmatrix} x_{12} \\ z_{12} \end{pmatrix}, \begin{pmatrix} u_{13} \\ u_{24} \end{pmatrix}\right)(t;\lambda) \\
&:= \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_1} u_1(s;\lambda) ds \\
&\quad + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_2} u_2(s;\lambda) ds \\
&\quad + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_3} u_3(s-h;\lambda) ds \\
&\quad + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial u_4} u_4(s-h;\lambda) ds \\
&\quad + \int_a^t \frac{\partial f(s, x(s;\lambda), z(s;\lambda), x(s-h;\lambda), z(s-h;\lambda);\lambda)}{\partial \lambda} ds,
\end{aligned}$$

for $t \in [a,b], \lambda \in J$. Here we use the notations $u_1(s;\lambda) := \frac{\partial x(s;\lambda)}{\partial \lambda}$, $u_2(s;\lambda) := \frac{\partial z(s;\lambda)}{\partial \lambda}$, $u_3(s-h;\lambda) := \frac{\partial x(s-h;\lambda)}{\partial \lambda}$ and $u_4(s-h;\lambda) := \frac{\partial z(s-h;\lambda)}{\partial \lambda}$.

In this way we have the triangular operator $D : X \times X \to X \times X$, $\left(\binom{x_{12}}{z_{12}}, \binom{u_{13}}{u_{24}}\right) \to \left(A\binom{x_{12}}{z_{12}}, C\left(\binom{x_{12}}{z_{12}}, \binom{u_{13}}{u_{24}}\right)\right)$, where A is Picard operator and $C(\binom{x_{12}}{z_{12}}, \binom{\cdot}{\cdot}) : X \to X$ is Q_C -contraction with $Q_C = (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$.

From Theorem 2.8 the operator D is Picard operator, i.e. the sequences

$$\begin{pmatrix} x_{12}^{n+1} \\ z_{12}^{n+1} \end{pmatrix} := A \begin{pmatrix} x_{12}^{n} \\ z_{12}^{n} \end{pmatrix},$$
$$\begin{pmatrix} u_{13}^{n+1} \\ u_{24}^{n+1} \end{pmatrix} := C\left(\begin{pmatrix} x_{12}^{n} \\ z_{12}^{n} \end{pmatrix}, \begin{pmatrix} u_{13}^{n} \\ u_{24}^{n} \end{pmatrix} \right),$$

$n \in \mathbb{N}$, converge uniformly, with respect to $t \in [a-h,b],\ \lambda \in J$, to $\left(\binom{x_{12}^n}{z_{12}^n}, \binom{u_{13}^n}{u_{24}^n} \right) \in F_D$, for all $\binom{x_{12}^0}{z_{12}^0},\ \binom{u_{13}^0}{u_{24}^0} \in X$.

If we take $\binom{x_{12}^0}{z_{12}^0} = \binom{0}{0}$ and $\binom{u_{13}^0}{u_{24}^0} = \begin{pmatrix} \frac{\partial x_{12}^0}{\partial \lambda} \\ \frac{\partial z_{12}^0}{\partial \lambda} \end{pmatrix} = \binom{0}{0}$ then $\binom{u_{13}^1}{u_{24}^1} = \begin{pmatrix} \frac{\partial x_{12}^1}{\partial \lambda} \\ \frac{\partial z_{12}^1}{\partial \lambda} \end{pmatrix}$.

By induction we prove that

$$\begin{pmatrix} u_{13}^{n} \\ u_{24}^{n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_{12}^n}{\partial \lambda} \\ \frac{\partial z_{12}^n}{\partial \lambda} \end{pmatrix},\ \forall n \in \mathbb{N}.$$

So, $\binom{x_{12}^n}{z_{12}^n} \overset{unif}{\rightarrow} \binom{x_{12}^*}{z_{12}^*}$, as $n \to \infty$ and $\begin{pmatrix} \frac{\partial x_{12}^n}{\partial \lambda} \\ \frac{\partial z_{12}^n}{\partial \lambda} \end{pmatrix} \overset{unif}{\rightarrow} \binom{u_{13}^*}{u_{24}^*}$, as $n \to \infty$. From a Weierstrass argument we get that there exists $\begin{pmatrix} \frac{\partial x_{12}^*}{\partial \lambda} \\ \frac{\partial z_{12}^*}{\partial \lambda} \end{pmatrix}$, $i = 1,2$ and $\begin{pmatrix} \frac{\partial x_{12}^*}{\partial \lambda} \\ \frac{\partial z_{12}^*}{\partial \lambda} \end{pmatrix} = \begin{pmatrix} u_{13}^* \\ u_{24}^* \end{pmatrix}$.

□

3.6. **Ulam-Hyers stability.** We start this section by presenting the Ulam-Hyers stability concept (see [16], [17]). For $f \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$, $\varepsilon > 0$ and $\psi \in C([a-h,b], \mathbb{R}_+)$, $h > 0$, we consider the system

$$\left\{ \begin{array}{l} x'(t) = z(t),\ t \in [a,b] \\ z'(t) = f(t, x(t), z(t), x(t-h), z(t-h)),\ t \in [a,b] \end{array} \right. \tag{3.7}$$

and the following inequations

$$\left\{ \begin{array}{l} |x'(t) - z(t)| \le \varepsilon,\ t \in [a,b] \\ |z'(t) - f(t, x(t), z(t), x(t-h), z(t-h))| \le \varepsilon,\ t \in [a,b], \end{array} \right. \tag{3.8}$$

$$\left\{ \begin{array}{l} |x'(t) - z(t)| \le \psi(t),\ t \in [a,b] \\ |z'(t) - f(t, x(t), z(t), x(t-h), z(t-h))| \le \psi(t),\ t \in [a,b]. \end{array} \right. \tag{3.9}$$

Definition 3.6. *The system (3.7) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $\binom{y}{w} \in$*

$C^2([a-h,b],\mathbb{R}^2)$ *of (3.8) there exists a solution* $\binom{x}{z} \in C^2([a-h,b],\mathbb{R}^2)$ *of (3.7) with*

$$\left|\begin{pmatrix} y \\ w \end{pmatrix}(t) - \begin{pmatrix} x \\ z \end{pmatrix}(t)\right| \le c\varepsilon, \quad \forall t \in [a-h,b].$$

Theorem 3.7. *We suppose that:*

(i) *the conditions* (C_1)*-*(C_3) *are satisfied;*

(ii) $Q^n \to 0$ *as* $n \to \infty$, *where* $Q = (b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$.

Then the system (1.2) is Ulam-Hyers stable.

Proof. The system (1.2) is equivalent with the functional integral system (1.5). We consider the operator $E : C([a-h,b],\mathbb{R}^2) \to C([a-h,b],\mathbb{R}^2)$, defined by $E\binom{x}{z}(t)$:=the right hand side of (1.5), for $t \in [a-h,b]$. So

$$\begin{pmatrix} x \\ z \end{pmatrix} = E\begin{pmatrix} x \\ z \end{pmatrix}(t), t \in [a-h,b].$$

From Theorem 3.1, E is c-WPO with $c = \left[1-(b-a)\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}\right]^{-1}$.
Applying Theorem 2.7 we obtain that (1.2) is Ulam-Hyers stable. □

Remark 3.8. *Another proof for the above theorem can be done using Gronwall lemma (*[7], [15]-[18]*).*

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