Systems of functional-differential equations with maxima, of mixed type

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Abstract

In this paper we study some properties of the solutions of a second order system of functional-differential equations with maxima, of mixed type, with “boundary” conditions. We use the Perov's fixed point theorem and the weakly Picard operator technique.

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1 Introduction

In the last few decades, much attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to the maximal deviation of the regulated quantity. A classical example is that of an electric generator. In this case, the mechanism becomes active when the maximum voltage variation that is permitted is reached in an interval of time $I_t = [t-h, t]$ with h a positive constant. The equation which describes the action of this regulator has the form

$$V'(t) = -\delta V(t) + p \max_{s \in I_t} V(s) + F(t),$$

where δ and p are constants that are determined by the characteristic of system, $V(t)$ is the voltage and $F(t)$ is the effect of the perturbation that appears associated to the change of voltage [1].

The use of the Perov's fixed point theorem [10, 11] generates an efficient technique to approach systems of functional-differential equations [5, 14].

In the study of existence and uniqueness of the solution of an operatorial equation, the notions of Picard and weakly Picard operators are very useful [11, 13], [15]-[17]. Some applications of the theory of weakly Picard operators can be found in [13]-[17], [3, 4] and [6]-[9]. Some problems concerning differential equations with maxima were studied in [1, 5, 8, 9] and in the monograph [2]. In [8] we have obtained conditions for existence and uniqueness, inequalities of Čaplygin type and data dependence for the solutions of functional-differential equations with maxima while in [9] we apply the technique of weakly Picard operators for the second order functional-differential equations with maxima, of mixed type. Here we continue the work from [8] and [9] with the study of systems of functional-differential equations with maxima, of mixed type.

We consider the following functional-differential system

$$-x''(t) = f\big(t, x(t), \max_{t-h_1\leq\xi\leq t} x(\xi), \max_{t\leq\xi\leq t+h_2} x(\xi)\big),\ t\in[a,b] \tag{1.1}$$

with the "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t\in[a-h_1,a], \\ x(t) = \psi(t),\ t\in[b,b+h_2]. \end{cases} \tag{1.2}$$

Suppose that:

(C$_1$) $h_1, h_2,\ a$ and $b\in\mathbb{R},\ a<b,\ h_1>0,\ h_2>0$;

(C$_2$) $f\in C([a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$;

(C$_3$) there exists a matrix $L_f\in M_{m\times m}(\mathbb{R}_+)$ such that

$$\big|f(t,u^1,u^2,u^3)-f(t,v^1,v^2,v^3)\big| \leq L_f \begin{pmatrix} \max\limits_{1\leq i\leq 3}|u_1^i-v_1^i| \\ \vdots \\ \max\limits_{1\leq i\leq 3}|u_m^i-v_m^i| \end{pmatrix},$$

for all $t\in[a,b]$ and $u^i=(u_1^i,\ldots,u_m^i), v^i=(v_1^i,\ldots,v_m^i)\in\mathbb{R}^m, i=1,2,3$ where

$$|w| := \begin{pmatrix} |w_1| \\ \vdots \\ |w_m| \end{pmatrix};$$

(C$_4$) $\varphi\in C([a-h_1,a],\mathbb{R}^m)$ and $\psi\in C([b,b+h_2],\mathbb{R}^m)$.

Let G be the Green function of the following problem

$$-x''(t) = \chi(t),\ t \in [a,b]$$
$$x(a) = 0,\ x(b) = 0,$$

where $\chi \in C([a,b],\mathbb{R})$. Denoting

$$w(\varphi,\psi)(t) := \tfrac{t-a}{b-a}\psi(b) + \tfrac{b-t}{b-a}\varphi(a),$$

the problem (1.1)–(1.2), with smoothness condition $x \in C([a-h_1,b+h_2],\mathbb{R}^m) \cap C^2([a,b],\mathbb{R}^m)$, is equivalent to the following equation

$$x(t) = \begin{cases} \varphi(t),\ t \in [a-h_1,a], \\ w(\varphi,\psi)(t) + \int_a^b G(t,s) f\big(s, x(s), \max\limits_{s-h_1 \le \xi \le s} x(\xi), \max\limits_{s \le \xi \le s+h_2} x(\xi)\big)ds,\ t \in [a,b], \\ \psi(t),\ t \in [b, b+h_2], \end{cases} \tag{1.3}$$

$x \in C([a-h_1,b+h_2],\mathbb{R}^m)$.

The equation (1.1) is equivalent to

$$x(t) = \begin{cases} x(t),\ t \in [a-h_1,a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t) + \\ \quad + \int_a^b G(t,s) f\big(s, x(s), \max\limits_{s-h_1 \le \xi \le s} x(\xi), \max\limits_{s \le \xi \le s+h_2} x(\xi)\big)ds, t \in [a,b], \\ x(t),\ t \in [b, b+h_2], \end{cases} \tag{1.4}$$

$x \in C([a-h_1,b+h_2],\mathbb{R}^m)$.

In what follows we consider the operators:

$$B_f, E_f : C([a-h_1,b+h_2],\mathbb{R}^m) \to C([a-h_1,b+h_2],\mathbb{R}^m)$$

defined by $B_f(x)(t) :=$ the right hand side of (1.3) and $E_f(x)(t) :=$ the right hand side of (1.4). Let $X := (C[a-h_1,b+h_2],\mathbb{R}^m)$ and $X_{\varphi,\psi} := \{x \in X|\ x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}$. It is clear that $X = \bigcup\limits_{\varphi,\psi} X_{\varphi,\psi}$ is a partition of X.

The following result is known.

Lemma 1.1 [13] *Suppose that the conditions* (C_1), (C_2) *and* (C_4) *are satisfied. Then*

(a) $B_f(X) \subset X_{\varphi,\psi}$ *and* $B_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$;

(b) $B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$.

Let $M_G := (\|G_{ij}\|)_{i,j=\overline{1,m}} \in M_{m\times m}(\mathbb{R}_+)$, where $\|G_{i,j}\| = \max\{|G_{i,j}(x,s)| : (x,s) \in [a,b]\times[a,b]\},\ i,j = \overline{1,m}$ and

$$Q := (b-a)M_G L_f \in M_{m\times m}(\mathbb{R}_+). \tag{1.5}$$

The following is a synopsis of the paper. In Section 2 we introduce notation, definitions and results from weakly Picard operator theory. In Section 3 we obtain existence and uniqueness result using Perov's fixed point theorem and the weakly Picard operator technique. Sections 4 and 5 present inequalities of Čaplygin type and data dependence results.

2 Picard and Weakly Picard operators

In this section, we introduce notation, definitions, and preliminary results which are used throughout this paper (see [12]-[17]). Let (X,d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$.

Definition 2.1 *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2 *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3 *If A is weakly Picard operator then we consider the operator A^∞ defined by $A^\infty : X \to X,\ A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$.*

Remark 2.4 *It is clear that $A^\infty(X) = F_A$.*

Throughout this paper we denote by $M_{m\times m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. A square matrix Q with nonnegative elements is said to be convergent to zero if $Q^k \to 0$ as $k \to \infty$. It is known that the property of being convergent to zero is equivalent to any of the following three conditions (see [12]):

(a) $I - Q$ is nonsingular and $(I-Q)^{-1} = I + Q + Q^2 + \cdots$ (where I stands for the unit matrix of the same order as Q);

(b) the eigenvalues of Q are located inside the unit open disc of the complex plane;

(c) $I - Q$ is nonsingular and $(I - Q)^{-1}$ has nonnegative elements.

We finish this section by recalling the following fundamental result (see, e.g., [10]).

Theorem 2.5 *(Perov's fixed point theorem) Let* (X, d) *with* $d(x, y) \in \mathbb{R}^m$*, be a complete generalized metric space and* $A : X \to X$ *an operator. We suppose that there exists a matrix* $Q \in M_{m\times m}(\mathbb{R}_+)$*, such that*

(i) $d(A(x), A(y)) \le Qd(x, y)$*, for all* $x, y \in X$*;*

(ii) $Q^n \to 0$*, as* $n \to \infty$*.*

Then

(a) $F_A = \{x^*\}$*,*

(b) $A^n(x) = x^*$*, as* $n \to \infty$ *and*

$$d(A^n(x), x^*) \le (I - Q)^{-1}Q^n d(x_0, A(x_0)).$$

3 Existence and uniqueness

Let us consider the problem (1.1)–(1.2). We obtain the following existence and uniqueness theorem.

Theorem 3.1 *Suppose that:*

(i) the conditions (C_1)*–*(C_4) *are satisfied;*

(ii) $Q^n \to 0$*, as* $n \to \infty$*, where* Q *is defined by* (1.5)*.*

Then

(a) the problem (1.1)*–*(1.2) *has a unique solution* $x^* = (x_1^*, \dots, x_m^*) \in C([a - h_1, b + h_2], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m)$*;*

(b) for all $x_0 \in C([a, b], \mathbb{R}^m)$*, the sequence* $(x^n)_{n\in\mathbb{N}}$*, defined by* $x^{n+1} = B_f(x^n)$*, converges uniformly to* x^**, for all* $t \in [a, b]$*, and, moreover*

$$\begin{pmatrix} |x_1^n(t) - x_1^*(t)| \\ \vdots \\ |x_m^n(t) - x_m^*(t)| \end{pmatrix} \le (I - Q)^{-1}Q^n \begin{pmatrix} |x_1^0(t) - x_1^1(t)| \\ \vdots \\ |x_m^0(t) - x_m^1(t)| \end{pmatrix}.$$

Proof. Consider the Banach space $(C([a-h_1,b+h_2],\mathbb{R}^m),\|\cdot\|)$ where $\|\cdot\|$ is the generalized Chebyshev norm,

$$\|u\| := \begin{pmatrix} \|u_1\| \\ \vdots \\ \|u_m\| \end{pmatrix}, \text{ where } \|u_i\| := \max_{a-h_1\le t\le b+h_2} |u_i(t)|\,,\ i=\overline{1,m}.$$

The problem (1.1)–(1.2) is equivalent to the fixed point equation

$$B_f(x)=x,\ x\in C([a-h_1,b+h_2],\mathbb{R}^m).$$

From the condition (C_3) we have, for any $t\in[a,b]$

$$\begin{aligned}
|B_f(x)(t)-B_f(y)(t)| &\le \int_a^b \Big|G(t,s)\Big[f\big(s,x(s),\max_{a-h_1\le\xi\le a} x(\xi),\max_{b\le\xi\le b+h_2} x(\xi)\big)- \\
&\quad - f\big(s,y(s),\max_{a-h_1\le\xi\le a} y(\xi),\max_{b\le\xi\le b+h_2} y(\xi)\big)\Big]\Big|\,ds \\
&\le \int_a^b M_G L_f \max\Big\{|x(s)-y(s)|\,,\Big|\max_{a-h_1\le\xi\le a} x(\xi)-\max_{a-h_1\le\xi\le a} y(\xi)\Big|\,, \\
&\quad \Big|\max_{b\le\xi\le b+h_2} x(\xi)-\max_{b\le\xi\le b+h_2} y(\xi)\Big|\Big\}\,ds \le \\
&\le \int_a^b M_G L_f \max_{a-h_1\le\xi\le b+h_2} |x(\xi)-y(\xi)|\,ds \le \\
&\le (b-a)M_G L_f\ \|x-y\| = Q\,\|x-y\|\,.
\end{aligned}$$

Then $\|B_f(x)-B_f(y)\|\le Q\,\|x-y\|\,,$ for all $x,y\in X$ and by (ii), the operator B_f is Q-contraction. From the Perov's fixed point theorem we have that the operator B_f is PO and has a unique fixed point $x^*=(x_1^*,\dots,x_m^*)\in X$. Since f is continuous, we have that $x^*\in C^2([a,b],\mathbb{R}^m)$ is the unique solution for the problem (1.1)–(1.2). ■

Remark 3.2 *From the proof of Theorem 3.1, it follows that the operator B_f is PO. Since $B_f|_{X_{\varphi,\psi}}=E_f|_{X_{\varphi,\psi}}$ and*

$$X := C([a-h_1,b+h_2],\mathbb{R}^m)=\bigcup_{\varphi,\psi} X_{\varphi,\psi},\ E_f(X_{\varphi,\psi})\subset X_{\varphi,\psi}$$

it follows that the operator E_f is WPO and, moreover $F_{E_f}\cap X_{\varphi,\psi}=\{x^*_{\varphi,\psi}\}$, $\forall\varphi\in C([a-h_1,a],\mathbb{R}^m)$, $\forall\psi\in C([b,b+h_2],\mathbb{R}^m)$, where $x^*_{\varphi,\psi}$ is the unique solution of the problem (1.1)–(1.2).

Example 3.3 *Consider the following system of differential equations with "maxima",*

$$-x''(t)= P^1x(t)+P^2 \max_{t-h_1\leq\xi\leq t} x(\xi)+P^3 \max_{t\leq\xi\leq t+h_2} x(\xi)+ g(t),\ t\in[a,b], \tag{3.1}$$

with the "boundary" conditions

$$\begin{cases} x(t)=\varphi(t),\ t\in[a-h_1,a], \\ x(t)=\psi(t),\ t\in[b,b+h_2], \end{cases} \tag{3.2}$$

where $P^i = \begin{pmatrix} a^i & a^i \\ b^i & b^i \end{pmatrix}$, $a^i, b^i \in \mathbb{N}_+, i=\overline{1,3},\ g\in C[a,b]$. *In this case* $f(t,u^1,u^2,u^3) = P^1u^1+P^2u^2+P^3u^3+g(t),\ t\in[a,b], u^1,u^2,u^3\in\mathbb{R}^2$, $L_f = \begin{pmatrix} a^1+a^2+a^3 & a^1+a^2+a^3 \\ b^1+b^2+b^3 & b^1+b^2+b^3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}_+),\ M_G = (\|G_{ij}\|)_{i,j=1,2} \in M_{2\times2}(\mathbb{R}_+)$, *where* $\|G_{i,j}\| = \max\{|G_{i,j}(x,s)| : (x,s)\in[a,b]\times[a,b]\},\ i,j=1,2$ *and* $Q=(b-a)M_GL_f\in M_{2\times2}(\mathbb{R}_+)$.

Suppose that:

(C_1'*)* $h_1,h_2,$ *a and* $b\in\mathbb{R},\ a<b,\ h_1>0,\ h_2>0$;

(C_2'*)* $a^1+a^2+a^3+b^1+b^2+b^3<1$;

(C_3'*)* $\varphi\in C([a-h_1,a],\mathbb{R}^2)$ *and* $\psi\in C([b,b+h_2],\mathbb{R}^2)$.

Theorem 3.1 can be now applied, since all its assumptions are verified.

4 Inequalities of Čaplygin type

In order to establish the Čaplygin type inequalities we need the following abstract result.

Lemma 4.1 *(see [15]) Let* $(X,d,\leq)$ *be an ordered metric space and* $A: X\to X$ *an operator. Suppose that* A *is increasing and WPO. Then the operator* A^∞ *is increasing.*

Now we consider the operators E_f and B_f on the ordered Banach space $(C([a-h_1,b+h_2],\mathbb{R}^m),\|\cdot\|,\leq)$ where we consider the following order relation on $\mathbb{R}^m$: $x\leq y \Leftrightarrow x_i\leq y_i,\ i=\overline{1,m}$.

Theorem 4.2 *Suppose that:*

(a) the conditions $(C_1)-(C_4)$ *are satisfied;*

(b) $Q^n \to 0$*, as* $n \to \infty$*, where* Q *is defined by (1.5);*

(c) $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ *is increasing,* $\forall t \in [a,b]$.

Let x *be a solution of equation* (1.1) *and* y *a solution of the inequality*

$$-y''(t) \leq f\big(t, y(t), \max_{t-h_1 \leq \xi \leq t} y(\xi), \max_{t \leq \xi \leq t+h_2} y(\xi)\big),\ t \in [a,b].$$

Then $y(t) \leq x(t), \forall t \in [a-h_1, a] \cup [b, b+h_2]$ *implies that* $y \leq x$.

Proof. Let us consider the operator $\widetilde{w} : C([a-h_1, b+h_2], \mathbb{R}^m) \to C([a-h_1, b+h_2], \mathbb{R}^m)$ defined by

$$\widetilde{w}(z)(t) := \begin{cases} z(t), & t \in [a-h_1, a], \\ w(z|_{[a-h_1,a]}, z|_{[b,b+h_2]})(t), & t \in [a,b], \\ z(t), & t \in [b, b+h_2], \end{cases}$$

for $z \in C([a-h_1, b+h_2], \mathbb{R}^m)$. First of all we remark that $w(y|_{[a-h_1,a]}, y|_{[b,b+h_2]}) \leq w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})$ and $\widetilde{w}(y) \leq \widetilde{w}(x)$.

In the terms of the operator E_f, we have $x = E_f(x)$ and $y \leq E_f(y)$. From Remark 3.2 we have that E_f is WPO. On the other hand, from the condition (c) and Lemma 4.1 we get that the operator E_f^∞ is increasing. Hence $y \leq E_f(y) \leq E_f^2(y) \leq \ldots \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x$. So, $y \leq x$. ■

5 Data dependence: monotony

In order to study the monotony of the solution of the problem (1.1)–(1.2) with respect to φ, ψ and f, we need the following result from the WPOs theory.

Lemma 5.1 *(Abstract comparison lemma, [16]) Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C : X \to X$ *be such that:*

(i) the operator A, B, C *are WPOs;*

(ii) $A \leq B \leq C$;

(iii) the operator B *is increasing.*

Then $x \leq y \leq z$ *implies that* $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

From this abstract result we obtain the following result:

Theorem 5.2 *Let $f^i \in C([a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m), i=\overline{1,3}$, and suppose that conditions (C_1)–(C_4) hold. Furthermore suppose that:*

(i) $f^1 \le f^2 \le f^3$;

(ii) $f^2(t,\cdot,\cdot,\cdot):\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m$ *is increasing.*

Let x^i be a solution of the equation

$$-(x^i)''(t) = f^i\big(t, x(t), \max_{t-h_1\le\xi\le t} x(\xi), \max_{t\le\xi\le t+h_2} x(\xi)\big),\ t\in[a,b] \text{ and } i=\overline{1,3}.$$

Then $x^1(t)\le x^2(t)\le x^3(t)$, $\forall t\in[a-h_1,a]\cup[b,b+h_2]$, implies $x^1\le x^2\le x^3$, i.e. the unique solution of the problem (1.1)–(1.2) *is increasing with respect to f, φ and ψ.*

Proof. From Remark 3.2, the operators $E_{f^i}, i=\overline{1,3}$, are WPOs. From the condition (ii) the operator E_{f^2} is monotone increasing. From the condition (i) it follows that $E_{f^1}\le E_{f^2}\le E_{f^3}$. On the other hand, we notice that $\widetilde{w}(x^1)\le\widetilde{w}(x^2)\le\widetilde{w}(x^3)$ and $x^i = E^\infty_{f^i}(\widetilde{w}(x^i)),\ i=\overline{1,3}$. So, the proof follows from Lemma 5.1. ∎

6 Data dependence: continuity

Consider the boundary value problem (1.1)–(1.2) and suppose that the conditions of the Theorem 3.1 are satisfied with the same Lipshitz constants. Denote by $x^*(\cdot;\varphi,\psi,f)$ the solution of this problem. We get the data dependence result.

Theorem 6.1 *Let $\varphi^i,\psi^i,f^i, i=1,2$ satisfy the conditions (C_1)–(C_4). Furthermore, we suppose that there exists $\eta_i\in\mathbb{R}^m_+, i=1,2$ such that*

(i) $|\varphi^1(t)-\varphi^2(t)|\le\eta_1,\ \forall t\in[a-h_1,a]$ *and* $|\psi^1(t)-\psi^2(t)|\le\eta_1,\ \forall t\in[b,b+h_2]$;

(ii) $|f^1(t,u^1,u^2,u^3)-f^2(t,u^1,u^2,u^3)|\le\eta_2,\ \forall t\in[a,b], u^i\in\mathbb{R}^m, i=1,2,3.$

Then

$$\big\|x^*(t;\varphi^1,\psi^1,f^1)-x^*(t;\varphi^2,\psi^2,f^2)\big\|\le(I-Q)^{-1}(2\eta_1+M_G(b-a)\eta_2),$$

where $x^*(t;\varphi^i,\psi^i,f^i)$ is the solution of the problem (1.1)–(1.2) *with respect to $\varphi^i,\psi^i,f^i,\ i=1,2$.*

Proof. Consider the operators B_{φ^i,ψ^i,f^i}, $i=1,2$. From Theorem 3.1 it follows that
$$\|B_{\varphi^1,\psi^1,f^1}(x) - B_{\varphi^1,\psi^1,f^1}(y)\| \leq Q\,\|x-y\|\,,\ \forall x,y \in X.$$
Additionally

$$\|B_{\varphi^1,\psi^1,f^1}(x) - B_{\varphi^2,\psi^2,f^2}(x)\| \leq 2\eta_1 + M_G(b-a)\eta_2,\ \forall x \in X.$$

We have

$$\begin{aligned}
&\left\|x^*(t;\varphi^1,\psi^1,f^1) - x^*(t;\varphi^2,\psi^2,f^2)\right\| = \\
&= \left\|B_{\varphi^1,\psi^1,f^1}(x^*(t;\varphi^1,\psi^1,f^1)) - B_{\varphi^2,\psi^2,f^2}(x^*(t;\varphi^2,\psi^2,f^2))\right\| \leq \\
&\leq \left\|B_{\varphi^1,\psi^1,f^1}(x^*(t;\varphi^1,\psi^1,f^1)) - B_{\varphi^1,\psi^1,f^1}(x^*(t;\varphi^2,\psi^2,f^2))\right\| + \\
&\quad + \left\|B_{\varphi^1,\psi^1,f^1}(x^*(t;\varphi^2,\psi^2,f^2)) - B_{\varphi^2,\psi^2,f^2}(x^*(t;\varphi^2,\psi^2,f^2))\right\| \leq \\
&\leq Q\left\|x^*(t;\varphi^1,\psi^1,f^1) - x^*(t;\varphi^2,\psi^2,f^2)\right\| + 2\eta_1 + M_G(b-a)\eta_2,
\end{aligned}$$

and since $Q^n \to 0$, as $n \to \infty$, implies that $(I-Q)^{-1} \in M_{m\times m}(\mathbb{R}_+)$, we finally obtain

$$\left\|x^*(t;\varphi^1,\psi^1,f^1) - x^*(t;\varphi^2,\psi^2,f^2)\right\| \leq (I-Q)^{-1}(2\eta_1 + M_G(b-a)\eta_2).$$

■

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