A HISTORY-DEPENDENT CONTACT PROBLEM WITH UNILATERAL CONSTRAINT

ANCA FARCAS* FLAVIUS PATRULESCU †

MIRCEA SOFONEA‡

Abstract

We consider a mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The contact is frictionless and is modelled with a new and nonstandard condition which involves both normal compliance, unilateral constraint and memory effects. We derive a variational formulation of the problem then we prove its unique weak solvability. The proof is based on arguments on history-dependent variational inequalities.

Keywords: viscoplastic material, frictionless contact, unilateral constraint, history-depdendent variational inequality, weak solution.

1 The model

We consider a viscoplastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that meas$(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1

*anca.farcas@ubbcluj.ro, University Babeş-Bolyai, 400110 Cluj-Napoca, Romania

†fpatrulescu@ictp.acad.ro, Tiberiu Popoviciu Institute of Numerical Analysis P.O. Box 68-1 and University Babeş-Bolyai, 400110 Cluj-Napoca, Romania

‡sofonea@univ-perp.fr, Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66 860 Perpignan, France

and d and an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $v_{i,j} = \partial v_i/\partial x_j$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$, is fixed on Γ_1, and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the problem is quasistatic and the time interval of interest is $\mathbb{R}_+ = [0,\infty)$. Everywhere in this paper the dot above a variable represents derivative with respect to the time variable, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ and r^+ is the positive part of r, i.e. $r^+ = \max\{0, r\}$. The classical formulation of the problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that, for all* $t \in \mathbb{R}_+$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\ \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ \xi(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{7}$$

Equation (1) represents the viscoplastic constitutive law of the material in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized stress tensor, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given constitutive function. Equation (2) is the equilibrium equation in which Div denotes the divergence operator for tensor valued functions. Conditions (3) and (4) are the displacement and traction boundary conditions, respectively, and condition (5) represents the contact condition with normal compliance, unilateral constraint and memory term, in which σ_ν denotes the normal stress, u_ν is the normal displacement, $g \ge 0$ and p, b

are given functions. In the case when b vanishes, this condition was used in [1, 3], for instance. Condition (6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

Quasistatic frictionless and frictional contact problems for viscoplastic materials with a constitutive law of the form (1) have been studied in various papers, see [2] for a survey. There, various models of contact were stated and their variational analysis, including existence and uniqueness results, was provided. The novelty of the current paper arises on the contact condition (5); it describes a deformable foundation which becomes rigid when the penetration reaches the critical bound g and which develops memory effects. Considering such condition leads to a new and nonstandard mathematical model which, in a variational formulation, is governed by a history-dependent variational inequality for the displacement field.

The rest of the paper is structured as follows. In Section 2 we list the assumptions on the data and introduce the variational formulation of the problem. Then, in Section 3 we state our main result, Theorem 1, and provide a sketch of the proof.

2 Variational formulation

In the study of problem $\mathcal{P}$ we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. Also, we denote by "$\cdot$" and $\|\cdot\|$ the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values on X and, for a subset $K\subset X$, we still use the symbol $C(\mathbb{R}_+;K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values on K. We also consider the spaces

$$V=\{\,\boldsymbol{v}\in H^1(\Omega)^d:\boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\,\},\ Q=\{\,\boldsymbol{\tau}=(\tau_{ij})\in L^2(\Omega)^d:\tau_{ij}=\tau_{ji}\,\}.$$

These are Hilbert spaces together with the inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$,

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,\qquad(\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx,$$

and the associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$, respectively. For an element $\boldsymbol{v}\in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν the normal component of $\boldsymbol{v}$ on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$.

We assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$ and the normal compliance function p satisfy the following conditions.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E}=(\mathcal{E}_{ijkl}):\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) } \mathcal{E}_{ijkl}=\mathcal{E}_{klij}=\mathcal{E}_{jikl}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d.\\ \text{(c) There exists } m_{\mathcal{E}}>0 \text{ such that}\\ \qquad \mathcal{E}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2\ \ \forall\,\boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. in } \Omega.\end{array}\right. \tag{8}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{G}}>0 \text{ such that}\\ \qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{G}}\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\right)\\ \qquad\quad \forall\,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \qquad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } Q.\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l}\text{(a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+.\\ \text{(b) There exists } L_p>0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p\,|r_1-r_2|\ \forall\,r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\ge 0\ \forall\,r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\ \qquad \text{for any } r\in\mathbb{R}.\\ \text{(e) } p(\boldsymbol{x},r)=0 \text{ for all } r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{10}$$

Moreover, the densities of body forces and surface tractions, the memory function and the initial data are such that

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d),\qquad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d). \tag{11}$$

$$b\in C(\mathbb{R}_+;L^\infty(\Gamma_3)),\qquad b(t,\boldsymbol{x})\ge 0\quad \text{a.e. } \boldsymbol{x}\in\Gamma_3, \tag{12}$$

$$\boldsymbol{u}_0\in V,\quad \boldsymbol{\sigma}_0\in Q. \tag{13}$$

Consider now the subset $U\subset V$, the operators $P:V\to V$, $\mathcal{B}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;L^2(\Gamma_3))$ and the function $\boldsymbol{f}:\mathbb{R}_+\to V$ defined by

$$U=\{\,\boldsymbol{v}\in V: v_\nu\le g \text{ on } \Gamma_3\,\}, \tag{14}$$

$$(P\boldsymbol{u},\boldsymbol{v})_V=\int_{\Gamma_3}p(u_\nu)v_\nu\,da\quad\forall\,\boldsymbol{u},\,\boldsymbol{v}\in V, \tag{15}$$

$$(\mathcal{B}\boldsymbol{u}(t),\xi)_{L^2(\Gamma_3)}=\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds,\xi\Big)_{L^2(\Gamma_3)} \tag{16}$$
$$\forall\,\boldsymbol{u}\in C(\mathbb{R}_+;V),\ \xi\in L^2(\Gamma_3),\ t\in\mathbb{R}_+,$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V=\int_\Omega\boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da\quad\forall\,\boldsymbol{v}\in V,\ t\in\mathbb{R}_+. \tag{17}$$

Then, the variational formulation of Problem $\mathcal{P}$ is the following.

Problem $\mathcal{P}_V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{18}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &\quad +(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{19}$$

Note that (18) is a consequence of (1) and (7), while (19) can be easily obtained by using integrations by parts, (2)–(6) and notation (14)–(17).

3 Existence and uniqueness

The unique solvability of Problem $\mathcal{P}_V$ is given by the following result.

Theorem 1 *Assume that* (8)–(13) *hold. Then Problem $\mathcal{P}_V$ has a unique solution, which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.*

Proof. The proof is carried out in several steps which we describe below.

(i) We use the Banach fixed point argument to prove that for each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ such that

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+.$$

(ii) Next, we note that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem $\mathcal{P}_V$ iff

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t) \quad \forall\, t \in \mathbb{R}_+, \tag{20}$$

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\quad +(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,\ \forall\, t \in \mathbb{R}_+. \end{aligned} \tag{21}$$

(iii) Let $A : V \to V$ and $\varphi : Q \times L^2(\Gamma_3) \times V \to \mathbb{R}$ be defined by equalities

$$\begin{aligned} &(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V, \\ &\varphi(x, \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\xi, v_\nu^+)_{L^2(\Gamma_3)} \end{aligned}$$

for all $\boldsymbol{u},\ \boldsymbol{v} \in V$, $x = (\boldsymbol{\sigma}, \xi) \in Q \times L^2(\Gamma_3)$. We prove that $A : V \to V$ is a strongly monotone and Lipschitz continuous operator and there exists $\beta \geq 0$ such that

$$\begin{aligned}&\varphi(x_1, \boldsymbol{u}_2) - \varphi(x_1, \boldsymbol{u}_1) + \varphi(x_2, \boldsymbol{u}_1) - \varphi(x_2, \boldsymbol{u}_2)\\ &\quad \leq \beta \, \|x_1 - x_2\|_{Q\times L^2(\Gamma_3)} \, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall\, x_1,\, x_2 \in Q \times L^2(\Gamma_3),\ \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in V.\end{aligned}$$

Moreover, we prove that for every $n \in \mathbb{N}$ there exists $s_n > 0$ such that

$$\begin{aligned}&\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q + \|\mathcal{B}\boldsymbol{u}_1(t) - \mathcal{B}\boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)}\\ &\quad \leq s_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \quad \forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].\end{aligned}$$

These properties allow to use Theorem 2 in [3]. In this way we prove the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ which satisfies the history-dependent variational inequality (21), for all $t \in \mathbb{R}_+$.

(iv) Let $\boldsymbol{\sigma}$ be the function given by (20); then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfies (20)–(21) for all $t \in \mathbb{R}_+$ and, moreover, it has the regularity $\boldsymbol{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. This concludes the existence part in Theorem 1. The uniqueness part follows from the uniqueness of the solution of the inequality (21), guaranteed by Theorem 2 in [3]. □

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4 Proofs

Variational formulation. Assume in what follows that $(\boldsymbol{u},\boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (1)–(7) and let $\boldsymbol{v}\in U$ and $t\in\mathbb{R}_+$ be given. We integrate equation (1) with the initial conditions (7) to obtain

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{22}$$

Next, we use Green formula and the equilibrium equation (2) to see that

$$\int_\Omega\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx=\int_\Omega\boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx+\int_\Gamma\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da.$$

We split the boundary integral over Γ_1, Γ_2 and Γ_3 and, since $\boldsymbol{v}-\boldsymbol{u}(t)=\boldsymbol{0}$ a.e. on Γ_1, and $\boldsymbol{\sigma\nu}=\boldsymbol{f}_2(t)$ a.e. on Γ_2 we deduce that

$$\int_\Omega\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx=\int_\Omega\boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx$$
$$+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da+\int_{\Gamma_3}\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da.$$

Moreover, since

$$\boldsymbol{\sigma\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))=\sigma_\nu(v_\nu-u_\nu(t))+\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\quad\text{a.e. on }\Gamma_3,$$

taking into account (6) we obtain

$$\int_\Omega\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx=\int_\Omega\boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx \tag{23}$$
$$+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da+\int_{\Gamma_3}\sigma_\nu(v_\nu-u_\nu(t))\,da.$$

We write now

$$\sigma_\nu(t)(v_\nu-u_\nu(t))=\big(\sigma_\nu(t)+p(u_\nu(t))+\xi(t)\big)(v_\nu-g) \tag{24}$$
$$+\big(\sigma_\nu(t)+p(u_\nu(t))+\xi(t))(g-u_\nu(t)\big)$$
$$-\big(p(u_\nu(t))+\xi(t)\big)(v_\nu-u_\nu(t))\quad\text{a.e. on }\Gamma_3,$$

then we use the contact conditions (5) and the definition (14) of the set U to see that

$$\big(\sigma_\nu(t)+p(u_\nu(t))+\xi(t)\big)(v_\nu-g)\geq 0, \tag{25}$$

$$\big(\sigma_\nu(t)+p(u_\nu(t))+\xi(t))(g-u_\nu(t)\big)=0 \tag{26}$$

$$\xi(t)(v_\nu-u_\nu(t))\leq\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds\Big)(v_\nu^+-u_\nu^+(t)) \tag{27}$$

a.e. on Γ_3. We combine (24)–(27) to deduce that

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \geq -p(u_\nu(t))(v_\nu - u_\nu(t))$$
$$-\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds\Big)(v_\nu^+ - u_\nu^+(t)) \quad \text{a.e. on } \Gamma_3$$

and, therefore,

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\,da \geq -\int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\,da \qquad (28)$$
$$-\int_{\Gamma_3}\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds\Big)(v_\nu^+ - u_\nu^+(t))\,da.$$

Then, combining (23) and (28) and using notation (15)–(17) we obtain

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \qquad (29)$$
$$+(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

The variational formulation $\mathcal{P}_V$ is now a consequence of (22) and (29).

Proof of Theorem 1. We need some preliminary results.

Theorem 2 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator with the following property: there exists $c > 0$ such that*

$$\|\Lambda u(t) - \Lambda v(t)\|_X \leq c \int_0^t \|u(s) - v(s)\|_X\,ds \qquad (30)$$

for all u, $v \in C(\mathbb{R}_+; X)$ and for all $t \in \mathbb{R}_+$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Theorem 2 represents a simplified version of Corollary 2.5 in the paper

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Note that in (30) and below, the notation $\Lambda\eta(t)$ represents the value of the function $\Lambda\eta$ at the point t, i.e. $\Lambda\eta(t) = (\Lambda\eta)(t)$.

Let X be a real Hilbert space with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$ and let Y be a normed space with the norm $\|\cdot\|_Y$. Assume given a set $K \subset X$, the operators $A : X \to X$, $\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$, the functional $\varphi : Y \times X \to \mathbb{R}$ and a function $f : \mathbb{R}_+ \to X$ such that:

$$K \text{ is a closed, convex, nonempty subset of } X. \tag{31}$$

$$\left\{\begin{array}{l} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m\, \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\ \text{(b) There exists } L > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq L\, \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X. \end{array}\right. \tag{32}$$

$$\left\{\begin{array}{l} \text{(a) For all } y \in Y,\ \varphi(y,\cdot) : X \to \mathbb{R} \text{ is convex and lsc.} \\ \text{(b) There exists } \alpha > 0 \text{ such that} \\ \qquad \varphi(y_1, u_2) - \varphi(y_1, u_1) + \varphi(y_2, u_1) - \varphi(y_2, u_2) \\ \qquad \leq \alpha\, \|y_1 - y_2\|_Y\, \|u_1 - u_2\|_X \quad \forall\, y_1,\, y_2 \in Y,\ \forall\, u_1,\, u_2 \in X. \end{array}\right. \tag{33}$$

$$\left\{\begin{array}{l} \text{For every } n \in \mathbb{N} \text{ there exists } r_n > 0 \text{ such that} \\ \qquad \|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_Y \leq r_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X\, ds \\ \qquad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{array}\right. \tag{34}$$

$$f \in C(\mathbb{R}_+; X). \tag{35}$$

We have the following result.

Theorem 3 *Assume that* (31)–(35) *hold. Then there exists a unique function* $u \in C(\mathbb{R}_+; K)$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$\begin{aligned} (Au(t), v - u(t))_X + \varphi(\mathcal{R}u(t), v) - \varphi(\mathcal{R}u(t), u(t)) \\ \geq (f(t), v - u(t))_X \qquad \forall\, v \in K. \end{aligned} \tag{36}$$

Theorem 3 represents a simplified version of Theorem 2 in [3].

We turn to the main steps of the proof of Theorem 1.

Lemma 1 *For each function* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ *such that*

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{37}$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ *satisfies the following property: for every* $n \in \mathbb{N}$ *there exists* $s_n > 0$ *such that*

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \leq s_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \tag{38}$$
$$\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Proof. Let $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and consider the operator $\Lambda : C(\mathbb{R}_+; Q) \to C(\mathbb{R}_+; Q)$ defined as follows

$$\Lambda\boldsymbol{\tau}(t) = \int_0^t \mathcal{G}(\boldsymbol{\tau}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \tag{39}$$
$$\forall\, \boldsymbol{\tau} \in C(\mathbb{R}_+; Q),\ \ t \in \mathbb{R}_+.$$

The operator Λ depends on $\boldsymbol{u}$ but, for simplicity, we do not indicate explicitly this dependence.

Let $\boldsymbol{\tau}_1,\, \boldsymbol{\tau}_2 \in C(\mathbb{R}_+; Q)$ and let $t \in \mathbb{R}_+$. Then, using (39) and (9) we have

$$\begin{aligned}
&\|\Lambda\boldsymbol{\tau}_1(t) - \Lambda\boldsymbol{\tau}_2(t)\|_Q \\
&= \Big\| \int_0^t \mathcal{G}(\boldsymbol{\tau}_1(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))ds - \int_0^t \mathcal{G}(\boldsymbol{\tau}_2(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \Big\|_Q \\
&\leq L_{\mathcal{G}} \int_0^t \|\boldsymbol{\tau}_1(s) - \boldsymbol{\tau}_2(s)\|_Q \, ds.
\end{aligned}$$

Next, we use Theorem 2 to see that Λ has a unique fixed point in $C(\mathbb{R}_+; Q)$, denoted $\mathcal{S}\boldsymbol{u}$. And, finally, we combine (39) with equality $\Lambda(\mathcal{S}\boldsymbol{u}) = \mathcal{S}\boldsymbol{u}$ to see that (37) holds.

To proceed, let $\boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V)$, $n \in \mathbb{N}$ and let $t \in [0, n]$. Then, using (37) and taking into account (8), (9) we write

$$\begin{aligned}
&\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \\
&\qquad \leq \mathcal{K}\Big(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_1(s)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2(s))\|_Q \, ds + \int_0^t \|\mathcal{S}\boldsymbol{u}_1(s) - \mathcal{S}\boldsymbol{u}_2(s)\|_Q \, ds \Big) \\
&\qquad = \mathcal{K}\Big(\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds + \int_0^t \|\mathcal{S}\boldsymbol{u}_1(s) - \mathcal{S}\boldsymbol{u}_2(s)\|_Q \, ds \Big),
\end{aligned}$$

where $\mathcal{K}$ is a positive constant which depends on $\mathcal{G}$ and $\mathcal{E}$. Using now a Gronwall argument we deduce that

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \leq \mathcal{K}\, e^{n\,\mathcal{K}} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds.$$

This inequality shows that (38) holds with $s_n = \mathcal{K}\, e^{n\mathcal{K}}$. □

Next, we use the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ defined in Lemma 1 to obtain the following equivalence result.

Lemma 2 *Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a couple of functions such that $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Then, $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem $\mathcal{P}_V$ if and only if* (20) *and* (21) *hold.*

Proof. Assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem $\mathcal{P}_V$ and let $t \in \mathbb{R}_+$. Using (18) we have

$$\boldsymbol{\sigma}(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)$$

and, using the definition (37) of the operator $\mathcal{S}$, we obtain (20). Moreover, combining (19) and (20) we deduce (21).

Conversely, assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfies (20) and (21). Then by (20) and the definition (37) of the operator $\mathcal{S}$ we obtain (18). Moreover, using (20) and (21) we find (19) which concludes the proof. □

We are now in position to provide the proof of Theorem 1.

Proof. We define the operator $A : V \to V$ and the form $\varphi : Q \times L^2(\Gamma_3) \times V \to \mathbb{R}$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V \tag{40}$$
$$\varphi(x, \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\xi, v_\nu^+)_{L^2(\Gamma_3)} \tag{41}$$

for all $\boldsymbol{u},\ \boldsymbol{v} \in V$, $x = (\boldsymbol{\sigma}, \xi) \in Q \times L^2(\Gamma_3)$. We also consider the operator $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Gamma_3))$ defined by

$$\mathcal{R}\boldsymbol{u}(t) = (\mathcal{S}\boldsymbol{u}(t), \mathcal{B}\boldsymbol{u}(t)) \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V) \tag{42}$$

where, recall, $\mathcal{S}$ and $\mathcal{B}$ are the operators given in (37) and (16), respectively.

With this notation we consider the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$\begin{aligned}(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \varphi(\mathcal{R}\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\mathcal{R}\boldsymbol{u}(t), \boldsymbol{u}(t))&\\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in V.&\end{aligned} \tag{43}$$

To solve (43) we employ Theorem 3 with $X = V$, $K = U$ and $Y = Q \times L^2(\Gamma_3)$. It is clear that (31) holds. Next, we use (8), (10) and the Sobolev trace theorem to see that the operator A is strongly monotone and Lipschitz continuous, i.e. it verifies condition (32). In addition, we note that the functional φ satisfies condition (33). We also use (38), assumption (12) and definition (42) to see that (34) holds, too. Finally, using (11) and (17) we deduce that $\boldsymbol{f}$ has the regularity expressed in (35). It follows now from Theorem 3 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which solves the inequality (43), for any $t \in \mathbb{R}_+$.

Based on the results above we deduce the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which satisfies (21). Let $\boldsymbol{\sigma}$ be the function given by (20); then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfies (20)–(21) for all $t \in \mathbb{R}_+$ and, moreover, it has the regularity $\boldsymbol{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. This concludes the existence part in Theorem 1. The uniqueness part follows from the uniqueness of the solution of the inequality (21), guaranteed by Theorem 3. □