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An Elastic Contact Problem with Normal Compliance and Memory Term

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Abstract

We consider a history-dependent problem which describes the contact between an elastic body and an obstacle, the so-called foundation. The contact is frictionless and is modeled with a version of the normal compliance condition in which the memory effects are taken into account. The mathematical analysis of the problem, including existence, uniqueness and convergence results, was provided in (Barboteu et al., in preparation). Here we present the analytic expression of the solution and numerical simulations, in the study of one and two-dimensional examples, respectively.

1. Introduction

Phenomena of contact between deformable bodies abound in industry and everyday life. Owing to their inherent complexity, they lead to nonlinear and nonsmooth mathematical models. To construct a mathematical model which describes a specific contact process we need to precise the material's behavior and the contact conditions, among others. In this paper we model the behavior of the material with an elastic constitutive law of the form

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}) \tag{1}$$

where $\mathbf{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor and $\mathcal{F}$ is a nonlinear constitutive function. Also, we use the contact condition

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$$-\sigma_\nu(t) = p(u_\nu(t)) + \int_0^t b(t-s)u_\nu^+(s)\mathrm{d}s \qquad (2)$$

in which σ_ν and u_ν are the normal stress and the normal displacement, respectively, p and b are given functions, and r^+ denotes the positive part of r, i.e. $r^+ = \max\{0, r\}$.

The contact condition (2) was first introduced by Petrulescu (2011). It shows that at each time moment $t \in \mathbb{R}_+$ the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^M(t)$$

in which the functions σ_ν^D and σ_ν^M are given by

$$-\sigma_\nu^D(t) = p(u_\nu(t)), \quad -\sigma_\nu^M(t) = \int b(t-s)u_\nu^+(s)\mathrm{d}s\,.$$

It follows from above that σ_ν^D satisfies a normal compliance contact condition and, therefore, it describes the deformability of the foundation. Also, σ_ν^M satisfies a history dependent condition and, therefore, it describes the memory properties of the foundation. Moreover, we conclude from (2) that at each moment t, the reaction of the foundation depends both on the current value of the penetration (represented by the term $p(u_\nu(t))$) as well as on the history of the penetration (represented by the integral term).

The analysis of various frictional and frictionless contact problems with elastic materials was provided in many papers, see for instance (Han and Sofonea, 2002; Shillor et al., 2004; Sofonea and Matei, 2012) and the references therein. In particular, results on the unique weak solvability can be found in (Shillor et al., 2004; Sofonea and Matei, 2012); fully discrete schemes for the numerical approximation of the models, including error estimates and numerical simulations can be found in (Han and Sofonea, 2002). In all these papers the mechanical process was studied in a finite interval of time and the contact was modeled either with the normal compliance condition or the Signorini condition, or was assumed to be bilateral.

A mathematical model constructed by using the constitutive law (1) and the contact condition (2) has been considered by Barboteu et al., (in preparation). There, the unique weak solvability of the model was proved by using arguments on history-dependent variational inequalities obtained by Sofonea and Matei (2011). Various convergence results were proved and their numerical validation was also provided. The current paper represents a continuation of (Barboteu et al., in preparation). Its aim is to present additional results in the study of the mathematical model considered there, which highlight the behavior of the solution of the corresponding history-dependent contact problem.

The manuscript is structured as follows. In section 2 we describe the physical setting and present the classical formulation of the problem. Then, in section 3 we present the analytic expression of the solution in the one-dimensional case and, in section 4, we present numerical simulations in the study of an academic two-dimensional example. Finally, we end this paper with section 5, in which we present some conclusions, comments and problems for further research.

2. The model

The physical setting is as follows. An elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (d = 1, 2, 3) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that meas $(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\mathbf{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\mathbf{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the contact process is history-dependent, and we study it in the interval of time $\mathbb{R}_+$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. We use the notation $\mathbf{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\mathbf{x}$, e.g. $u_{i,j} = \partial u_i/\partial x_j$. Also, $\boldsymbol{\varepsilon}$ and Div will represent the deformation and the divergence operators, respectively, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}) = (\varepsilon_{ij}(\boldsymbol{\upsilon})), \quad \varepsilon_{ij}(\boldsymbol{\upsilon}) = 0.5(\upsilon_{i,j} + \upsilon_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

We denote by υ_ν and $\boldsymbol{\upsilon}_\tau$ the normal and tangential components of $\boldsymbol{\upsilon}$ on Γ given by $\upsilon_\nu = \boldsymbol{\upsilon}\cdot\boldsymbol{\nu}$, $\boldsymbol{\upsilon}_\tau = \boldsymbol{\upsilon} - \upsilon_\nu\boldsymbol{\nu}$. Also, for a regular stress function $\boldsymbol{\sigma}$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces on the boundary, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

With these preliminaries, the classical formulation of the contact problem is as follows.

Problem $\mathcal{P}$

Find a displacement field $\mathbf{u} : \Omega\times\mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega\times\mathbb{R}_+ \to \mathbb{S}^d$ *such that, for each* $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{3}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega \times \mathbb{R}_+ \tag{4}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1 \times \mathbb{R}_+ \tag{5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2 \times \mathbb{R}_+ \tag{6}$$

$$\sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)u_\nu^+(s)\mathrm{d}s = 0 \quad \text{on } \Gamma_3 \times \mathbb{R}_+ \tag{7}$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \quad \text{on } \Gamma_3 \times \mathbb{R}_+ \tag{8}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x}$. Equation (3) represents the elastic constitutive law of the material, see (1). Equation (4) is the equation of equilibrium, conditions (5), (6) represent the displacement and traction boundary conditions, respectively, and condition (7) is the contact condition with normal compliance and memory term, see (2). Finally, (8) represents the frictionless condition.

The well-posedness of Problem $\mathcal{P}$ was obtained by Barboteu et al. (in preparation), see also (Petrulescu, 2012) for details and complements. In the next two sections we turn to the study of one and two-dimensional examples which underline the influence of the memory term on the solution.

3. One dimensional example

We consider a cantilever elastic rod which is fixed at its left end $x = 0$, is in contact at its right end $x = 1$, and is subjected to the action of a body force of density f_0 in the x-direction, as shown in Fig. 1. This problem corresponds to problem (3)–(8) with $\Omega = (0,1)$, $\Gamma_1 = \{0\}$, $\Gamma_2 = \emptyset$, $\Gamma_3 = \{1\}$. We use the linearly elastic constitutive law $\sigma = E\varepsilon(u)$ where $E > 0$ represents the Young modulus of the material and $\varepsilon(u) = \partial u/\partial x$. Moreover, we assume that f_0 and b are constant functions, i.e. $f_0(x,t) = f$ and $b(t) = b$ for all $x \in (0,1)$ and $t \in \mathbb{R}_+$ where f, $b > 0$. Finally, we suppose that the compliance function has the form $p(r) = cr^+$, where $c > 0$. With these assumptions, the one-dimensional problem we consider is as follows.

Problem $\mathcal{P}_1$

Find a displacement field $u : (0,1) \times \mathbb{R}_+ \to \mathbb{R}$ and a stress field $\sigma : (0,1) \times \mathbb{R}_+ \to \mathbb{R}$ such that, for each $t \in \mathbb{R}_+$,

$$\sigma(x,t) = E\frac{\partial u}{\partial x}(x,t) \quad \text{in } (0,1) \times \mathbb{R}_+ \tag{9}$$

$$\frac{\partial \sigma}{\partial x}(x,t) = f = 0 \quad \text{in } (0,1) \times \mathbb{R}_+ \tag{10}$$

$$u(0,t) = 0 \quad \text{for all } t \in \mathbb{R}_+ \tag{11}$$

$$\sigma(1,t) + p(u(1,t)) + b\int_0^t u^+(1,s)\mathrm{d}s = 0 \quad \text{for all } t \in \mathbb{R}_+ \tag{12}$$

The exact solution of Problem $\mathcal{P}_1$ can be obtained through an elementary calculus which we present in what follows. Let $t \in \mathbb{R}_+$. First, we integrate (10) with respect to x and obtain

$$\sigma(x,t) = -fx + \alpha(t) \quad \forall x \in (0,1) \tag{13}$$

where $\alpha(t)$ is a constant of integration. Then we combine (9) and (13) to deduce that

$$\frac{\partial u}{\partial x}(x,t) = -\frac{f}{E}x + \frac{\alpha(t)}{E} \quad \forall x \in (0,1) \tag{14}$$

And, finally, we integrate (14) with respect to x and use condition (11) to see that

$$u(x,t)(x,t) = -\frac{f}{2E}x^2 + \frac{\alpha(t)}{E}x \ldots \forall x \in (0,1) \tag{15}$$

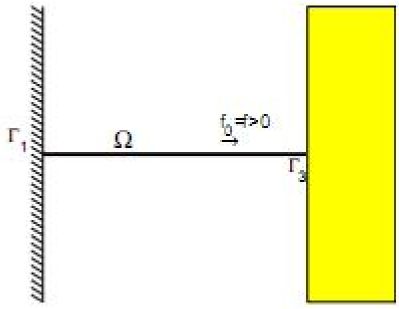

Fig. 1. Contact rod with a foundation

We turn now to determinate $\alpha(t)$ and, to this end, we use the contact condition (12). It follows from (12), (13) and (15) that

$$\alpha(t) - f + \frac{c}{E}\left(\alpha(t) - \frac{f}{2}\right)^+ + \frac{b}{E}\int_0^t \left(\alpha(t) - \frac{f}{2}\right)^+ \mathrm{d}s = 0 \quad \forall t \in \mathbb{R}_+ \tag{16}$$

The existence of a unique solution to the nonlinear integral equation (16) follows from a general abstract result, as shown in (Petrulescu, 2012). Moreover, an elementary calculus shows that the function

$$\alpha(t) = \frac{fE}{2(E+c)} e^{-\frac{b}{E+c}t} + \frac{f}{2} \quad \forall t \in \mathbb{R}_+ \tag{17}$$

satisfies (16). Therefore, using (13) and (15) we obtain

$$\sigma(x,t) = -fx + \left(\frac{E}{E+c} e^{-\frac{b}{E+c}t} + 1\right) + \frac{f}{2} \quad \forall (x,t) \in (0,1) \times \mathbb{R}_+ \tag{18}$$

and

$$u(x,t) = -\frac{f}{2E}x^2 + \left(\frac{E}{E+c}e^{-\frac{b}{E+c}t} + 1\right) + \frac{f}{2E}x \quad \forall (x,t) \in (0,1) \times \mathbb{R}_+ \tag{19}$$

Assume now that E = 1, f = 2, b = c = 1. Then, the stress $\sigma(1,t)$ and the displacement $u(1,t)$ are plotted in Fig. 2. Moreover, the stress function (18) and the displacement function (19) are plotted in Figs. 3 and 4, respectively. Finally, we note that

$$\lim_{t\to\infty} u(1,t) = 0, \quad \lim_{t\to\infty} \sigma(1,t) = -\frac{f}{2} \tag{20}$$

We deduce that in x = 1 the displacement vanishes and the stress reach the residual value $-f/2$, as t tends to infinity. This behavior of the solution represents one of the effects of the memory term in the contact condition.

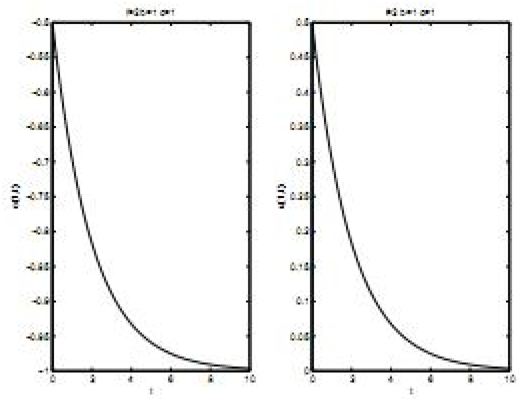

Fig. 2. The stress $\sigma(1,t)$ and the displacement $u(1,t)$ for $t \in [0,10]$

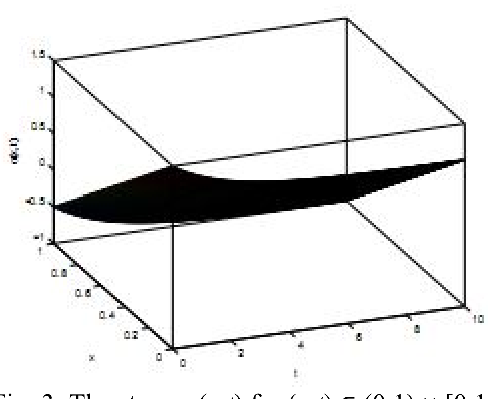

Fig. 3. The stress $\sigma(x,t)$ for $(x,t) \in (0,1) \times [0,10]$

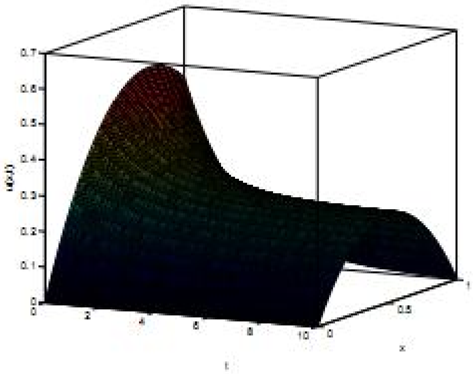

Fig. 4. The displacement $u(x,t)$ for $(x,t) \in (0,1) \times [0,10]$

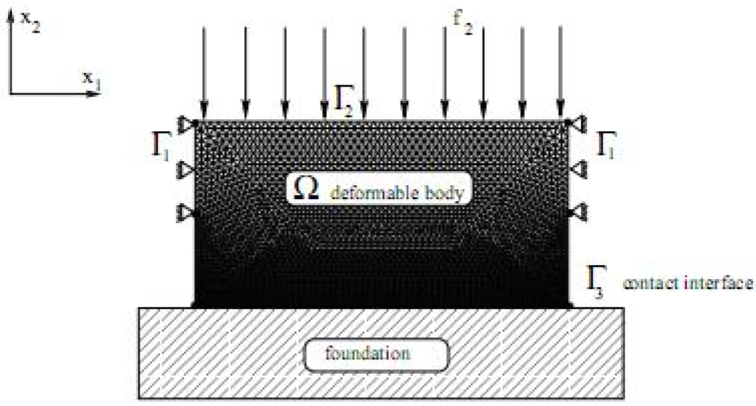

Fig. 5. Initial configuration of the two-dimensional example

4. Two dimensional example

The numerical solution of the contact problem $\mathcal{P}$ is based on a classical Newton method combined with a penalized method for the compliance contact and a trapezoidale rule for the time integration of the memory term. More details on this discretization step and the corresponding numerical method can be found in (Alart and Curnier, 1991; Laursen, 2002; Wriggers, 2002). In order to keep this paper in a reasonable length, we skip the presentation of the numerical algorithm and we describe in what follows a two-dimensional numerical example.

We consider the physical setting depicted in Fig. 5. There, $\Omega = (0,2) \times (0,1) \subset \mathbb{R}^2$ with $\Gamma_1 = (\{0\} \times [0.5,1]) \cup (\{2\} \times [0.5,1])$, $\Gamma_2 = ([0,2] \times \{1\}) \cup (\{0\} \times [0,0.5]) \cup (\{2\} \times [0,0.5])$, $\Gamma_3 = [0,2] \times \{0\}$. The domain Ω represents the cross section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part Γ_1 the body is clamped and, therefore, the displacement field vanishes there. Vertical tractions act on the part

$[0,2]\times\{1\}$ of the boundary Γ_2 and the part $(\{0\} \times [0,0.5]) \cup (\{2\} \times [0,0.5])$ is traction and displacement free. No body forces are assumed to act on the body during the process. The body is in frictionless contact with an obstacle on the part $\Gamma 3 = [0,2] \times \{0\}$ of the boundary. For the discretization we use 7935 elastic finite elements and 129 contact elements. The total number of degrees of freedom is equal to 8326 and we take a time step equal to 0.1 s. We model the material's behavior with a constitutive law of the form (1) in which the elasticity tensor $\mathcal{F}$ satisfies

$$(\mathcal{F}\varepsilon)_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\varepsilon_{11}+\varepsilon_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\varepsilon_{\alpha\beta}, \quad 1 \le \alpha, \beta \le 2 \tag{21}$$

Here E is the Young modulus, κ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

For the computation below we used the following data:

$t \in [0,T]$, $T = 3.6$ s, $E = 10000$ N/m^2, $\kappa = 0.3$, $\mathbf{f}_0 = (0,0)$ N/m^2,

$$\mathbf{f}_2 = \begin{cases} (0,0)\ \text{N/m} & \text{on } (\{0\}\times[0,0.5])\cup(\{2\}\times[0,0.5] \\ (\text{-}2000,0)\ \text{N/m} & \text{on } [0,2]\times\{1\} \end{cases},$$

$p(r) = c_v r^+$, $c_v = 200$ N/m^2.

Our results are presented in Figs. 6–9 and are described in what follows.

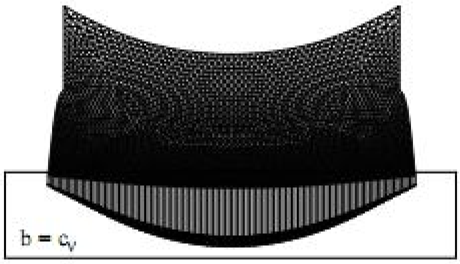

Fig. 6. Deformed mesh and contact interface forces for $b = c_v$

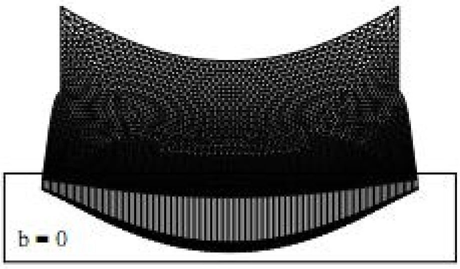

Fig. 7. Deformed mesh and contact interface forces for $b = 0$

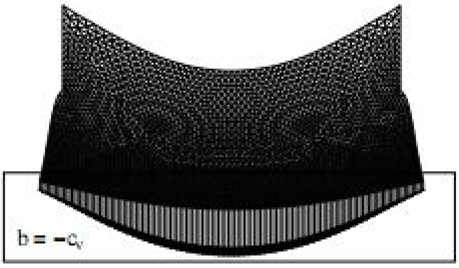

Fig. 8. Deformed mesh and contact interface forces for $b = -c_v$

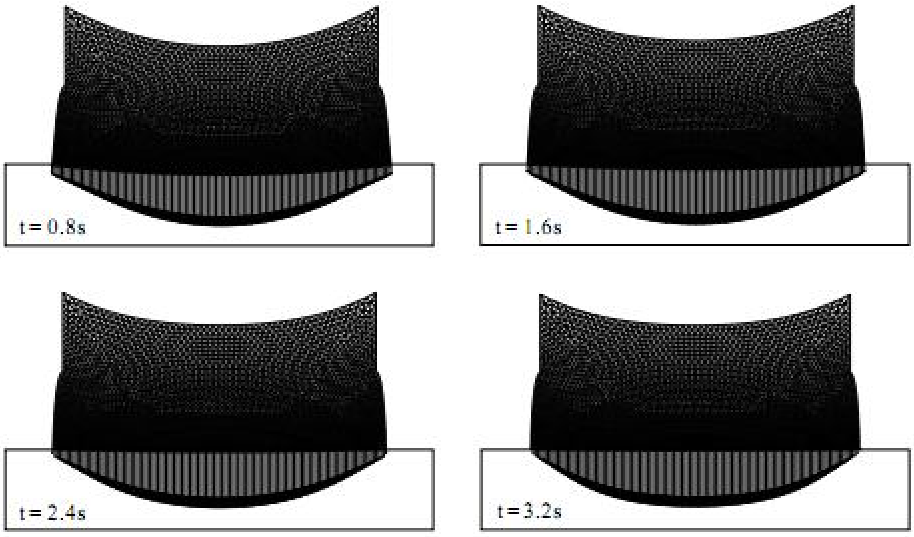

Fig. 9. Deformed meshes and contact interface forces for t = 0.8, 1.6, 2.4 and 3.6 s

First, in Figs. 6, 7 and 8 the deformed mesh and the contact interface forces are presented in the case $b = c_v > 0$, $b = c_v = 0$ and $b = -c_v < 0$, respectively. We note that when $b = c_v > 0$ the penetration is less important than in the case when the memory term vanishes (i.e. $b = 0$). Also, when $b = -c_v < 0$ the penetration is more important than in the case when $b = 0$. We conclude from here that in function of the sign of b, the memory term describes either the hardness or the softness of the foundation.

In Fig. 9 we present the evolution of the contact in time, for $b = c_v$. There we present the deformed meshes as well as the associated contact forces for t = 0.8, 1.6, 2.4 and 3.2 s, respectively. We note that the penetration is smaller at t = 3.2 s than at t = 0.8 s and it decreases in time. These results suggest that for a large time interval the penetration converges to zero, which represents a similar behavior to that observed in the one-dimensional case, see the first equality in (20).

5. Conclusions

In this paper we presented a model for the history-dependent process of frictionless contact of an elastic body. The contact was governed by a normal compliance condition with memory. Our aim was to study the behaviour of the solution and to underline the effects of the memory terms in the process of contact. To this end we considered a one-dimensional example and we provided an analytic expression of the solution. Then, we provided numerical simulations in the study of a two-dimensional model. We used an algorithm based on a Newton method combined with a penalized method for the compliance contact and a trapezoidal rule for the time integration of the contact memory term. Based on the simulations, we found that the algorithm worked well and the convergence was rapid. Subsequent stages of the research presented in this paper will consist to add friction in the model and to extend these results to dynamic frictional or frictionless contact processes.

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References

Alart, P., Curnier, A., 1991, A mixed formulation for frictional contact problems prone to Newton like solution methods, *Computer Methods in Applied Mechanics and Engineering*, **92**, 353–375.

Barboteu, M., Patrulescu, F., Sofonea, M., Analysis of a contact problem with normal compliance and memory term, in preparation.

Han, W., Sofonea M., 2002, Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics*, **30**, American Mathematical Society–International Press, Sommerville, MA.

Laursen, T., 2002, *Computational Contact and Impact Mechanics*, Springer, Berlin.

Patrulescu, F., 2012, Ordinary Differential Equations and Contact Problems: Modeling, Analysis and Numerical Methods, *Ph.D. Thesis*, University Babes-Bolyai Cluj-Napoca and University of Perpignan Via Domitia.

Shillor, M., Sofonea, M., Telega, J.J., 2004, Models and Analysis of Quasistatic Contact, *Lecture Notes in Physics*, **655**, Springer, Berlin.

Sofonea, M., Matei, A., 2011, History-dependent quasivariational inequalities arising in Contact Mechanics, *European Journal of Applied Mathematics*, **22**, 471–491.

Sofonea, M., Matei A., 2012, Mathematical Models in Contact Mechanics, *London Mathematical Society Lecture Note Series*, **398**, Cambridge University Press, Cambridge.

Wriggers, P., 2002, *Computational Contact Mechanics*, Wiley, Chichester.