ON THE BEHAVIOR OF THE SOLUTION TO A VISCOPLASTIC CONTACT PROBLEM

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Abstract

The present paper represents a continuation of [2]. There, a quasistatic contact problem for viscoplastic materials was considered, in which the contact was assumed to be frictionless and was described with normal compliance and unilateral constraint; the unique weak solvability of the problem was proved, a fully discrete scheme for the numerical approximation of the problem was described and numerical simulations were presented. In the present paper we analyse the dependence of the solution of the viscoplastic contact problem in [2] with respect to the data. We state and prove a convergence result, Theorem 3.1, then we illustrate its validity in the study of a two-dimensional numerical example.

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1 Indroduction

The aim of this paper is to study the continuous dependence of the solution to a frictionless contact problems for rate-type viscoplastic materials. We model the behavior of the material with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1.1}$$

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where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor. Here $\mathcal{E}$ is a fourth order tensor which describes the elastic properties of the material and $\mathcal{G}$ is a nonlinear constitutive function which describes its visco-plastic behavior. In (1.1) and everywhere in this paper the dot above a variable represents the derivative with respect to the time variable t.

Various results, examples and mechanical interpretations in the study of viscoplastic materials of the form (1.1) can be found in [3, 5] and the references therein. Displacement-traction boundary value problems with such materials were considered in [5], both in the dynamic and quasistatic case. Quasistatic frictionless and frictional contact problems for materials of the form (1.1) were studied in various papers, see [10] for a survey. There, various models of contact were stated and their variational analysis, including existence and uniqueness results, was provided. The numerical analysis of the corresponding models can be found in [4] and the references therein.

A quasistatic frictionless contact problem for viscoplastic materials of the form (1.1) was recently considered in [2]. There, the process was assumed to be quasistatic and the contact was modelled by using the normal compliance condition with unilateral constraint; the unique solvability of the solution was obtained by using new arguments on history-dependent variational inequalities obtained in [11]; a convergence result was provided, which shows that the weak solution of the problem may be approached as closely as one wishes by the solution of the viscoplastic contact problem with normal compliance and infinite penetration, with a sufficiently small deformability coefficient; finally, a fully discrete scheme for the numerical approximation of the problem was implemented and numerical simulations were presented. In the present paper we analyse the dependence of the solution of the viscoplastic contact problem in [2] with respect to the data. We state and prove a convergence result, Theorem 3.1, then we illustrate its validity in the study of a two-dimensional numerical example.

The rest of the paper is structured as follows. In Section 2 we introduce the contact problem and resume the results on its unique weak solvability obtained in [2]. In Section 3 we state and prove our converge result, Theorem 3.1, which represents the main result of this paper. And, finally, in Section 4 we present a numerical validation of this convergence result.

Everywhere in this paper we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of non negative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

For each Banach space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values in X and, for a subset $K \subset X$, we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms; moreover, the convergence of a sequence $(x_m)_m$ to the element x, in the space $C(\mathbb{R}_+;X)$, can be described as follows:

$$\begin{cases} x_m \to x \quad \text{in } C(\mathbb{R}_+;X) \ \text{ as } \ m\to\infty \ \text{ if and only if} \\ \\ \max\limits_{r\in[0,n]} \|x_m(r)-x(r)\|_X \to 0 \ \text{ as } m\to\infty, \ \text{ for each } \ n\in\mathbb{N}^*. \end{cases} \tag{1.2}$$

Equivalence (1.2) will be used several times in Section 3 of the paper.

2 The model and preliminaries

The physical setting is as follows. A viscoplastic body occupies the domain $\Omega\subset\mathbb{R}^d$ ($d=1,2,3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\,(\Gamma_1)>0$. We use the notation $\boldsymbol{x}=(x_i)$ for a typical point in $\Omega\cup\Gamma$ and we denote by $\boldsymbol{\nu}=(\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $v_{i,j}=\partial v_i/\partial x_j$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the problem is quasistatic, and we study the contact process in the interval of time $\mathbb{R}_+=[0,\infty)$. The contact is modelled with normal compliance and unilateral constraint. Therefore, the classical formulation of the problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u}:\Omega\times\mathbb{R}_+\to\mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}:\Omega\times\mathbb{R}_+\to\mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}=\mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})+\mathcal{G}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{in} \quad \Omega\times(0,\infty), \tag{2.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}+\boldsymbol{f}_0=\boldsymbol{0} \qquad \text{in} \quad \Omega\times(0,\infty), \tag{2.2}$$

$$\boldsymbol{u}=\boldsymbol{0} \qquad \text{on} \quad \Gamma_1\times(0,\infty), \tag{2.3}$$

$$\boldsymbol{\sigma\nu}=\boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2\times(0,\infty), \tag{2.4}$$

$$\left.\begin{array}{l} u_\nu\le g, \quad \sigma_\nu+p(u_\nu)\le 0, \\ (u_\nu-g)(\sigma_\nu+p(u_\nu))=0 \end{array}\right\} \qquad \text{on} \quad \Gamma_3\times(0,\infty), \tag{2.5}$$

$$\boldsymbol{\sigma}_\tau=\boldsymbol{0} \qquad \text{on} \quad \Gamma_3\times(0,\infty), \tag{2.6}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\ \boldsymbol{\sigma}(0)=\boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{2.7}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}$ or t. Equation (2.1) represents the viscoplastic constitutive law of the material introduced in Section 1. Equation (2.2) is the equilibrium equation in which Div denotes the divergence operator for tensor valued functions. Conditions (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, and condition (2.5) represents the contact condition with normal compliance and unilateral constraint, in which σ_ν denotes the normal stress, u_ν is the normal displacement, $g \geq 0$ and p is a given function. This condition was first introduced in [6] and then it was used in various papers, see [11, 12] and the references therein. Condition (2.6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (2.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

In the study of problem $\mathcal{P}$ we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. Moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}, \qquad Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.8}$$

We turn now to the assumptions on the data. First, we assume that the elasticity tensor $\mathcal{E}$ and the nonlinear constitutive function $\mathcal{G}$ satisfy the following conditions.

$$\left\{\begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \leq i, j, k, l \leq d. \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{array}\right. \tag{2.9}$$

$$\left\{\begin{array}{l} \text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad\quad \leq L_{\mathcal{G}}\,(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|) \\ \qquad\quad\ \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \qquad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } Q. \end{array}\right. \tag{2.10}$$

Next, we assume that the normal compliance function p is such that

$$\left\{\begin{array}{l}\text{(a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+.\\ \text{(b) There exists } L_p>0 \text{ such that}\\ \qquad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p\,|r_1-r_2|\\ \qquad\quad \forall\, r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\ge 0\\ \qquad\quad \forall\, r_1,\,r_2\in\mathbb{R},\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\ \qquad \text{for any } r\in\mathbb{R}.\\ \text{(e) } p(\boldsymbol{x},r)=0 \text{ for all } r\le 0,\ \text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{2.11}$$

Finally, we assume that the body forces and the tractions have the regularity

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d),\quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d), \tag{2.12}$$

and the initial data satisfy

$$\boldsymbol{u}_0\in V,\quad \boldsymbol{\sigma}_0\in Q. \tag{2.13}$$

Consider now the subset $U\subset V$, the operator $P:V\to V$ and the function $\boldsymbol{f}:\mathbb{R}_+\to V$ defined by equalities

$$U=\{\,\boldsymbol{v}\in V:v_\nu\le g \text{ on } \Gamma_3\,\}, \tag{2.14}$$

$$(P\boldsymbol{u},\boldsymbol{v})_V=\int_{\Gamma_3}p(u_\nu)v_\nu\,da\quad\forall\,\boldsymbol{u},\,\boldsymbol{v}\in V, \tag{2.15}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da\quad\forall\,\boldsymbol{v}\in V,\ t\in\mathbb{R}_+. \tag{2.16}$$

Then, the variational formulation of Problem $\mathcal{P}$, derived in [2], is the following.

Problem $\mathcal{P}^V$. *Find a displacement field $\boldsymbol{u}:\mathbb{R}_+\to U$ and a stress field $\boldsymbol{\sigma}:\mathbb{R}_+\to Q$ such that, for all $t\in\mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{2.17}$$

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q+(P\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\\ &\qquad\ge(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\quad\forall\,\boldsymbol{v}\in U.\end{aligned} \tag{2.18}$$

The unique solvability of Problem $\mathcal{P}^V$ is given by the following result.

Theorem 2.1 *Assume that (2.9)–(2.13) hold. Then Problem $\mathcal{P}^V$ has a unique solution, which satisfies $\boldsymbol{u}\in C(\mathbb{R}_+;U)$, $\boldsymbol{\sigma}\in C(\mathbb{R}_+;Q)$.*

The proof of Theorem 2.1 can be found in [2] as well as in [12, Ch. 6]. It is based on arguments of history-dependent variational inequalities obtained in [11].

3 A convergence result

We now study the dependence of the solution of Problem $\mathcal{P}^V$ with respect to perturbations of the data. To this end, we assume in what follows that (2.9)–(2.13) hold and we denote by $(\boldsymbol{u},\boldsymbol{\sigma})$ the solution of Problem $\mathcal{P}^V$ obtained in Theorem 2.1. For each $\rho > 0$ let $\mathcal{G}_\rho$, p_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\boldsymbol{u}_{0\rho}$ and $\boldsymbol{\sigma}_{0\rho}$ be perturbations of $\mathcal{G}$, p, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$, respectively, which satisfy conditions (2.10)–(2.13). We define the operator $P_\rho : V \to V$ and the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{3.1}$$

$$(\boldsymbol{f}_\rho(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t) \cdot \boldsymbol{v} \, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+, \tag{3.2}$$

and we consider the following perturbation of the variational Problem $\mathcal{P}^V$.

Problem $\mathcal{P}^V_\rho$. *Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \int_0^t \mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)))\, ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}), \tag{3.3}$$

$$\begin{aligned}(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \\ \geq (\boldsymbol{f}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{3.4}$$

It follows from Theorem 2.1 that, for each $\rho > 0$ Problem $\mathcal{P}^V_\rho$ has a unique solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho)$ with regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q)$. Consider now the following assumptions.

$$\left\{\begin{array}{l}\text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } |\mathcal{G}_\rho(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon})| \leq G(\rho) \\ \qquad \forall\, \boldsymbol{\sigma},\ \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega,\ \text{for each } \rho > 0. \\ \text{(b) } G(\rho) \to 0 \quad \text{as} \quad \rho \to 0.\end{array}\right. \tag{3.5}$$

$$\left\{\begin{array}{l}\text{There exists } F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \leq F(\rho) \\ \qquad \forall\, r \leq g,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{for each } \rho > 0. \\ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0.\end{array}\right. \tag{3.6}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \tag{3.7}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \tag{3.8}$$

$$\boldsymbol{u}_{0\rho} \to \boldsymbol{u}_0 \quad \text{in} \quad V \quad \text{as} \quad \rho \to 0. \tag{3.9}$$

$$\boldsymbol{\sigma}_{0\rho} \to \boldsymbol{\sigma}_0 \quad \text{in} \quad Q \quad \text{as} \quad \rho \to 0. \tag{3.10}$$

We have the following convergence result.

Theorem 3.1 *Under assumptions* (3.5)–(3.10)*, the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho)$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *of Problem* $\mathcal{P}^V$*, i.e.,*

$$\boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V), \qquad \boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C(\mathbb{R}_+; Q) \tag{3.11}$$

as $\rho \to 0$.

Proof. Let $\rho > 0$, $n \in \mathbb{N}^*$ and let $t \in [0, n]$. We take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (3.4) and $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in (2.18) and add the resulting inequalities to obtain

$$\begin{aligned}&(\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\quad \le (P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V + (\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V.\end{aligned} \tag{3.12}$$

On the other hand, using (3.3) and (2.17) we find that

$$\begin{aligned}\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t) &= \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \\ &\quad + \int_0^t \big(\mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))) - \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\big)\, ds \\ &\quad + \boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0).\end{aligned} \tag{3.13}$$

We substitute equality (3.13) in (3.12) and use assumption (2.9) to deduce that

$$\begin{aligned}&m_{\mathcal{E}}\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 \\ &\le (P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V + (\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \\ &+ \Big(\int_0^t \big(\mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))) - \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\big)\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t))\Big)_Q \\ &+ (\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q.\end{aligned} \tag{3.14}$$

Next, we use the definitions (3.1) and (2.15), the monotonicity of the function p_ρ and assumption (3.6) to see that

$$\begin{aligned}&(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V = \int_{\Gamma_3} \big(p_\rho(u_{\rho\nu}(t)) - p(u_\nu(t))\big)(u_\nu(t) - u_{\rho\nu}(t))\, da \\ &\le \int_{\Gamma_3} \big(p_\rho(u_\nu(t)) - p(u_\nu(t))\big)(u_\nu(t) - u_{\rho\nu}(t))\, da \\ &\le \int_{\Gamma_3} \big|p_\rho(u_\nu(t)) - p(u_\nu(t))\big|\, |u_\nu(t) - u_{\rho\nu}(t)|\, da \\ &\le \int_{\Gamma_3} F(\rho)\, |u_\nu(t) - u_{\rho\nu}(t)|\, da \le F(\rho)(meas\, \Gamma_3)^{\frac{1}{2}}\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}\|_{L^2(\Gamma_3)^d}\end{aligned}$$

Therefore, using the trace inequality (2.8), we find that

$$(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \le cF(\rho)\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{3.15}$$

where, here and below in this section, c represents a positive constant which may depend on the data but it is independent on ρ, t and n, and whose value may change from line to line.

Finally, let $\delta_{\rho n} > 0$, $H_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega_{0\rho} > 0$ be defined by

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V, \tag{3.16}$$

$$H_\rho(s) = \|\mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))) - \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\|_Q \quad \forall\, s \in \mathbb{R}_+, \tag{3.17}$$

$$\omega_{0\rho} = \|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V. \tag{3.18}$$

Then, it is easy to see that the following inequalities hold:

$$(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \le \delta_{\rho n}\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V, \tag{3.19}$$

$$\begin{aligned}&\Big(\int_0^t \big(\mathcal{G}_\rho(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s))) - \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\big)\, ds, \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_Q \\ &\quad \le \Big(\int_0^t H_\rho(s)\, ds\Big)\, \|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V,\end{aligned} \tag{3.20}$$

$$\begin{aligned}&(\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ &\qquad +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\qquad\qquad \le c\, \omega_{0\rho}\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.\end{aligned} \tag{3.21}$$

We now combine inequalities (3.14), (3.15), (3.19)–(3.21) to deduce that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le c\,(F(\rho) + \delta_{\rho n} + \omega_{0\rho}) + \int_0^t H_\rho(s)\, ds. \tag{3.22}$$

On the other hand, using equality (3.13), assumptions (2.9), (2.10) and notation (3.17), (3.18) we find that

$$\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \le c\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \int_0^t H_\rho(s)\, ds + \omega_{0\rho}$$

and, using (3.22) yields

$$\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \le c\,(F(\rho) + \delta_{\rho n} + \omega_{0\rho}) + c\int_0^t H_\rho(s)\, ds. \tag{3.23}$$

We now add inequalities (3.22) and (3.23) to obtain

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \le c\,(F(\rho) + \delta_{\rho n} + \omega_{0\rho}) + c\int_0^t H_\rho(s)\, ds \tag{3.24}$$

and, using (3.17) and (2.10) we deduce that

$$H_\rho(s) \le G(\rho) + L_{\mathcal{G}}\,(\|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V + \|\boldsymbol{\sigma}_\rho(s) - \boldsymbol{\sigma}(s)\|_Q) \quad \forall\, s \in [0, t]. \tag{3.25}$$

Next, we combine (3.24) and (3.25) then we use the Gronwall inequality to see that

$$\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q\le c\left(G(\rho)+F(\rho)+\delta_{\rho n}+\omega_{0\rho}\right)e^{ct}. \tag{3.26}$$

We pas to the upper bound as $t\in[0,n]$ in (3.26) to obtain

$$\begin{aligned}&\max_{t\in[0,n]}\left(\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q\right)\\&\qquad\le c\left(G(\rho)+F(\rho)+\delta_{\rho n}+\omega_{0\rho}\right)e^{cn}\quad\text{for all}\quad\rho>0.\end{aligned} \tag{3.27}$$

We now use equalities (3.2) and (2.16) to see that

$$\|\boldsymbol{f}_\rho(s)-\boldsymbol{f}(s)\|_V\le c\,\|\boldsymbol{f}_{0\rho}(s)-\boldsymbol{f}_0(s)\|_{L^2(\Omega)^d}+c\,\|\boldsymbol{f}_{2\rho}(s)-\boldsymbol{f}_2(s)\|_{L^2(\Gamma_2)^d}$$

for all $s\in[0,n]$ and, therefore, (3.7), (3.8) and (1.2) imply that

$$\delta_{\rho n}\to 0\quad\text{as }\rho\to 0. \tag{3.28}$$

On the other hand, (3.9) and (3.10) yield

$$\omega_{0\rho}\to 0\quad\text{as }\rho\to 0. \tag{3.29}$$

We now combine the convergences (3.5)(b), (3.6)(b), (3.28) and (3.29) with inequality (3.27) to obtain that

$$\max_{t\in[0,n]}\left(\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q\right)\to 0\quad\text{as }\rho\to 0. \tag{3.30}$$

Since the convergence (3.30) holds for each $n\in\mathbb{N}^*$, we deduce from (1.2) that (3.11) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence result (3.11), it is of importance from mechanical point of view, since it states that the weak solution of the problem (2.1)–(2.7) depends continuously on the viscoplastic constitutive function, the normal compliance function, the densities of body forces and surface tractions and the initial data, as well.

4 Numerical validation

We proceed with the numerical validation of the convergence result in Theorem 3.1. Details on the numerical approximation of Problem $\mathcal{P}^V$ can be found in [2]. Here we restrict ourselves to recall that for the numerical treatment of the contact condition we use the penalized method for the compliance contact combined with the augmented Lagrangean approach for the unilateral constraint. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface Γ_3. In the case of the numerical example presented below, the discretization is based on "node-to-rigid" contact element, which is composed by one node of Γ_3 and one Lagrange multiplier node. More details on this discretization step and the corresponding numerical method

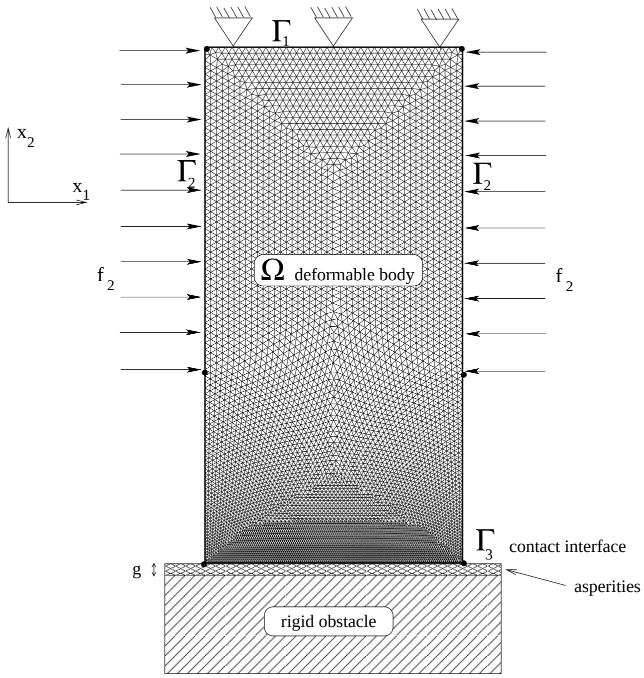

Figure 1: Initial configuration of the two-dimensional example.

can be found in [1, 7, 8, 9, 13]. For the numerical simulations we consider the physical setting depicted in Figure 1 and described below. There, $\Omega = (0,2) \times (0,4) \subset \mathbb{R}^2$ with $\Gamma_1 = [0,2] \times \{4\}$, $\Gamma_2 = (\{0\} \times [0,4]) \cup (\{2\} \times [0,4])$, $\Gamma_3 = [0,2] \times \{0\}$. The domain Ω represents the cross section of a three-dimensional deformable body subjected to the action of tractions in such a way that the plane stress hypothesis is assumed. On the part $\Gamma_1 = [0,2] \times \{4\}$ the body is clamped and, therefore, the displacement field vanishes there. Horizontal tractions act on the part $(\{0\} \times [1.5,4]) \cup (\{2\} \times [1.5,4])$ of the boundary Γ_2 and the part $(\{0\} \times [0,1.5]) \cup (\{2\} \times [0,1.5])$ is traction free. No body forces are assumed to act on the body during the process. The body is in frictionless contact with an obstacle on the part $\Gamma_3 = [0,2] \times \{0\}$ of the boundary. For the discretization we use 13223 elastic finite elements and 129 contact elements. The total number of degrees of freedom is equal to 13712 and we take a time step k equal to $0.01s$.

We model the material's behavior with a constitutive law of the form (1.1) in which elasticity tensor $\mathcal{E}$ satisfies

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2, \tag{4.1}$$

where E is the Young modulus, κ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol. Moreover, in order to facilitate the numerical implementation, we assume that $\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathcal{G}_\rho(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$, where the tensor $\mathcal{C}$ satisfies

$$(\mathcal{C}\boldsymbol{\tau})_{\alpha\beta} = \gamma_1(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \gamma_2\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2. \tag{4.2}$$

For the computation below we use the following data:

$$t \in [0,T], \quad T = 1s,$$

$$E = 20000N/m^2, \quad \kappa = 0.4, \quad \gamma_1 = 20N/m^2, \quad \gamma_2 = 30N/m^2,$$

$$\boldsymbol{f}_0 = \boldsymbol{f}_{0\rho} = (0,0)N/m^2,$$

$$\boldsymbol{f}_2 = \begin{cases} (0,0)\,N/m & \text{on } (\{0\}\times[0,1.5])\cup(\{2\}\times[0,1.5]), \\ (8000,0)\,N/m & \text{on } \{0\}\times[1.5,4], \\ (-8000,0)\,N/m & \text{on } \{2\}\times[1.5,4], \end{cases}$$

$$\boldsymbol{f}_{2\rho} = \begin{cases} (0,0)\,N/m & \text{on } (\{0\}\times[0,1.5])\cup(\{2\}\times[0,1.5]), \\ (8000+\rho,0)\,N/m & \text{on } \{0\}\times[1.5,4], \\ (-8000-\rho),0)\,N/m & \text{on } \{2\}\times[1.5,4], \end{cases}$$

$$p(r) = c_\nu r_+, \quad p_\rho(r) = (c_\nu + \rho)r_+, \quad c_\nu = 200\,N/m^2, \quad g = 0.1\,m,$$

$$\boldsymbol{u}_0 = \boldsymbol{u}_{0\rho} = \boldsymbol{0}\,m, \quad \boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_{0\rho} = \boldsymbol{0}\,N/m^2.$$

Our results are presented in Figures 2–3 and are described in what follows.

First, in Figure 2, the deformed configuration as well as the contact interface forces at $t = 1s$ are plotted in the case $\rho = 0$, which corresponds to problem $\mathcal{P}^V$. We recall that the contact follows a normal compliance condition as far as the penetration is less than the limit $g = 0.1\,m$ and, when this limit is reached, it follows a unilateral condition. As it results from the zoom depicted in Figure 2, for part of the contact nodes the complete flattening of the asperities of size $g = 0.1\,m$ was reached; therefore, these nodes (situated in the central part of the boundary Γ_3) are into unilateral contact. In contrast, the nodes on the extremities of the boundary Γ_3 remain in the status of a contact with normal compliance.

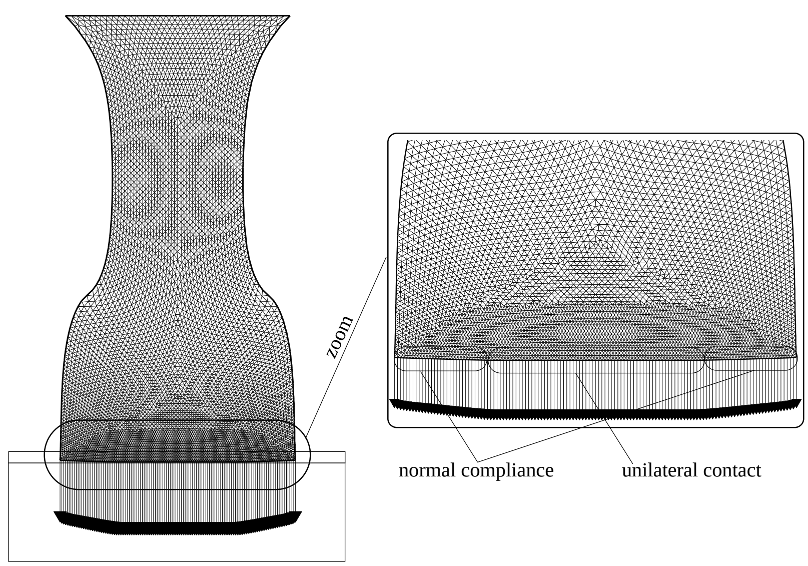

Figure 2: Deformed mesh and contact interface forces for Problem $\mathcal{P}^V$.

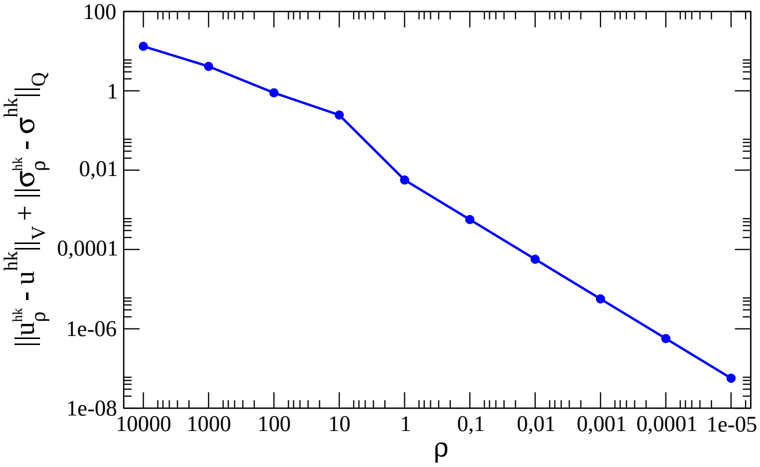

Figure 3: Numerical validation of the convergence result in Theorem 3.1.

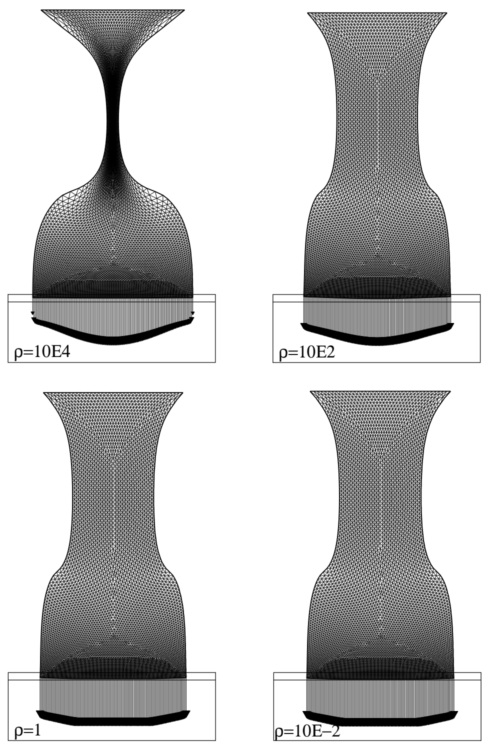

Figure 4: Deformed meshes and contact interface forces for $\rho = 10^4$, $\rho = 10^2$, $\rho = 1$ and $\rho = 10^{-2}$.

Next, we denote by $(\boldsymbol{u}_\rho^{hk}, \boldsymbol{\sigma}_\rho^{hk})$ and $(\boldsymbol{u}^{hk}, \boldsymbol{\sigma}^{hk})$ the discrete solution of the contact problems $\mathcal{P}_\rho^V$ and $\mathcal{P}^V$, respectively, for a given $\rho > 0$. The numerical estimations of the difference

$$\|\boldsymbol{u}_\rho^{hk} - \boldsymbol{u}^{hk}\|_V + \|\boldsymbol{\sigma}_\rho^{hk} - \boldsymbol{\sigma}^{hk}\|_Q$$

at the time $t = 1\,s$, for various values of the parameter ρ, is presented in Figure 3. It results from here that this difference converges to zero as ρ tends to zero. To highlight this study, we plot 4 deformed meshes and the associated contact forces at $t = 1s$, for $\rho = 10^4,\ 10^2,\ 1,\ 10^{-2}$, respectively. One can see that for $\rho = 10^4$ both the deformed mesh and the contact interface forces are very different from those in Figure 2 which, recall, corresponds to the case $\rho = 0$. These differences progressively disappear as $\rho \to 0$ in such a way that, for $\rho = 10^{-2}$, both the deformed mesh and the contact interface forces are very close to that obtained for $\rho = 0$. We conclude that our results in Figures 3–4 represent a numerical validation of the theoretical convergence result obtained in Theorem 3.1.

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