A Viscoplastic Contact Problem with Normal Compliance, Unilateral Constraint and Memory Term

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Abstract

We consider a mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The material's behavior is modelled with a rate-type constitutive law with internal state variable. The contact is frictionless and is modelled with normal compliance, unilateral constraint and memory term. We present the classical formulation of the problem, list the assumptions on the data and derive a variational formulation of the model. Then we prove its unique weak solvability. The proof is based on arguments of history-dependent quasivariational inequalities. We also study the dependence of the solution with respect to the data and prove a convergence result.

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1 Introduction

The aim of this paper is to study a frictionless contact problem for rate-type viscoplastic materials within the framework of the Mathematical Theory of Contact Mechanics. We model the material's behavior with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and $\boldsymbol{\kappa}$ denotes an internal state variable. Here $\mathcal{E}$ is a linear operator which describes the elastic properties of the material and $\mathcal{G}$ is a nonlinear constitutive function which describes its viscoplastic behavior. In (1.1) and everywhere in this paper the dot above a variable represents the derivative with respect to the time variable t. Following [3, 7], the internal state variable $\boldsymbol{\kappa}$ is a vector-valued function whose evolution is governed by the differential equation

$$\dot{\boldsymbol{\kappa}}(t) = \boldsymbol{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)), \tag{1.2}$$

in which $\boldsymbol{G}$ is a nonlinear constitutive function with values in $\mathbb{R}^m$, m being a positive integer.

Various results, examples and mechanical interpretations in the study of viscoplastic materials of the form (1.1), (1.2) can be found in [3, 7] and the references therein. Quasistatic contact problems for such materials have been considered in [5, 6] and the references therein. There, the contact was assumed to be frictionless and was modelled with normal compliance; the unique weak solvability of the corresponding problems was proved by using arguments of nonlinear equations with monotone operators and fixed point; semi-discrete and fully discrete scheme were considered, error estimates and convergence results were proved and numerical simulation in the study of two-dimensional test problems were presented. The normal compliance contact condition was first introduced in [14] and since then used in many publications, see, e.g., [9, 10, 11, 13] and references therein. The term *normal compliance* was first introduced in [10, 11].

In the particular case without internal state variable the constitutive equation (1.1) reads

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{1.3}$$

and was used in the literature in order to model the behaviour of various materials like rubbers, rocks, metals, pastes and polymers. Quasistatic frictionless contact problems for materials of the form (1.3) have been considered in [1, 6, 15, 18] and the references therein, under various contact conditions. In [6, 15] both the Signorini and the normal compliance condition were used which, recall, describe a contact with a rigid and elastic foundation, respectively. In [1, 18] the contact was modelled with normal compliance and unilateral constraint condition. This condition, introduced for the first time in [8], models an elastic-rigid behavior of the foundation.

With respect to the papers above mentioned, the current paper has three traits of novelties that we describe in what follows. First, the model we consider involves a contact condition with normal compliance, unilateral constraint and memory term. This condition takes into account both the deformability, the rigidity, and the memory effects of the foundation. Second, in contrast with the short note [4], we model the behavior of the material with a viscoplastic constitutive law with internal state variable. And, finally, we study the contact process on an unbounded interval of time which implies the use of the framework of Fréchet spaces of continuous functions, instead of that of the classical Banach spaces of continuous functions defined on a bounded interval of time. The three ingredients above lead to a new and interesting mathematical model. The aim of this work is to prove the unique weak solvability of this model and to study the dependence of the weak solution with respect to the data.

The rest of the paper is structured as follows. In Section 2 we present the notation we shall use as well as some preliminary material. In Section 3 we describe the model of the contact process. In Section 4 we list the assumptions on the data and derive the variational formulation of the problem. Then we state and prove our main existence and uniqueness result, Theorem 4.1. In Section 5 we state and prove our converge result, Theorem 5.1. It states the continuous dependence of the solution with respect to the data.

2 Notations and preliminaries

Everywhere in this paper we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a given $r \in \mathbb{R}$ we denote by r^+ its positive part, i.e. $r^+ = max\,\{r, 0\}$. Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that $\text{meas}\,(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Also, we use the notation $\|\boldsymbol{\kappa}\|$ for the Euclidean norm of the element $\boldsymbol{\kappa} \in \mathbb{R}^m$. In addition, we use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}, \quad Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $\text{meas}(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.1}$$

Also, for a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\text{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$ and, finally, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \text{Div}\,\boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{2.2}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) \ :\ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i, j, k, l \le d\ \},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \qquad \forall\, \mathcal{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{2.3}$$

For each Banach space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Details can be found in [2] and [12], for instance. Here we restrict ourseleves to recall that the convergence of a sequence $(x_k)_k$ to the element x, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} x_k \to x \quad \text{in } C(\mathbb{R}_+; X) \;\text{ as }\; k \to \infty \;\text{ if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0 \;\text{ as } k \to \infty, \;\text{ for all }\; n \in \mathbb{N}^*. \end{cases} \tag{2.4}$$

The following fixed-point result will be used in Section 4 of the paper.

Theorem 2.1 *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator with the following property: there exists $c > 0$ such that*

$$\|\Lambda u(t) - \Lambda v(t)\|_X \le c \int_0^t \|u(s) - v(s)\|_X ds \tag{2.5}$$

for all u, $v \in C(\mathbb{R}_+; X)$ and for all $t \in \mathbb{R}_+$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Theorem 2.1 represents a simplified version of Corollary 2.5 in [16]. We underline that in (2.5) and below, the notation $\Lambda\eta(t)$ represents the value of the function $\Lambda\eta$ at the point t, i.e. $\Lambda\eta(t) = (\Lambda\eta)(t)$.

Consider now a real Hilbert space X with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$ as well as a normed space Y with norm $\|\cdot\|_Y$. Let K be a subset of X and consider the operators $A : K \to X$, $\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ as well as the functions $\varphi : Y \times K \to \mathbb{R}$, $f : \mathbb{R}_+ \to X$ such that:

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.6}$$

$$\left.\begin{array}{l} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \ge m\,\|u_1 - u_2\|_X^2 \quad \forall\, u_1,\, u_2 \in K. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \le M\,\|u_1 - u_2\|_X \quad \forall\, u_1,\, u_2 \in K. \end{array}\right\} \tag{2.7}$$

$$\left.\begin{array}{l} \text{For every } n \in \mathbb{N}^* \text{ there exists } r_n > 0 \text{ such that} \\ \quad \|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_Y \le r_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X\, ds \\ \quad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{array}\right\} \tag{2.8}$$

$$\left.\begin{array}{l}\text{(a) The function } \varphi(u,\cdot):K\to\mathbb{R} \text{ is convex and}\\ \quad\text{lower semicontinuous, for all } u\in Y.\\ \text{(b) There exists } \alpha\geq 0 \text{ such that}\\ \quad \varphi(u_1,v_2)-\varphi(u_1,v_1)+\varphi(u_2,v_1)-\varphi(u_2,v_2)\\ \qquad \leq \alpha\,\|u_1-u_2\|_Y\|v_1-v_2\|_X \quad \forall\, u_1,\,u_2\in Y,\,\forall\, v_1,\,v_2\in K.\end{array}\right\} \tag{2.9}$$

$$f\in C(\mathbb{R}_+;X). \tag{2.10}$$

The following result, proved in [17], will be used in Section 4 of this paper.

Theorem 2.2 *Assume that* (2.6)–(2.10) *hold. Then there exists a unique function* $u\in C(\mathbb{R}_+;K)$ *such that, for all* $t\in\mathbb{R}_+$*, the inequality below holds:*

$$\begin{aligned}(Au(t),v-u(t))_X+\varphi(\mathcal{R}u(t),v)-\varphi(\mathcal{R}u(t),u(t))&\\ \geq (f(t),v-u(t))_X \qquad \forall\, v\in K.&\end{aligned} \tag{2.11}$$

Following the terminology introduced in [17] we refer to an operator which satisfies condition (2.8) as a *history-dependent operator*. Moreover, (2.11) represents a *history-dependent quasivariational inequality.*

Finally, assume that X and Y represent two real Hilbert spaces with the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, and associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Then, we denote by $X\times Y$ the product of these spaces. We recall that $X\times Y$ is a real Hilbert space with the canonical inner product $(\cdot,\cdot)_{X\times Y}$ defined by

$$(z_1,z_2)_{X\times Y}=(x_1,x_2)_X+(y_1,y_2)_Y \quad \forall\, z_1=(x_1,y_1),\; z_2=(x_2,y_2)\in X\times Y.$$

The associated norm of the space $X\times Y$, denoted $\|\cdot\|_{X\times Y}$, satisfies the inequality

$$\|z\|_{X\times Y}\leq \|x\|_X+\|y\|_Y\leq \sqrt{2}\,\|z\|_{X\times Y} \quad \forall\, z=(x,y)\in X\times Y.$$

This inequality will be used several times in Sections 4 and 5 of this manuscript.

3 The model

The physical setting is as follows. A viscoplastic body occupies a bounded domain $\Omega\subset\mathbb{R}^d$ ($d=1,2,3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that meas$(\Gamma_1)>0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the contact process is quasistatic and we study it in the interval of time $\mathbb{R}_+=[0,\infty)$. Then, the classical formulation of the contact problem we consider in this paper is the following.

Problem $\mathcal{P}$. *Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ and an internal state variable $\boldsymbol{\kappa} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)) \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\dot{\boldsymbol{\kappa}}(t) = \boldsymbol{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\kappa}(t)) \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.3}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.4}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{3.6}$$

for all $t \in \mathbb{R}_+$, there exists $\xi : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ which satisfies

$$\left.\begin{array}{l} u_\nu(t) \le g, \;\; \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\ \xi(t) = 0 \;\; \text{if} \;\; u_\nu(t) < 0, \\ \xi(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \;\; \text{if} \;\; u_\nu(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{3.7}$$

for all $t \in \mathbb{R}_+$ and, moreover,

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \quad \boldsymbol{\kappa}(0) = \boldsymbol{\kappa}_0 \quad \text{in} \quad \Omega. \tag{3.8}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Equations (3.1), (3.2) represent the rate-type viscoplastic constitutive law with internal state variable introduced in Section 1. Equation (3.3) represents the equation of equilibrium in which Div denotes the divergence operator for tensor valued functions. Conditions (3.4) and (3.5) are the displacement boundary condition and the traction boundary condition, respectively. Condition (3.6) is the frictionless condition and it shows that the tangential stress on the contact surface vanishes. Finally, (3.8) represents the initial conditions in which $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, $\boldsymbol{\kappa}_0$ denote the initial displacement, the initial stress field and the initial state variable, respectively.

We now describe the contact condition (3.7) in which our main interest is. Here σ_ν denotes the normal stress, u_ν is the normal displacement and u_ν^+ may be interpreted as the penetration of the body's surface asperities and those of the foundation. Moreover, p is a Lipschitz continuous increasing function which vanishes for a negative argument, b is a positive function and $g > 0$. This condition can be derived in the following way. Let $t \in \mathbb{R}_+$ be a given time moment. First, we assume that the penetration is limited by the bound g and, therefore, at each time moment $t \in \mathbb{R}_+$, the normal displacement satisfies the inequality

$$u_\nu(t) \le g \quad \text{on } \Gamma_3. \tag{3.9}$$

Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^R(t) + \sigma_\nu^M(t) \quad \text{on } \Gamma_3 \tag{3.10}$$

in which the functions $\sigma_\nu^D(t)$, $\sigma_\nu^R(t)$ and $\sigma_\nu^M(t)$ describe the deformability, the rigidity and the memory properties of the foundation. We assume that $\sigma_\nu^D(t)$ satisfies a normal compliance contact condition, that is

$$-\sigma_\nu^D(t) = p(u_\nu(t)) \quad \text{on } \Gamma_3. \tag{3.11}$$

The part $\sigma_\nu^R(t)$ of the normal stress satisfies the Signorini condition in the form with a gap function, i.e.

$$\sigma_\nu^R(t) \le 0, \quad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_3. \tag{3.12}$$

And, finally, the function $\sigma_\nu^M(t)$ satisfies the condition

$$\begin{cases} |\sigma_\nu^M(t)| \le \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \quad \sigma_\nu^M(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ -\sigma_\nu^M(t) = \displaystyle\int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0 \end{cases} \tag{3.13}$$

on Γ_3. We combine (3.10), (3.11) and denote $-\sigma_\nu^M(t) = \xi(t)$ to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \quad \text{on } \Gamma_3. \tag{3.14}$$

Then we substitute equality (3.14) in (3.12) and use (3.9), (3.13) to obtain the contact condition (3.6).

We now present additional details of the contact condition (3.7). The inequalities and equalities below in this section are valid in an arbitrary point $\boldsymbol{x} \in \Gamma_3$. First, we recall that (3.7) describes a condition with unilateral constraint, since inequality (3.9) holds at each time moment. Next, assume that at a given moment t there is penetration which did not reach the bound g, i.e. $0 < u_\nu(t) < g$. Then (3.7) yields

$$-\sigma_\nu(t) = p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds. \tag{3.15}$$

This equality shows that at the moment t, the reaction of the foundation depend both on the current value of the penetration (represented by the term $p(u_\nu(t))$) as well as on the history of the penetration (represented by the integral term in (3.15)). Assume now that at a given moment t there is separation between the body and the foundation, i.e. $u_\nu(t) < 0$. Then, since $p(u_\nu(t)) = 0$, (3.7) shows that $\sigma_\nu(t) = 0$, i.e. the reaction of the foundation vanishes. Note that the same behavior of the normal stress is described both in the classical normal compliance condition and in the Signorini contact condition, when separation arises.

In conclusion, condition (3.7) shows that when there is separation then the normal stress vanishes; when there is penetration the contact follows a normal compliance condition with memory term of the form (3.15) but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. For this reason we refer to this condition as to a normal compliance contact condition with unilateral constraint and memory term. It can be interpreted physically as follows. The foundation is assumed to be made of a hard material covered by a thin layer of a soft material with thickness g. The soft material has a viscoelastic behaviour, i.e. is deformable, allows penetration and presents memory effects; the contact with this layer is modelled with normal compliance and memory term, as shown in equality (3.15). The hard material is perfectly rigid and, therefore, it does not allow penetration; the contact with this material is modelled with the Signorini contact condition. To resume, the foundation has a rigid-viscoelastic behavior; its viscoelastic behavior is given by the layer of the soft material while its rigid behavior is given by the hard material.

4 Existence and uniqueness

In this section we list the assumptions on the data, derive the variational formulation of the problem $\mathcal{P}$ and then we state and prove its unique weak solvability. To this end we assume that the elasticity tensor $\mathcal{E}$ and the constitutive functions $\mathcal{G}$ and $\boldsymbol{G}$ satisfy the following conditions.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that}\\ \qquad \mathcal{E}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2\ \ \forall\,\boldsymbol{\tau}\in\mathbb{S}^d,\ \text{a.e. in } \Omega.\end{array}\right. \tag{4.1}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\ \qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\boldsymbol{\kappa}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{\kappa}_2)\|\\ \qquad\quad \le L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2\|)\\ \qquad\qquad \forall\,\boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \boldsymbol{\kappa}_1,\ \boldsymbol{\kappa}_2 \in \mathbb{R}^m,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\kappa}) \text{ is measurable on } \Omega,\\ \qquad \text{for any } \boldsymbol{\sigma},\ \boldsymbol{\varepsilon}\in\mathbb{S}^d \text{ and } \boldsymbol{\kappa}\in\mathbb{R}^m.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } Q.\end{array}\right. \tag{4.2}$$

$$\begin{cases} \text{(a) } \boldsymbol{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{R}^m. \\ \text{(b) There exists } L_G > 0 \text{ such that} \\ \qquad \|\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{\kappa}_1) - \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{\kappa}_2)\| \\ \qquad\quad \le L_G(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2\|) \\ \qquad\qquad \forall\, \boldsymbol{\sigma}_1,\, \boldsymbol{\sigma}_2,\, \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \boldsymbol{\kappa}_1,\, \boldsymbol{\kappa}_2 \in \mathbb{R}^m,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \text{ is measurable on } \Omega, \\ \qquad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \boldsymbol{\kappa} \in \mathbb{R}^m. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } L^2(\Omega)^m. \end{cases} \tag{4.3}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d) \tag{4.4}$$

and the normal compliance function p satisfies

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p\, |r_1 - r_2| \\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0 \\ \qquad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{4.5}$$

Also, the surface memory function and the initial data verify

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \quad b(t, \boldsymbol{x}) \ge 0 \quad \text{for all } t \in \mathbb{R}_+,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{4.6}$$

$$\boldsymbol{u}_0 \in U, \quad \boldsymbol{\sigma}_0 \in Q, \quad \boldsymbol{\kappa}_0 \in L^2(\Omega)^m, \tag{4.7}$$

where U denotes the set of admissible displacements defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}. \tag{4.8}$$

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ are sufficiently regular functions which satisfy (3.1)–(3.8) and let $\boldsymbol{v} \in U$ and $t > 0$ be given. First, we integrate equations (3.1), (3.2) with the initial conditions (3.8) to obtain

$$\boldsymbol{\sigma}(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \tag{4.9}$$

and

$$\boldsymbol{\kappa}(t) = \int_0^t \boldsymbol{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s))\, ds + \boldsymbol{\kappa}_0. \tag{4.10}$$

Next, we use Green formula (2.2) and the equilibrium equation (3.3) to see that

$$\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx=\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx+\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da.$$

We split the surface integral over Γ_1, Γ_2 and Γ_3 and, since $\boldsymbol{v}-\boldsymbol{u}(t)=\boldsymbol{0}$ a.e. on Γ_1, $\boldsymbol{\sigma}(t)\boldsymbol{\nu}=\boldsymbol{f}_2(t)$ on Γ_2, we deduce that

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx&=\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx\\&+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da+\int_{\Gamma_3}\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da.\end{aligned}$$

Moreover, since

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{v}-\boldsymbol{u}(t))=\sigma_\nu(t)(v_\nu-u_\nu(t))+\boldsymbol{\sigma}_\tau(t)\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\quad\text{on }\Gamma_3,$$

taking into account the frictionless condition (3.6) we obtain

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,dx&=\int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,dx\\&+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,da+\int_{\Gamma_3}\sigma_\nu(t)(v_\nu-u_\nu(t))\,da.\end{aligned}\tag{4.11}$$

We write now

$$\begin{aligned}\sigma_\nu(t)(v_\nu-u_\nu(t))&=(\sigma_\nu(t)+p(u_\nu(t))+\xi(t))(v_\nu-g)\\&+(\sigma_\nu(t)+p(u_\nu(t))+\xi(t))(g-u_\nu(t))\\&-(p(u_\nu(t))+\xi(t))(v_\nu-u_\nu(t))\quad\text{on }\Gamma_3,\end{aligned}$$

then we use the contact conditions (3.7) and the definition (4.8) of the set U to see that

$$\sigma_\nu(t)(v_\nu-u_\nu(t))\ge-(p(u_\nu(t))+\xi(t))(v_\nu-u_\nu(t))\quad\text{on }\Gamma_3.\tag{4.12}$$

We use (3.7), again, and the hypothesis (4.6) on function b to deduce that

$$\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds\Big)(v_\nu^+-u_\nu^+(t))\ge\xi(t)(v_\nu-u_\nu(t))\quad\text{on }\Gamma_3.\tag{4.13}$$

Then we add the inequalities (4.12) and (4.13) and integrate the result on Γ_3 to find that

$$\begin{aligned}&\int_{\Gamma_3}\sigma_\nu(t)(v_\nu-u_\nu(t))\,da\\&\qquad\ge-(p(u_\nu(t)),v_\nu-u_\nu(t))_{L^2(\Gamma_3)}-\Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds,v_\nu^+-u_\nu^+(t)\Big)_{L^2(\Gamma_3)}.\end{aligned}\tag{4.14}$$

Finally, we combine (4.11) and (4.14) to deduce that

$$
\begin{aligned}
&(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (p(u_\nu(t)), v_\nu - u_\nu(t))_{L^2(\Gamma_3)} \\
&\quad + \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, v_\nu^+ - u_\nu^+(t) \Big)_{L^2(\Gamma_3)} \\
&\quad\quad \geq (\boldsymbol{f}_0(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v} - \boldsymbol{u}(t))_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in U.
\end{aligned}
\tag{4.15}
$$

We gather the results above to obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to U$*, a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ *and an internal state variable* $\boldsymbol{\kappa} : \mathbb{R}_+ \to L^2(\Omega)^m$ *such that* (4.9), (4.10) *and* (4.15) *hold, for all* $t \in \mathbb{R}_+$.

In the study of the problem $\mathcal{P}^V$ we have the following existence and uniqueness result.

Theorem 4.1 *Assume that* (4.1)–(4.7) *hold. Then, Problem* $\mathcal{P}^V$ *has a unique solution which satisfies*

$$
\boldsymbol{u} \in C(\mathbb{R}_+; U), \quad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q) \quad \text{and} \quad \boldsymbol{\kappa} \in C(\mathbb{R}_+; L^2(\Omega)^m). \tag{4.16}
$$

We now turn to the proof of the theorem. We start with the following existence and uniqueness result.

Lemma 4.2 *Assume that* (4.1)–(4.3) *and* (4.7) *hold. Then, for each* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\mathcal{S}\boldsymbol{u} = (\mathcal{S}_1\boldsymbol{u}, \mathcal{S}_2\boldsymbol{u}) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *such that*

$$
\mathcal{S}_1\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_2\boldsymbol{u}(s))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.17}
$$

$$
\mathcal{S}_2\boldsymbol{u}(t) = \int_0^t \boldsymbol{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_2\boldsymbol{u}(s))ds + \boldsymbol{\kappa}_0 \tag{4.18}
$$

for all $t \in \mathbb{R}_+$*. Moreover, the operator* $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *is a history-dependent operator, i.e. it satisfies the following property: for every* $n \in \mathbb{N}^*$ *there exists* $s_n > 0$ *which depends only on* n*,* d*,* $\mathcal{G}$*,* $\boldsymbol{G}$ *and* $\mathcal{E}$*, such that*

$$
\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t)\|_{Q \times L^2(\Omega)^m} \leq s_n \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V\, ds \tag{4.19}
$$

$$
\forall\, \boldsymbol{u},\, \boldsymbol{v} \in C(\mathbb{R}_+; V) \quad \forall\, t \in [0, n].
$$

Proof. Let $\boldsymbol{u} \in C(\mathbb{R}_+; V)$. We consider the operator $\Lambda : C(\mathbb{R}_+; Q \times L^2(\Omega)^m) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ defined by

$$\Lambda\boldsymbol{\tau}(t) = (\Lambda_1\boldsymbol{\tau}(t), \Lambda_2\boldsymbol{\tau}(t)), \tag{4.20}$$

$$\Lambda_1\boldsymbol{\tau}(t) = \int_0^t \mathcal{G}(\boldsymbol{\alpha}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\beta}(s))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.21}$$

$$\Lambda_2\boldsymbol{\tau}(t) = \int_0^t \boldsymbol{G}(\boldsymbol{\alpha}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\beta}(s))ds + \boldsymbol{\kappa}_0 \tag{4.22}$$

for all $\boldsymbol{\tau} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ and $t \in \mathbb{R}_+$. Note that the operator Λ depends on $\boldsymbol{u}$ but, for simplicity, we do not indicate explicitly this dependence.

Let $\boldsymbol{\tau}_1 = (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$, $\boldsymbol{\tau}_2 = (\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ and let $t \in \mathbb{R}_+$. Then, using definition (4.20)–(4.22) and assumptions (4.2), (4.3), we deduce that

$$\begin{aligned}
&\|\Lambda\boldsymbol{\tau}_1(t) - \Lambda\boldsymbol{\tau}_2(t)\|_{Q\times L^2(\Omega)^m} \\
&\quad \le (L_{\mathcal{G}} + L_G)\int_0^t \Big(\|\boldsymbol{\alpha}_1(s) - \boldsymbol{\alpha}_2(s)\|_Q + \|\boldsymbol{\beta}_1(s) - \boldsymbol{\beta}_2(s)\|_{L^2(\Omega)^m}\Big)ds \\
&\quad = \sqrt{2}\,(L_{\mathcal{G}} + L_G)\int_0^t \|\boldsymbol{\tau}_1(s) - \boldsymbol{\tau}_2(s)\|_{Q\times L^2(\Omega)^m} ds.
\end{aligned}$$

This inequality combined with Theorem 2.1 shows that the operator Λ has a unique fixed point in $C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$, denoted $\boldsymbol{\mathcal{S}}\boldsymbol{u} = (\mathcal{S}_1\boldsymbol{u}, \mathcal{S}_2\boldsymbol{u})$. Moreover, combining (4.20)–(4.22) with equality $\Lambda(\boldsymbol{\mathcal{S}}\boldsymbol{u}) = \boldsymbol{\mathcal{S}}\boldsymbol{u}$ we deduce that (4.17)–(4.18) hold.

To proceed, let $\boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+; V), n \in \mathbb{N}^*$ and let $t \in [0, n]$. Then using (4.17)–(4.18) and taking into account (4.1)–(4.3) and (2.3) we obtain that

$$\begin{aligned}
\|\mathcal{S}_1\boldsymbol{u}(t) - \mathcal{S}_1\boldsymbol{v}(t)\|_Q &= \Big\| \int_0^t \mathcal{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_2\boldsymbol{u}(s))\, ds \\
&\qquad - \int_0^t \mathcal{G}(\mathcal{S}_1\boldsymbol{v}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \boldsymbol{\varepsilon}(\boldsymbol{v}(s)), \mathcal{S}_2\boldsymbol{v}(s))\, ds \Big\|_Q \\
&\le L_{\mathcal{G}} \int_0^t \Big(\|\mathcal{S}_1\boldsymbol{u}(s) - \mathcal{S}_1\boldsymbol{v}(s)\|_Q + \|\mathcal{S}_2\boldsymbol{u}(s) - \mathcal{S}_2\boldsymbol{v}(s)\|_{L^2(\Omega)^m} \Big) ds \\
&\qquad + L_{\mathcal{G}}(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1) \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \boldsymbol{\varepsilon}(\boldsymbol{v}(s))\|_Q\, ds \\
&\le \sqrt{2}\, L_{\mathcal{G}} \int_0^t \|\boldsymbol{\mathcal{S}}\boldsymbol{u}(s) - \boldsymbol{\mathcal{S}}\boldsymbol{v}(s)\|_{Q\times L^2(\Omega)^m}\, ds \\
&\qquad + L_{\mathcal{G}}(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + 1) \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V\, ds,
\end{aligned}$$

$$
\begin{aligned}
\|\mathcal{S}_2\boldsymbol{u}(t)-\mathcal{S}_2\boldsymbol{v}(t)\|_{L^2(\Omega)^m} &= \Big\| \int_0^t \boldsymbol{G}(\mathcal{S}_1\boldsymbol{u}(s)+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\mathcal{S}_2\boldsymbol{u}(s))\,ds \\
&\qquad - \int_0^t \boldsymbol{G}(\mathcal{S}_1\boldsymbol{v}(s)+\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}(s)),\boldsymbol{\varepsilon}(\boldsymbol{v}(s)),\mathcal{S}_2\boldsymbol{v}(s))ds\Big\|_{L^2(\Omega)^m} \\
&\le L_G \int_0^t \Big(\|\mathcal{S}_1\boldsymbol{u}(s)-\mathcal{S}_1\boldsymbol{v}(s)\|_Q + \|\mathcal{S}_2\boldsymbol{u}(s)-\mathcal{S}_2\boldsymbol{v}(s)\|_{L^2(\Omega)^m}\Big)\,ds \\
&\qquad + L_G(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\boldsymbol{\varepsilon}(\boldsymbol{v}(s))\|_Q\,ds \\
&\le \sqrt{2}\,L_G\int_0^t \|\boldsymbol{\mathcal{S}}\boldsymbol{u}(s)-\boldsymbol{\mathcal{S}}\boldsymbol{v}(s)\|_{Q\times L^2(\Omega)^m}\,ds \\
&\qquad + L_G(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)\int_0^t \|\boldsymbol{u}(s)-\boldsymbol{v}(s)\|_V\,ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|\boldsymbol{\mathcal{S}}\boldsymbol{u}(t)-\boldsymbol{\mathcal{S}}\boldsymbol{v}(t)\|_{Q\times L^2(\Omega)^m} &\le \|\mathcal{S}_1\boldsymbol{u}(t)-\mathcal{S}_1\boldsymbol{v}(t)\|_Q + \|\mathcal{S}_2\boldsymbol{u}(t)-\mathcal{S}_2\boldsymbol{v}(t)\|_{L^2(\Omega)^m} \\
&\le \mathcal{K}\Big(\int_0^t \|\boldsymbol{\mathcal{S}}\boldsymbol{u}(s)-\boldsymbol{\mathcal{S}}\boldsymbol{v}(s)\|_{Q\times L^2(\Omega)^m}\,ds + \int_0^t \|\boldsymbol{u}(s)-\boldsymbol{v}(s)\|_V\,ds\Big),
\end{aligned}
$$

where

$$
\mathcal{K}=\max\,\{\sqrt{2}\,(L_{\mathcal{G}}+L_G),(L_{\mathcal{G}}+L_G)(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}+1)\}. \tag{4.23}
$$

Using now a Gronwall argument we deduce that

$$
\|\boldsymbol{\mathcal{S}}\boldsymbol{u}(t)-\boldsymbol{\mathcal{S}}\boldsymbol{v}(t)\|_{Q\times L^2(\Omega)^m} \le \mathcal{K}e^{n\mathcal{K}}\int_0^t \|\boldsymbol{u}(s)-\boldsymbol{v}(s)\|_V ds. \tag{4.24}
$$

This inequality shows that inequality (4.19) holds with $s_n=\mathcal{K}e^{n\mathcal{K}}$, which concludes the proof. □

Next, using the Riesz representation Theorem we define the operators $P:V\to V$, $\mathcal{B}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;L^2(\Gamma_3))$ and the function $\boldsymbol{f}:\mathbb{R}_+\to V$ by equalities

$$
(P\boldsymbol{u},\boldsymbol{v})_V=\int_{\Gamma_3} p(u_\nu)v_\nu\,da \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V, \tag{4.25}
$$

$$
\begin{aligned}
(\mathcal{B}\boldsymbol{u}(t),\xi)_{L^2(\Gamma_3)} &= \Big(\int_0^t b(t-s)\,u_\nu^+(s)\,ds,\xi\Big)_{L^2(\Gamma_3)} \\
&\forall\,\boldsymbol{u}\in C(\mathbb{R}_+;V),\ \xi\in L^2(\Gamma_3),\ t\in\mathbb{R}_+,
\end{aligned} \tag{4.26}
$$

$$
(\boldsymbol{f}(t),\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v}\in V,\ t\in\mathbb{R}_+. \tag{4.27}
$$

We use the operator $\boldsymbol{\mathcal{S}}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;Q\times L^2(\Omega)^m)$ defined in Lemma 4.2 to obtain the following equivalence result.

Lemma 4.3 *Let $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ be a triple of functions which satisfy* (4.16)*. Then $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ is a solution of $\mathcal{P}^V$ if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1(\boldsymbol{u}(t)), \tag{4.28}$$

$$\boldsymbol{\kappa}(t) = \mathcal{S}_2\boldsymbol{u}(t), \tag{4.29}$$

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &\quad +(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + (\mathcal{B}\boldsymbol{u}(t)\,, v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \\ &\qquad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V, \quad \forall\, \boldsymbol{v} \in U,\end{aligned} \tag{4.30}$$

for all $t \in \mathbb{R}_+$.

Proof. First we suppose that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ is solution for Problem $\mathcal{P}^V$ and let $t \in \mathbb{R}_+$. Using (4.9) and (4.10) we obtain

$$\begin{aligned}&\boldsymbol{\sigma}(t) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \\ &\quad = \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s)\big)ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),\end{aligned} \tag{4.31}$$

$$\boldsymbol{\kappa}(t) = \int_0^t \boldsymbol{G}\big(\boldsymbol{\sigma}(s) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\kappa}(s)\big)ds + \boldsymbol{\kappa}_0. \tag{4.32}$$

We now use the definitions of $\mathcal{S}_1$ and $\mathcal{S}_2$ in Lemma 4.2 to obtain (4.28) and (4.29). Then we combine (4.15), (4.28) and use notation (4.25)–(4.27) to see that (4.30) holds.

Conversely, assume that $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ satisfies (4.28)–(4.30) and let $t \in \mathbb{R}_+$. We use (4.28), (4.29) and the definitions (4.17), (4.18) of the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ to obtain (4.31) and (4.32), which show that (4.9) and (4.10) hold. Moreover, using (4.28), (4.30) and the definitions (4.25)–(4.27) we find (4.15), which concludes the proof. □

We are now in position to provide the proof for Theorem 4.1.

Proof. We first define the operators $A : V \to V$, $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Gamma_3))$ and the functional $\varphi : Q \times L^2(\Gamma_3) \times V \to \mathbb{R}$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v}) = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.33}$$

$$\mathcal{R}\boldsymbol{u}(t) = \big(\mathcal{S}_1\boldsymbol{u}(t), \mathcal{B}\boldsymbol{u}(t)\big) \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V), \tag{4.34}$$

$$\varphi(\boldsymbol{\sigma}, \xi, \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\xi^+, v_\nu^+)_{L^2(\Gamma_3)} \quad \forall\, \boldsymbol{\sigma} \in Q,\ \xi \in L^2(\Gamma_3),\ \boldsymbol{v} \in V. \tag{4.35}$$

With these notation we consider the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the following inequality holds:

$$\begin{aligned}&\boldsymbol{u}(t) \in U, \quad (A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \varphi(\mathcal{R}\boldsymbol{u}(t), \boldsymbol{v}) - \varphi(\mathcal{R}\boldsymbol{u}(t), \boldsymbol{u}(t)) \\ &\qquad \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.\end{aligned} \tag{4.36}$$

In order to solve (4.36) we employ Theorem 2.2 with $X = V$, $K = U$ and $Y = Q \times L^2(\Gamma_3)$. To this end we use the definition (4.33) and inequalities (2.1), (2.3) to obtain that

$$\begin{aligned}|(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{w})_V| &\le |(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q| + |(P\boldsymbol{u} - P\boldsymbol{v}, \boldsymbol{w})_V| \\ &\le d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{u} - \boldsymbol{v}\|_V\|\boldsymbol{w}\|_V + L_p\|\boldsymbol{w}\|_{L^2(\Gamma_3)^d}\|\boldsymbol{u} - \boldsymbol{v}\|_{L^2(\Gamma_3)^d} \\ &\le \big(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + c_0^2 L_p\big)\|\boldsymbol{u} - \boldsymbol{v}\|_V\|\boldsymbol{w}\|_V \qquad \forall\,\boldsymbol{u},\,\boldsymbol{v},\,\boldsymbol{w} \in V.\end{aligned}$$

Then we take $\boldsymbol{w} = A\boldsymbol{u} - A\boldsymbol{v}$ in the previous inequality to find that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \le \big(d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} + c_0^2 L_p\big)\|\boldsymbol{u} - \boldsymbol{v}\|_V \qquad \forall\,\boldsymbol{u},\,\boldsymbol{v} \in V. \tag{4.37}$$

On the other hand, from (4.1) and the monotonicity of the function p we deduce that

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \ge m_{\mathcal{E}}\|\boldsymbol{u} - \boldsymbol{v}\|_V^2. \tag{4.38}$$

Inequalities (4.37) and (4.38) imply that the operator A satisfies assumption (2.7).

Let $n \in \mathbb{N}^*$ and let $t \in [0, n]$. Then, using (4.34), (4.19) and the trace inequality (2.1) we find that

$$\begin{aligned}&\|\mathcal{R}\boldsymbol{u}(t) - \mathcal{R}\boldsymbol{v}(t)\|_{Q\times L^2(\Gamma_3)} \\ &\qquad \le \big(s_n + c_0 \cdot \max_{r\in[0,n]}\|b(r)\|_{L^\infty(\Gamma_3)}\big)\int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V ds\end{aligned}$$

which shows that (2.8) holds with $r_n = s_n + c_0 \cdot \max_{r\in[0,n]}\|b(r)\|_{L^\infty(\Gamma_3)}$.

We now take into account (4.35) and (2.1) to deduce that

$$\begin{aligned}&\varphi\big((\boldsymbol{\sigma}_1, \xi_1), \boldsymbol{u}_2\big) - \varphi\big((\boldsymbol{\sigma}_1, \xi_1), \boldsymbol{u}_1\big) + \varphi\big((\boldsymbol{\sigma}_2, \xi_2), \boldsymbol{u}_1\big) - \varphi\big((\boldsymbol{\sigma}_2, \xi_2), \boldsymbol{u}_2\big) \\ &\qquad = \big(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_2) - \boldsymbol{\varepsilon}(\boldsymbol{u}_1)\big)_Q + \big(\xi_1^+ - \xi_2^+, u_{2\nu}^+ - u_{1\nu}^+\big)_{L^2(\Gamma_3)} \\ &\le (\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_Q + c_0\|\xi_1 - \xi_2\|_{L^2(\Gamma_3)})\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \\ &\qquad \le \sqrt{2}\,\max\{1,\,c_0\}\|(\boldsymbol{\sigma}_1, \xi_1) - (\boldsymbol{\sigma}_2, \xi_2)\|_{Q\times L^2(\Gamma_3)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V, \\ &\qquad\qquad \forall\,\boldsymbol{\sigma}_1,\,\boldsymbol{\sigma}_1 \in Q,\ \xi_1,\,\xi_2 \in L^2(\Gamma_3),\ \boldsymbol{u}_1,\,\boldsymbol{u}_2 \in V,\end{aligned}$$

which shows that (2.9) (b) holds with $\alpha = \sqrt{2}\,\max\{1,\,c_0\}$. In addition, we note that the function $\varphi((\boldsymbol{\sigma}, \xi), \cdot) : V \to \mathbb{R}$ is convex and lower semi-continuous for all $(\boldsymbol{\sigma}, \xi) \in Q \times L^2(\Gamma_3)$ and, therefore, (2.9) (a) holds, too.

Finally, using assumption (4.4) and definition (4.27) we deduce that $\boldsymbol{f}$ has the regularity expressed in (2.10). It follows now from Theorem 2.2 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which solves the inequality (4.36). And, using notation (4.33)–(4.35), we deduce the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; U)$

which satisfies (4.30) for any $t \in \mathbb{R}_+$. Let $\boldsymbol{\sigma}$, $\boldsymbol{\kappa}$ be the functions defined by (4.28) and (4.29). Then, it follows that the triple $(\boldsymbol{u},\, \boldsymbol{\sigma},\, \boldsymbol{\kappa})$ is the unique triple of functions with regularity (4.16) which satisfies (4.28)–(4.30). Theorem 4.1 is now a consequence of Lemma 4.3. □

We refer in the rest of the paper to solution of Problem $\mathcal{P}^V$ as a *weak solution* to the contact problem $\mathcal{P}$. We conclude by Theorem 4.1 that, Problem $\mathcal{P}$ has a unique weak solution solution with regularity (4.16), provided that (4.1)–(4.7) hold.

5 A convergence result

We now study the dependence of the solution of Problem $\mathcal{P}^V$ with respect to perturbations of the data. To this end, we assume in what follows that (4.1)–(4.7) hold and we denote by $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ the solution of Problem $\mathcal{P}^V$ obtained in Theorem 4.1. For each $\rho > 0$ let p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\boldsymbol{u}_{0\rho}$, $\boldsymbol{\sigma}_{0\rho}$ and $\boldsymbol{\kappa}_{0\rho}$ represent perturbations of p, b, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\kappa}_0$, respectively, which satisfy conditions (4.4)–(4.7). With these data, we consider the following perturbation of Problem $\mathcal{P}^V$.

Problem $\mathcal{P}^V_\rho$. *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U$*, a stress field* $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$ *and an internal state variable* $\boldsymbol{\kappa}_\rho : \mathbb{R}_+ \to L^2(\Omega)^m$ *such that*

$$\boldsymbol{\sigma}_\rho(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)), \boldsymbol{\kappa}_\rho(s))\, ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \tag{5.1}$$

$$\boldsymbol{\kappa}_\rho(t) = \int_0^t \boldsymbol{G}(\boldsymbol{\sigma}_\rho(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(s)), \boldsymbol{\kappa}_\rho(s))\, ds + \boldsymbol{\kappa}_{0\rho}, \tag{5.2}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (p_\rho(u_{\rho\nu}(t)), v_\nu - u_{\rho\nu}(t))_{L^2(\Gamma_3)} \\ &\quad + \Big(\int_0^t b_\rho(t-s)\, u^+_{\rho\nu}(s)\, ds, v^+_\nu - u^+_{\rho\nu}(t) \Big)_{L^2(\Gamma_3)} \\ &\quad\quad \ge (\boldsymbol{f}_{0\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Omega)^d} + (\boldsymbol{f}_{2\rho}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in U, \end{aligned} \tag{5.3}$$

for all $t \in \mathbb{R}_+$.

Here and below $u_{\rho\nu}$ represents the normal component of the function $\boldsymbol{u}_\rho$. It follows from Theorem 4.1 that, for each $\rho > 0$, Problem $\mathcal{P}^V_\rho$ has a unique solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\kappa}_\rho)$ with the regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q)$ and $\boldsymbol{\kappa}_\rho \in C(\mathbb{R}_+; L^2(\Omega)^m)$. Consider now the following assumptions:

$$\left\{ \begin{array}{l} \text{There exists } F : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \alpha \in \mathbb{R}_+ \text{ such that} \\ \ \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \le F(\rho)(|r| + \alpha) \\ \qquad \forall\, r \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3,\ \text{for each } \rho > 0. \\ \ \text{(b) } F(\rho) \to 0 \quad \text{as} \quad \rho \to 0. \end{array} \right. \tag{5.4}$$

$$b_\rho \to b \quad \text{in} \quad C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0. \tag{5.5}$$

$$\boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0. \tag{5.6}$$

$$\boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0. \tag{5.7}$$

$$\boldsymbol{u}_{0\rho} \to \boldsymbol{u}_0 \quad \text{in} \quad V \quad \text{as} \quad \rho \to 0. \tag{5.8}$$

$$\boldsymbol{\sigma}_{0\rho} \to \boldsymbol{\sigma}_0 \quad \text{in} \quad Q \quad \text{as} \quad \rho \to 0. \tag{5.9}$$

$$\boldsymbol{\kappa}_{0\rho} \to \boldsymbol{\kappa}_0 \quad \text{in} \quad L^2(\Omega)^m \quad \text{as} \quad \rho \to 0. \tag{5.10}$$

We have the following convergence result.

Theorem 5.1 *Assume that* (5.4)–(5.10) *hold. Then the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\kappa}_\rho)$ *of Problem* $\mathcal{P}_\rho^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\kappa})$ *of Problem* $\mathcal{P}^V$*, i.e.*

$$\begin{cases} \boldsymbol{u}_\rho \to \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V), \\ \boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C(\mathbb{R}_+; Q), \\ \boldsymbol{\kappa}_\rho \to \boldsymbol{\kappa} \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^m) \end{cases} \tag{5.11}$$

as $\rho \to 0$.

Proof. Let $\rho > 0$. We define the operators $P_\rho : V \to V$, $\mathcal{B}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ and the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{5.12}$$

$$(\mathcal{B}_\rho \boldsymbol{u}(t), \xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b_\rho(t-s)\, u_\nu^+(s)\, ds, \xi \Big)_{L^2(\Gamma_3)} \tag{5.13}$$
$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \xi \in L^2(\Gamma_3),\ t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}_\rho(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0\rho}(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2\rho}(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{5.14}$$

Also, we use Lemma 4.2 to define the operator $\mathcal{S}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ by equalities

$$\mathcal{S}_\rho \boldsymbol{u}(t) = (\mathcal{S}_{1\rho} \boldsymbol{u}(t), \mathcal{S}_{2\rho} \boldsymbol{u}(t)), \tag{5.15}$$

$$\mathcal{S}_{1\rho} \boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_{1\rho}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_{2\rho}\boldsymbol{u}(s)) ds + \boldsymbol{\sigma}_{0\rho} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}), \tag{5.16}$$

$$\mathcal{S}_{2\rho} \boldsymbol{u}(t) = \int_0^t \boldsymbol{G}(\mathcal{S}_{1\rho}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \mathcal{S}_{2\rho}\boldsymbol{u}(s)) ds + \boldsymbol{\kappa}_{0\rho} \tag{5.17}$$

for all $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and $t \in \mathbb{R}_+$. Finally, we recall Lemma 4.3 which shows that the solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho, \boldsymbol{\kappa}_\rho)$ satisfies

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \mathcal{S}_{1\rho}(\boldsymbol{u}_\rho(t)), \tag{5.18}$$

$$\boldsymbol{\kappa}_\rho(t) = \mathcal{S}_{2\rho}\boldsymbol{u}_\rho(t), \tag{5.19}$$

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (\mathcal{S}_{1\rho}\boldsymbol{u}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ +(P_\rho\boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V + \Big(\mathcal{B}_\rho\boldsymbol{u}_\rho(t), v_\nu^+ - u_{\rho\nu}^+(t)\Big)_{L^2(\Gamma_3)} \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V,\end{aligned} \tag{5.20}$$

for all $t \in \mathbb{R}_+$.

Let $\rho > 0$, $n \in \mathbb{N}^*$ and let $t \in [0, n]$. We take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (5.20) and $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in (4.30) and add the resulting inequalities to obtain

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ &\leq (\mathcal{S}_{1\rho}\boldsymbol{u}_\rho(t) - \mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ &\quad +(P_\rho\boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\ &\quad +(\mathcal{B}_\rho\boldsymbol{u}_\rho(t) - \mathcal{B}\boldsymbol{u}(t), u_\nu^+(t) - u_{\rho\nu}^+(t))_{L^2(\Gamma_3)} \\ &\quad +(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V.\end{aligned} \tag{5.21}$$

We now estimate each term in the previous inequality. First, we use assumption (4.1) to deduce that

$$m_\mathcal{E}\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2 \leq (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q. \tag{5.22}$$

Next, using the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}&(\mathcal{S}_{1\rho}\boldsymbol{u}_\rho(t) - \mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ &\leq \|\boldsymbol{\mathcal{S}}_\rho\boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m}\|\boldsymbol{u}(t) - \boldsymbol{u}_\rho(t)\|_V.\end{aligned} \tag{5.23}$$

Moreover, by arguments similar to those used in the proof of (4.24) we deduce that

$$\|\boldsymbol{\mathcal{S}}_\rho\boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m} \leq \Big(\mathcal{K}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V ds + \tau_{0\rho}\Big)e^{n\mathcal{K}}. \tag{5.24}$$

where $\mathcal{K}$ is given by (4.23) and

$$\tau_{0\rho} = \|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + d\,\|\mathcal{E}\|_{\mathbf{Q}^\infty}\|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V + \|\boldsymbol{\kappa}_{0\rho} - \boldsymbol{\kappa}_0\|_{L^2(\Omega)^m}. \tag{5.25}$$

We combine now (5.23) and (5.24) and use the notation $s_n = \mathcal{K}e^{n\mathcal{K}}$ introduced in the proof of Lemma 4.2 to deduce that

$$\begin{aligned}&(\mathcal{S}_{1\rho}\boldsymbol{u}_\rho(t) - \mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q \\ &\leq \Big(s_n\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\,ds + \tau_{0\rho}e^{n\mathcal{K}}\Big)\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.\end{aligned} \tag{5.26}$$

To proceed, we use the definitions (5.12) and (4.25), the monotonicity of the function p_ρ and assumption (5.4) to see that

$$
\begin{aligned}
&(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\
&\quad = \int_{\Gamma_3} \big(p_\rho(u_{\rho\nu}(t)) - p(u_\nu(t))\big)(u_\nu(t) - u_{\rho\nu}(t))\, da \\
&\quad \le \int_{\Gamma_3} \big(p_\rho(u_\nu(t)) - p(u_\nu(t))\big)(u_\nu(t) - u_{\rho\nu}(t))\, da \\
&\quad \le \int_{\Gamma_3} |p_\rho(u_\nu(t)) - p(u_\nu(t))|\, |u_\nu(t) - u_{\rho\nu}(t)|\, da \\
&\quad \le \int_{\Gamma_3} F(\rho)(|u_\nu(t)| + \alpha)\, |u_\nu(t) - u_{\rho\nu}(t)|\, da.
\end{aligned}
$$

Therefore, using the trace inequality (2.1), after some elementary calculus we find that

$$
\begin{aligned}
&(P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \\
&\quad \le F(\rho)\big(c_0^2\, \|\boldsymbol{u}(t)\|_V + c_0\alpha\, \mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}}\big)\, \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.
\end{aligned} \tag{5.27}
$$

Next, using definitions (5.13), (4.26) and condition (4.6) we have

$$
\begin{aligned}
&(\mathcal{B}_\rho \boldsymbol{u}_\rho(t) - \mathcal{B}\boldsymbol{u}(t), u_\nu^+(t) - u_{\rho\nu}^+(t))_{L^2(\Gamma_3)} \\
&\quad = \Big(\int_0^t \Big(b_\rho(t-s)u_{\rho\nu}^+(s) - b(t-s)u_\nu^+(s)\Big)ds, u_\nu^+(t) - u_{\rho\nu}^+(t)\Big)_{L^2(\Gamma_3)} \\
&\quad \le \Big(\int_0^t \|b_\rho(t-s)(u_{\rho\nu}^+(s) - u_\nu^+(s))\|_{L^2(\Gamma_3)} ds \\
&\quad + \int_0^t \|b_\rho(t-s)u_\nu^+(s) - b(t-s)u_\nu^+(s)\|_{L^2(\Gamma_3)} ds\Big) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_{L^2(\Gamma_3)^d}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\mathcal{B}_\rho \boldsymbol{u}_\rho(t) - \mathcal{B}\boldsymbol{u}(t), u_\nu^+(t) - u_{\rho\nu}^+(t))_{L^2(\Gamma_3)} \\
&\quad \le \Big(\theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V\, ds + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|_V\, ds\Big) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V,
\end{aligned} \tag{5.28}
$$

where

$$
\theta_{\rho n} = c_0^2 \max_{r \in [0,n]} \|b_\rho(r)\|_{L^\infty(\Gamma_3)}, \tag{5.29}
$$

$$
\omega_{\rho n} = c_0^2 \max_{r \in [0,n]} \|b_\rho(r) - b(r)\|_{L^\infty(\Gamma_3)}. \tag{5.30}
$$

Finally, it is easy to see that

$$(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \le \delta_{\rho n} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V, \tag{5.31}$$

where

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V. \tag{5.32}$$

We now combine (5.21), (5.22), (5.26), (5.27), (5.28) and (5.31) to deduce that

$$\begin{aligned} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le \frac{s_n}{m_\mathcal{E}} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds + \frac{\tau_{0\rho} e^{n\mathcal{K}}}{m_\mathcal{E}} \\ + \frac{F(\rho)}{m_\mathcal{E}} \big(c_0^2 \, \|\boldsymbol{u}(t)\|_V + c_0 \alpha \, \mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}} \big) \\ + \frac{\theta_{\rho n}}{m_\mathcal{E}} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds + \frac{\omega_{\rho n}}{m_\mathcal{E}} \int_0^t \|\boldsymbol{u}(s)\|_V \, ds + \frac{\delta_{\rho n}}{m_\mathcal{E}}. \end{aligned} \tag{5.33}$$

Let

$$\begin{aligned} \xi_{n,u} = \max \Big\{ \frac{e^{n\mathcal{K}}}{m_\mathcal{E}}, \frac{1}{m_\mathcal{E}} \big(c_0^2 \max_{t\in[0,n]} \|\boldsymbol{u}(t)\|_V + c_0 \, \alpha \, \mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}} \big), \\ \frac{1}{m_\mathcal{E}} \int_0^n \|\boldsymbol{u}(s)\|_V \, ds, \frac{1}{m_\mathcal{E}} \Big\} \end{aligned}$$

and note that $\xi_{n,u}$ depends on n, $\boldsymbol{u}$, d, $\mathcal{E}$, $\mathcal{G}$, G, c_0, α and Γ_3 but does not depends neither on ρ nor on t. Then, (5.33) yields

$$\begin{aligned} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le (F(\rho) + \omega_{\rho n} + \delta_{\rho n} + \tau_{0\rho}) \xi_{n,u} \\ + \frac{\theta_{\rho n} + s_n}{m_\mathcal{E}} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V ds. \end{aligned} \tag{5.34}$$

Next, we use assumption (5.5) and equivalence (2.4) to see that the sequence $(\theta_{\rho n})_\rho$ defined by (5.29) is bounded. Therefore, there exists $\zeta_n > 0$ which depends on n and is independent of ρ such that

$$0 \le \frac{\theta_{\rho n} + s_n}{m_\mathcal{E}} \le \zeta_n \quad \text{for all} \quad \rho > 0$$

and, using this inequality in (5.34) we obtain that

$$\begin{aligned} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le (F(\rho) + \omega_{\rho n} + \delta_{\rho n} + \tau_{0\rho}) \xi_{n,u} \\ + \zeta_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V ds. \end{aligned} \tag{5.35}$$

Then, we use the Gronwall inequality to see that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le (F(\rho) + \omega_{\rho n} + \delta_{\rho n} + \tau_{0\rho}) \xi_{n,u} e^{t\zeta_n}$$

and, passing to the upper bound as $t \in [0,n]$ we find that

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq (F(\rho) + \omega_{\rho n} + \delta_{\rho n} + \tau_{0\rho})\xi_{n,u} e^{n\zeta_n}. \tag{5.36}$$

Note that (5.5), (2.4) and (5.30) imply that

$$\omega_{\rho n} \to 0 \quad \text{as} \quad \rho \to 0. \tag{5.37}$$

Moreover, (5.6), (5.7), (2.4) and (5.32) yield

$$\delta_{\rho n} \to 0 \quad \text{as} \quad \rho \to 0 \tag{5.38}$$

and, finally, (5.8)–(5.10) and (5.25) show that

$$\tau_{0\rho} \to 0 \quad \text{as} \quad \rho \to 0. \tag{5.39}$$

We use now the convergences (5.4) (b), (5.37)–(5.39) and inequality (5.36) to obtain that

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0. \tag{5.40}$$

On the other hand using equalities (5.18), (5.19) and (4.28), (4.29) we find that

$$\begin{aligned} &\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q + \|\boldsymbol{\kappa}_\rho(t) - \boldsymbol{\kappa}(t)\|_{L^2(\Omega)^m} \\ &\quad \leq d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \sqrt{2}\,\|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m}. \end{aligned} \tag{5.41}$$

We write

$$\begin{aligned} &\|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m} \\ &\quad \leq \|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m} + \|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m}, \end{aligned}$$

then we use inequalities (4.19) and (5.24) to see that

$$\begin{aligned} \|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m} &\leq s_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V ds \\ &+ \Big(\mathcal{K}\int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V ds + \tau_{0\rho}\Big) e^{n\mathcal{K}}. \end{aligned}$$

This inequality combined with convergences (5.39) and (5.40) implies that

$$\max_{t\in[0,n]} \|\boldsymbol{\mathcal{S}}_\rho \boldsymbol{u}_\rho(t) - \boldsymbol{\mathcal{S}}\boldsymbol{u}(t)\|_{Q\times L^2(\Omega)^m} \to 0 \quad \text{as } \rho \to 0. \tag{5.42}$$

Therefore, using equality (5.41) and convergences (5.40), (5.42) we deduce that

$$\max_{t\in[0,n]} \|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \quad \text{as } \rho \to 0, \tag{5.43}$$

$$\max_{t\in[0,n]} \|\boldsymbol{\kappa}_\rho(t) - \boldsymbol{\kappa}(t)\|_{L^2(\Omega)^m} \to 0 \quad \text{as } \rho \to 0. \tag{5.44}$$

The convergence (5.11) is now a direct consequence of the convergences (5.40), (5.43) and (5.44). □

In addition to the mathematical interest in the convergence result (5.11) it is of importance from mechanical point of view, since it states that the weak solution of the problem (3.1)–(3.8) depends continuously on the normal compliance function, the surface memory function, the densities of body forces and surface tractions and the initial data, as well.

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