Penalization of History-Dependent Variational Inequalities

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Abstract

The present paper represents a continuation of [21]. There, a new class of variational inequalities involving history-dependent operators was considered, an abstract existence and uniqueness result was proved and it was completed with a regularity result. Moreover, these results were used in the analysis of various frictional and frictionless models of contact. In this current paper we present a penalization method in the study of such inequalities. We start with an example which motivates our study; it concerns a mathematical model which describes the quasistatic contact between a viscoelastic body and a foundation; the material's behaviour is modelled with a constitutive law with long memory, the contact is frictionless and is modelled with a multivalued normal compliance condition and unilateral constraint. Then, we introduce the abstract variational inequalities together with their penalizations. We prove the unique solvability of the penalized problems and the convergence of their solutions to the solution of the original problem, as the penalization parameter converges to zero. Finally, we turn back to our a contact model, apply our abstract results in the study of this problem and provide their mechanical interpretation.

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1 Introduction

The theory of variational inequalities plays an important role in the study of nonlinear boundary value problems arising in mechanics, physics and engineering science. At the heart of this theory is the intrinsic inclusion of free boundaries in an elegant mathematical formulation. General results on the analysis of the variational inequalities, including existence and uniqueness results, can be found in [1, 2, 11, 13, 17, 22], for instance. Details concerning the numerical analysis of variational inequalities, including solution algorithms and error estimates, can be found in [6, 10]. References in the study of mathematical and numerical analysis of variational inequalities arising in hardening plasticity include [7, 8].

Phenomena of contact between deformable bodies abound in industry and everyday life. For this reason, considerable progress has been achieved recently in modelling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general mathematical theory of contact mechanics is currently emerging. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws, i.e. materials, varied geometries and different contact conditions. To this end, it uses various mathematical concepts which include both variational and hemivariational inequalities and multivalued inclusions. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [4]. An excellent reference on analysis and numerical approximations of contact problems involving elastic materials with or without friction is [12]. The variational analysis of various contact problems can be found in the monographs [5, 9, 10, 12, 16, 17, 20]. The state of the art in the field can be found in the proceedings [14, 18, 24] and in the special issue [19], as well.

Existence, uniqueness and regularity results in the study of a new class of variational inequalities were proved in [21]. There, the first trait of novelty lies in the fact that, unlike the results obtained in literature, the variational inequalities considered were defined on an unbounded interval of time. The second novelty was related to their special structure, which involves two nondifferentiable convex functionals, one of them depending on the history of the solution. This class of variational inequalities represents a general framework in which a large number of quasistatic contact problems, associated with various constitutive laws and frictional contact conditions, can be cast, as exemplified in [22].

Our intention in this current paper is to present a penalization method in the study of the variational inequalities introduced in [21] and to apply it to a new model of contact. Penalization methods in the study of elliptic variational inequalities were used by many authors, mainly for numerical reasons. Details can be found in [6] and the references therein. The main ingredient of these methods arises from the fact that they remove the constraints by considering penalized problems defined on the whole space; these approximative problems have unique solutions which converge to the solutions of the original problems, as the penalization parameter converges to zero.

The rest of the paper is structured as follows. In Section 2 we present a new mathematical model of contact which is of applied interest and which motivates the abstract study we present in this paper. In Section 3 we state the abstract problem and recall its unique solvability, obtained in [21]. Then we state the penalized problems and prove our main result, Theorem 3.2. The proof of this theorem is given in Section 4. Further, we illustrate the use of the abstract results in the study of the contact model introduced in Section 2. To this end, in Section 5 we list the assumptions on the data and derive the variational formulation. Then we state and prove Theorem 5.1 which concerns the unique weak solvability of the model. Next, in Section 6, we use our abstract penalization method. Our main result in this section is given by Theorem 6.1 which states the existence of a unique weak solution of the penalized contact problems and its convergence to the weak solution of the original contact model. Finally, in Section 7, we present some concluding remarks.

2 A viscoelastic contact problem

The physical setting is as follows. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ which is divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the contact process is quasistatic, we study it in the interval of time $\mathbb{R}_+ = [0, \infty)$, and we denote by $\boldsymbol{\nu}$ and $\mathbb{S}^d$ the outward unit normal at Γ and the space of second order symmetric tensors on $\mathbb{R}^d$, respectively. Then, the classical formulation of the contact problem we consider in the rest of this paper is the following.

Problem $\mathcal{Q}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{2.2}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on} \quad \Gamma_1, \tag{2.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{2.4}$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \quad \text{on} \quad \Gamma_3, \tag{2.5}$$

for all $t \in \mathbb{R}_+$, *and there exists* $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{aligned} & u_\nu(t) \le g,\ \ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ & (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ & 0 \le \xi(t) \le F, \\ & \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ & \xi(t) = F \ \text{ if } \ u_\nu(t) > 0 \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{2.6}$$

for all $t \in \mathbb{R}_+$.

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Equation (2.1) represents the viscoelastic constitutive law with long memory in which $\mathcal{A}$ is the elasticity operator, $\mathcal{B}$ represents the relaxation tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor. Equation (2.2) represents the equation of equilibrium in which Div denotes the divergence operator for tensor valued functions. Conditions (2.3) and (2.4) are the displacement boundary condition and the traction boundary condition, respectively. Condition (2.5) is the frictionless condition and it shows that the tangential stress on the contact surface, denoted by $\boldsymbol{\sigma}_\tau$, vanishes. More details on the equations and conditions (2.1)–(2.5) can be found in [22].

We now describe the contact condition (2.6) in which our main interest lies and which represents the main novelty of the model. Here σ_ν denotes the normal stress, u_ν is the normal displacement and u_ν^+ may be interpreted as the penetration of the body's surface asperities and those of the foundation. Moreover, p is a Lipschitz continuous increasing function which vanishes for a negative argument, F is a positive function and $g > 0$. This condition can be derived in the following way. Let $t \in \mathbb{R}_+$ be given. First, we assume that the penetration is limited by the bound g and, therefore, the normal displacement satisfies the inequality

$$u_\nu(t) \le g \quad \text{on } \Gamma_3. \tag{2.7}$$

Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^R(t) + \sigma_\nu^M(t) \quad \text{on } \Gamma_3, \tag{2.8}$$

in which the function $\sigma_\nu^D(t)$ describes the deformability of the foundation and the functions $\sigma_\nu^R(t)$, $\sigma_\nu^M(t)$ describe the rigidity and the memory properties of the foundation, respectively. We assume that $\sigma_\nu^D(t)$ satisfies a normal compliance contact condition, that is

$$-\sigma_\nu^D(t) = p(u_\nu(t)) \quad \text{on } \Gamma_3. \tag{2.9}$$

The part $\sigma_\nu^R(t)$ of the normal stress satisfies the Signorini condition in the form with a gap function, i.e.

$$\sigma_\nu^R(t) \le 0, \quad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_3. \tag{2.10}$$

Finally, the function $\sigma_\nu^M(t)$ satisfies the condition

$$\left\{\begin{array}{l} |\sigma_\nu^M(t)| \le F, \quad \sigma_\nu^M(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ -\sigma_\nu^M(t) = F \ \text{ if } \ u_\nu(t) > 0 \end{array}\right. \qquad \text{on } \Gamma_3. \tag{2.11}$$

We combine (2.8), (2.9) and write $-\sigma_\nu^M(t) = \xi(t)$ to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \quad \text{on } \Gamma_3. \tag{2.12}$$

Then we substitute equality (2.12) in (2.10) and use (2.7), (2.11) to obtain the contact condition (2.6).

We now present additional details of the contact condition (2.6). The inequalities and equalities below in this section are valid at an arbitrary point $\boldsymbol{x} \in \Gamma_3$. First, we recall that (2.6) describes a condition with unilateral constraint, since inequality (2.7) holds at each moment of time. Next, assume that at a given moment t there is separation between the body and the foundation, i.e. $u_\nu(t) < 0$. Then, since $p(u_\nu(t)) = 0$, (2.6) shows that $\sigma_\nu(t) = 0$, i.e. the reaction of the foundation vanishes. Note that the same behaviour of the normal stress is described both in the classical normal compliance condition and in the Signorini contact condition, when there is separation. Assume now that at the moment t there is penetration which did not reach the bound g, i.e. $0 < u_\nu(t) < g$. Then (2.6) yields

$$-\sigma_\nu(t) = p(u_\nu(t)) + F. \tag{2.13}$$

This equality shows that, at the moment t, the reaction of the foundation depends on the penetration and represents a normal compliance-type condition. Note that (2.6) also shows that if at the moment t we have penetration which satisfies $0 < u_\nu(t) < g$ then $-\sigma_\nu(t) \ge F$. Indeed, if $0 < u_\nu(t) < g$ then (2.13) holds and this implies that $-\sigma_\nu(t) \ge F$. We conclude from above that if $-\sigma_\nu(t) < F$ then there is no penetration and, therefore, F represents a yield limit of the normal pressure, under which the penetration is not possible. This kind of behaviour characterizes a rigid-elastic foundation.

In conclusion, condition (2.6) shows that when there is separation between the body's surface and the foundation then the normal stress vanishes; the penetration arises only if the normal stress reaches the critical value F; when there is penetration the contact follows a normal compliance condition of the form (2.13) but up to the limit g and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap g. For this reason we refer to this condition as to a *multivalued normal compliance contact condition with unilateral constraint*. It can be interpreted physically as follows. The foundation is assumed to be made of a hard material covered by a thin layer of a soft material with thickness g. The soft material has a rigid-elastic behaviour, i.e. is deformable, allows penetration, but only if the normal stress arrives to the yield value F; the contact with this layer is modelled with normal compliance, as shown in equality (2.13). The hard material is perfectly rigid and, therefore, it does not allow penetration; the contact with this material is modelled with the Signorini contact condition.

Two questions arise in the study of the unilateral contact problem $\mathcal{Q}$. The first one concerns its unique solvability; the second one concerns the approach of the solution by the solution of a contact model with normal compliance without unilateral constraint. The answers to the questions above are provided by the variational analysis of this contact problem, presented in Section 5 and 6. This analysis is carried out based on the abstract existence, uniqueness and convergence result that we present in the next section.

3 Abstract problem and main result

Everywhere below we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+ = [0, \infty)$. For each normed space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that, if X is a Banach space, then $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Details can be found in [3] and [15], for instance. Here we only need to recall that the convergence of a sequence $(x_k)_k$ to the element x, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} x_k \to x \quad \text{in } C(\mathbb{R}_+; X) \ \text{ as } \ k \to \infty \ \text{ if and only if} \\ \max\limits_{r \in [0,n]} \|x_k(r) - x(r)\|_X \to 0 \ \text{ as } k \to \infty, \quad \text{for all } \ n \in \mathbb{N}^*. \end{cases} \tag{3.1}$$

Consider now a real Hilbert space X with inner product $(\cdot, \cdot)_X$ and associated norm $\|\cdot\|_X$. Also, let K be a subset of X, let $A : X \to X$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be two operators, and let $j : X \to \mathbb{R}$, $f : \mathbb{R}_+ \to X$ be two functions. We assume in what follows that

$$K \text{ is a nonempty closed convex subset of } X, \tag{3.2}$$

and A is strongly monotone and Lipschitz continuous operator, i.e.

$$\begin{cases} \text{(a) There exists } m > 0 \text{ such that} \\ \qquad (Au_1 - Au_2, u_1 - u_2)_X \geq m \, \|u_1 - u_2\|_X^2 \\ \qquad\quad \forall \, u_1, \, u_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \qquad \|Au_1 - Au_2\|_X \leq M \, \|u_1 - u_2\|_X \quad \forall \, u_1, \, u_2 \in X. \end{cases} \tag{3.3}$$

Moreover, we assume that the operator $\mathcal{S}$ satisfies the following condition:

$$\left\{\begin{array}{l}\text{For every } n \in \mathbb{N}^* \text{ there exists } d_n > 0 \text{ such that} \\ \qquad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \le d_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \qquad\qquad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n].\end{array}\right. \tag{3.4}$$

Following the terminology in [21, 22] we refer to an operator $\mathcal{S}$ which satisfies (3.4) as a *history-dependent operator*. Finally, we suppose that

$$j : X \to \mathbb{R} \text{ is a proper convex lower semicontinuous function.} \tag{3.5}$$

$$f \in C(\mathbb{R}_+; X). \tag{3.6}$$

With the data above, we consider the following problem.

Problem $\mathcal{P}$. *Find a function $u : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$\begin{array}{ll} u(t) \in K, & (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \\ & +j(v) - j(u(t)) \ge (f(t), v - u(t))_X \quad \forall\, v \in K. \end{array} \tag{3.7}$$

Following [21, 22] we refer to (3.7) as a *history-dependent variational inequality*. It represents the framework in which the variational formulation of a large number of contact problems can be cast, with the appropriate choice of spaces and operators. Details can be found in [9, 21, 22, 23] and the references therein. The solvability of Problem $\mathcal{P}$ is provided by the following existence and uniqueness result, proved in [21].

Theorem 3.1 *Let X be a Hilbert space and assume that* (3.2)–(3.6) *hold. Then, Problem $\mathcal{P}$ has a unique solution $u \in C(\mathbb{R}_+; K)$.*

In order to formulate the penalized problems associated to Problem $\mathcal{P}$ we consider an operator $G : X \to X$ which satisfies the following conditions:

$$\left\{\begin{array}{ll} \text{(a)} & (Gu - Gv, u - v)_X \ge 0 \quad \forall\, u,\, v \in X. \\ \text{(b)} & \text{There exists } L > 0 \text{ such that} \\ & \|Gu - Gv\|_X \le L\, \|u - v\|_X \quad \forall\, u,\, v \in X. \\ \text{(c)} & (Gu, v - u)_X \le 0 \quad \forall\, u \in X,\, v \in K. \\ \text{(d)} & Gu = 0_X \text{ iff } u \in K. \end{array}\right. \tag{3.8}$$

Note that conditions (3.8)(a) and (b) show that G is a monotone Lipschtz continuous operator. Also, note that such an operator G always exists. For example consider the operator $G : X \to X$ defined by

$$Gu = u - P_K u, \qquad \forall\, u \in K,$$

where $P_K : X \to K$ represents the projection operator onto K. Then, using the properties of the projections, it is easy to see that the operator G satisfies condition (3.8).

Next, for each $\mu > 0$ we consider the following problem.

Problem $\mathcal{P}_\mu$. *Find a function* $u_\mu : \mathbb{R}_+ \to X$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$(Au_\mu(t), v - u_\mu(t))_X + (\mathcal{S}u_\mu(t), v - u_\mu(t))_X + \frac{1}{\mu}\,(Gu_\mu(t), v - u_\mu(t))_X \tag{3.9}$$
$$+j(v) - j(u_\mu(t)) \geq (f(t), v - u_\mu(t))_X \qquad \forall\, v \in X.$$

Note that, in contrast to Problem $\mathcal{P}$, in Problem $\mathcal{P}_\mu$ the constraint $u(t) \in K$ is removed and is replaced with an additional term which contains the penalization parameter μ. For this reason, we refer to Problem $\mathcal{P}_\mu$ as a *penalized* problem associated to Problem $\mathcal{P}$.

We have the following existence, uniqueness and convergence result, which represents the main result of this section.

Theorem 3.2 *Let X be a Hilbert space and assume that* (3.2)–(3.6), (3.8) *hold. Then:*

1) *For each $\mu > 0$ Problem $\mathcal{P}_\mu$ has a unique solution which satisfies $u_\mu \in C(\mathbb{R}_+; X)$.*

2) *The solution u_μ of Problem $\mathcal{P}_\mu$ converges to the solution u of Problem $\mathcal{P}$, that is*

$$\|u_\mu(t) - u(t)\|_X \to 0 \qquad \text{as} \quad \mu \to 0, \tag{3.10}$$

for each $t \in \mathbb{R}_+$.

Note that the convergence (3.10) above is understood in the following sense: for each $t \in \mathbb{R}_+$ and for every sequence $\{\mu_n\} \subset \mathbb{R}_+$ converging to 0 as $n \to \infty$ we have $u_{\mu_n}(t) \to u(t)$ as $n \to \infty$.

4 Proof of Theorem 3.2

The proof of Theorem 3.2 will be carried out in several steps that we present in what follows. To this end, below in this section we assume that (3.2)–(3.6), (3.8) hold and we denote by c a positive constant which may depend on t, A, $\mathcal{S}$, j, f and u, but is independent of μ, and whose value may change from line to line. The following lemma shows the unique solvability of the nonlinear inequality (3.9).

Lemma 4.1 *For each $\mu > 0$ there exists a unique function $u_\mu \in C(\mathbb{R}_+; X)$ which satisfies the inequality* (3.9) *for all $t \in \mathbb{R}_+$.*

Proof. Let $\mu > 0$. Using (3.3) and (3.8) it is easy to show that the operator

$$v \longmapsto Av + \frac{1}{\mu}\, Gv$$

is a strongly monotone Lipschitz continuous operator on X. Lemma 4.1 is now a consequence of Theorem 3.1 used with $K = X$. □

Next, we consider the following intermediate problem.

Problem $\widetilde{\mathcal{P}}_\mu$. *Find a function $\widetilde{u}_\mu : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$(A\widetilde{u}_\mu(t), v - \widetilde{u}_\mu(t))_X + (\mathcal{S}u(t), v - \widetilde{u}_\mu(t))_X + \frac{1}{\mu}\,(G\widetilde{u}_\mu(t), v - \widetilde{u}_\mu(t))_X \qquad (4.1)$$
$$+ j(v) - j(\widetilde{u}_\mu(t)) \geq (f(t), v - \widetilde{u}_\mu(t))_X \quad \forall\, v \in X.$$

Note that inequality (3.9) is a history-dependent variational inequality, since the operator $\mathcal{S}$ is applied to the unknown u_μ. In contrast, the variational inequality (4.1) is a time-dependent variational inequality, since here $\mathcal{S}u$ is a given function. The following lemma shows the unique solvability of the nonlinear inequality (4.1).

Lemma 4.2 *For each $\mu > 0$ there exists a unique function $\widetilde{u}_\mu \in C(\mathbb{R}_+; X)$ which satisfies the inequality* (4.1), *for all $t \in \mathbb{R}_+$.*

Proof. The proof is obtained by similar arguments to those used in the proof of Lemma 4.1. □

Next we investigate the properties of the sequence $\{\widetilde{u}_\mu(t)\}$ for a fixed $t \in \mathbb{R}_+$.

Lemma 4.3 *For each $t \in \mathbb{R}_+$ there exists a subsequence of the sequence $\{\widetilde{u}_\mu(t)\}$, again denoted $\{\widetilde{u}_\mu(t)\}$, which converges weakly to $u(t)$, i.e.*

$$\widetilde{u}_\mu(t) \rightharpoonup u(t) \quad \text{in} \;\; X \quad \text{as} \;\; \mu \to 0. \qquad (4.2)$$

Proof. Let $t \in \mathbb{R}_+$, $\mu > 0$ and let $v_0 \in K$. We use (4.1) to obtain

$$(A\widetilde{u}_\mu(t), v_0 - \widetilde{u}_\mu(t))_X + (\mathcal{S}u(t), v_0 - \widetilde{u}_\mu(t))_X + \frac{1}{\mu}\,(G\widetilde{u}_\mu(t), v_0 - \widetilde{u}_\mu(t))_X$$
$$+j(v_0) - j(\widetilde{u}_\mu(t)) \geq (f(t), v_0 - \widetilde{u}_\mu(t))_X$$

and, therefore,

$$(A\widetilde{u}_\mu(t) - Av_0, \widetilde{u}_\mu(t) - v_0)_X \leq (Av_0, v_0 - \widetilde{u}_\mu(t))_X \tag{4.3}$$
$$+(\mathcal{S}u(t), v_0 - \widetilde{u}_\mu(t))_X + \frac{1}{\mu}\,(G\widetilde{u}_\mu(t), v_0 - \widetilde{u}_\mu(t))_X$$
$$+j(v_0) - j(\widetilde{u}_\mu(t)) + (f(t), \widetilde{u}_\mu(t) - v_0)_X.$$

We use (3.5) to see that there exist $\omega \in X$ and $\alpha \in \mathbb{R}$, which do not depend on t, such that

$$j(v) \geq (\omega, v)_X + \alpha \qquad \forall\, v \in V$$

and, therefore,

$$j(\widetilde{u}_\mu(t)) \geq (\omega, \widetilde{u}_\mu(t))_X + \alpha. \tag{4.4}$$

Then, we combine (4.3), (3.3), (3.8)(c) and (4.4) to find that

$$m\,\|\widetilde{u}_\mu(t) - v_0\|_X^2 \tag{4.5}$$
$$\leq \Big(\|Av_0\|_X + \|\mathcal{S}u(t)\|_X + \|f(t)\|_X + \|\omega\|_X\Big)\|\widetilde{u}_\mu(t) - v_0\|_X$$
$$+|j(v_0)| + |\alpha| + \|\omega\|_X\|v_0\|_X.$$

We use now (4.5), the elementary inequality

$$x,\ a,\ b \geq 0 \quad \text{and} \quad x^2 \leq ax + b \implies x^2 \leq a^2 + 2b$$

and the triangle inequality

$$\|\widetilde{u}_\mu(t)\|_X \leq \|\widetilde{u}_\mu(t) - v_0\|_X + \|v_0\|_X.$$

As a result we deduce that there exists $c > 0$ which depends on v_0 but does not depend on μ such that

$$\|\widetilde{u}_\mu(t)\|_X \leq c. \tag{4.6}$$

Inequality (4.6) shows that the sequence $\{\widetilde{u}_\mu(t)\}$ is bounded in X. Therefore, it follows that there exists a subsequence of the sequence $\{\widetilde{u}_\mu(t)\}$, again denoted $\{\widetilde{u}_\mu(t)\}$ and an element $\widetilde{u}(t) \in X$ such that

$$\widetilde{u}_\mu(t) \rightharpoonup \widetilde{u}(t) \quad \text{in } X \quad \text{as } \mu \to 0. \tag{4.7}$$

Next, we investigate the properties of the element $\widetilde{u}(t) \in X$. First of all, we show that $\widetilde{u}(t) \in K$. To this end, we use (4.1) to deduce that

$$\frac{1}{\mu}\,(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \leq (A\widetilde{u}_\mu(t), v - \widetilde{u}_\mu(t))_X + (\mathcal{S}u(t), v - \widetilde{u}_\mu(t))_X \tag{4.8}$$
$$+ j(v) - j(\widetilde{u}_\mu(t)) + (f(t), \widetilde{u}_\mu(t) - v)_X \qquad \forall\, v \in X.$$

We now write

$$A\widetilde{u}_\mu(t) = A\widetilde{u}_\mu(t) - A0_X + A0_X,$$

then we use the Lipschitz continuity of the operator A and inequality (4.4) to obtain that

$$\frac{1}{\mu}\,(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \leq (A\widetilde{u}_\mu(t) - A0_X, v - \widetilde{u}_\mu(t))_X + (A0_X, v - \widetilde{u}_\mu(t))_X$$
$$+ (\mathcal{S}u(t), v - \widetilde{u}_\mu(t))_X + j(v) - j(\widetilde{u}_\mu(t)) + (f(t), \widetilde{u}_\mu(t) - v)_X$$
$$\leq \Big(M\|\widetilde{u}_\mu(t)\|_X + \|A0_X\|_X + \|\mathcal{S}u(t)\|_X + \|f(t)\|_X\Big)\Big(\|v\|_X + \|\widetilde{u}_\mu(t)\|_X\Big)$$
$$+ |j(v)| + \|\widetilde{u}_\mu(t)\|_X\|\omega\|_X + |\alpha|.$$

We combine now this inequality and (4.6) to see that there exists a positive constant c which depends on t, A, f, $\mathcal{S}$, j, u and v, but is independent on μ, such that

$$(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \leq c\mu \qquad \forall\, v \in X. \tag{4.9}$$

We take now $v = \widetilde{u}(t)$ in (4.9), then we pass to the upper limit as $\mu \to 0$ in the resulting inequality to obtain

$$\limsup_{\mu \to 0}\,(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X \leq 0.$$

Therefore, using assumption (3.8)(a), (b) the convergence (4.7) and standard arguments on pseudomonotone operators (see Proposition 1.23 in [22], for instance) we deduce that

$$\liminf_{\mu \to 0}\,(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \geq (G\widetilde{u}(t), \widetilde{u}(t) - v)_X \qquad \forall\, v \in X. \tag{4.10}$$

On the other hand, the inequality (4.9) implies that

$$\liminf_{\mu \to 0}\,(G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \leq 0 \qquad \forall\, v \in X. \tag{4.11}$$

We combine the inequalities (4.10) and (4.11) to see that

$$(G\widetilde{u}(t), \widetilde{u}(t) - v)_X \leq 0 \qquad \forall\, v \in X$$

and, taking $v = \widetilde{u}(t) - G\widetilde{u}(t)$ in this inequality yields $\|G\widetilde{u}(t)\|_X^2 \leq 0$. We conclude that $G\widetilde{u}(t) = 0_X$ and, using assumption (3.8)(d) it follows that

$$\widetilde{u}(t) \in K. \tag{4.12}$$

Next, from inequality (4.1) and assumption (3.8)(c) we find that

$$(A\widetilde{u}_\mu(t), v - \widetilde{u}_\mu(t))_X + (\mathcal{S}u(t), v - \widetilde{u}_\mu(t))_X \qquad (4.13)$$
$$+j(v) - j(\widetilde{u}_\mu(t)) \geq (f(t), v - \widetilde{u}_\mu(t))_X \qquad \forall\, v \in K.$$

We now take $v = \widetilde{u}(t) \in K$ in (4.13) and obtain

$$(A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X \leq (\mathcal{S}u(t), \widetilde{u}(t) - \widetilde{u}_\mu(t))_X$$
$$+j(\widetilde{u}(t)) - j(\widetilde{u}_\mu(t)) + (f(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X,$$

then we pass to the upper limit as $\mu \to 0$ in this inequality and use the weak convergence (4.7) and the assumption (3.5). As a result we obtain

$$\limsup_{\mu\to 0}\,(A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X \leq 0 \qquad (4.14)$$

and, using again the argument on pseudomonotonicity employed in the proof of Lemma 4.3, it follows that

$$\liminf_{\mu\to 0}\,(A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \geq (A\widetilde{u}(t), \widetilde{u}(t) - v)_X \qquad \forall\, v \in X. \qquad (4.15)$$

On the other hand, passing to the lower limit as $\mu \to 0$ in (4.13) and using (4.7) yields

$$\liminf_{\mu\to 0}\,(A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - v)_X \leq (\mathcal{S}u(t), v - \widetilde{u}(t))_X \qquad (4.16)$$
$$+j(v) - j(\widetilde{u}(t)) + (f(t), \widetilde{u}(t) - v)_X \qquad \forall\, v \in K.$$

We combine now the inequalities (4.15) and (4.16) to see that

$$(A\widetilde{u}(t), v - \widetilde{u}(t))_X + (\mathcal{S}u(t), v - \widetilde{u}(t))_X \qquad (4.17)$$
$$+j(v) - j(\widetilde{u}(t)) \geq (f(t), v - \widetilde{u}(t))_X \qquad \forall\, v \in K.$$

Next, we take $v = u(t)$ in (4.17) and $v = \widetilde{u}(t)$ in (3.7). Then, adding the resulting inequalities and using the strong monotonicity of the operator A we obtain that

$$\widetilde{u}(t) = u(t), \qquad (4.18)$$

which concludes the proof. □

The next step is provided by the following weak convergence result.

Lemma 4.4 *For each $t \in \mathbb{R}_+$ the whole sequence $\{\widetilde{u}_\mu(t)\}$ converges weakly in X to $u(t)$ as $\mu \to 0$.*

Proof. Let $t \in \mathbb{R}_+$. A carefully examination of the proof of Lemma 4.3 shows that any weak convergent subsequence of the sequence $\{\widetilde{u}_\mu(t)\} \subset X$ converges weakly to $u(t)$, where, recall, $u(t)$ is the element of X which solves the variational inequality (3.7) at the moment t. This inequality has a unique solution and, moreover, estimate (4.6) shows that the sequence $\{\widetilde{u}_\mu(t)\}$ is bounded in X. Lemma 4.4 is now a consequence of a standard compactness argument. □

We proceed with the following strong convergence result.

Lemma 4.5 *For each $t \in \mathbb{R}_+$ the sequence $\{\widetilde{u}_\mu(t)\}$ converges strongly in X to $u(t)$, that is*

$$\widetilde{u}_\mu(t) \to u(t) \quad \text{in} \quad X \quad \text{as} \quad \mu \to 0. \tag{4.19}$$

Proof. Let $\mu > 0$ and $t \in \mathbb{R}_+$. We take $v = \widetilde{u}(t)$ in (4.15) to see that

$$\liminf_{\mu \to 0} \, (A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X \geq 0,$$

then we combine this inequality with (4.14) to obtain that

$$\lim_{\mu \to 0} \, (A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - \widetilde{u}(t))_X = 0.$$

Finally, we use (4.18) to find that

$$\lim_{\mu \to 0} \, (A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - u(t))_X = 0. \tag{4.20}$$

On the other hand, from the weak convergence of the sequence $\{\widetilde{u}_\mu(t)\}$ to $u(t)$, guaranteed by Lemma 4.4, it follows that

$$\lim_{\mu \to 0} \, (Au(t), \widetilde{u}_\mu(t) - u(t))_X = 0. \tag{4.21}$$

Next, from the strong monotonicity of the operator A we have

$$\begin{aligned} m \, \|\widetilde{u}_\mu(t) - u(t)\|^2 &\leq (A\widetilde{u}_\mu(t) - Au(t), \widetilde{u}_\mu(t) - u(t))_X \\ &= (A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - u(t))_X - (Au(t), \widetilde{u}_\mu(t) - u(t))_X. \end{aligned} \tag{4.22}$$

The strong convergence (4.19) is now a consequence of (4.20)–(4.22). □

The last step is provided by the following strong convergence result.

Lemma 4.6 *For each $t \in \mathbb{R}_+$ the sequence $\{u_\mu(t)\}$ converges strongly in X to $u(t)$, that is*

$$u_\mu(t) \to u(t) \quad \text{in} \quad X \quad \text{as} \quad \mu \to 0. \tag{4.23}$$

Proof. Let $t \in \mathbb{R}_+$ and $n \in \mathbb{N}^*$ be such that $t \in [0, n]$. Let also $\mu > 0$. We take $v = u_\mu(t)$ in (4.1) and $v = \widetilde{u}_\mu(t)$ in (3.9). Then, adding the resulting inequalities we deduce that

$$(Au_\mu(t) - A\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - u_\mu(t))_X + (\mathcal{S}u_\mu(t) - \mathcal{S}u(t), \widetilde{u}_\mu(t) - u_\mu(t))_X$$
$$+\frac{1}{\mu}\,(Gu_\mu(t) - G\widetilde{u}_\mu(t), \widetilde{u}_\mu(t) - u_\mu(t))_X \geq 0.$$

Next, we use the monotony of the operator G, (3.8)(a), to obtain that

$$(Au_\mu(t) - A\widetilde{u}_\mu(t), u_\mu(t) - \widetilde{u}_\mu(t))_X \leq (\mathcal{S}u_\mu(t) - \mathcal{S}u(t), \widetilde{u}_\mu(t) - u_\mu(t))_X.$$

Therefore, using (3.3)(a) yields

$$\|u_\mu(t) - \widetilde{u}_\mu(t)\|_X \leq \frac{1}{m}\,\|\mathcal{S}u_\mu(t) - \mathcal{S}u(t)\|_X. \tag{4.24}$$

We now combine (4.24) and (3.4) to find that

$$\|u_\mu(t) - \widetilde{u}_\mu(t)\|_X \leq \frac{d_n}{m}\int_0^t \|u_\mu(s) - u(s)\|_X\,ds.$$

It follows from here that

$$\|u_\mu(t) - u(t)\|_X \leq \|\widetilde{u}_\mu(t) - u(t)\|_X + \frac{d_n}{m}\int_0^t \|u_\mu(s) - u(s)\|_X ds$$

and, using a Gronwall's argument, we obtain that

$$\|u_\mu(t) - u(t)\|_X \leq \|\widetilde{u}_\mu(t) - u(t)\|_X + \frac{d_n}{m}\int_0^t e^{\frac{d_n}{m}(t-s)}\|\widetilde{u}_\mu(s) - u(s)\|_X\,ds. \tag{4.25}$$

Note that $e^{\frac{d_n}{m}(t-s)} \leq e^{\frac{d_n}{m}t} \leq e^{\frac{nd_n}{m}}$ for all $s \in [0, n]$ and, therefore, (4.25) yields

$$\|u_\mu(t) - u(t)\|_X \leq \|\widetilde{u}_\mu(t) - u(t)\|_X + \frac{d_n}{m}e^{\frac{nd_n}{m}}\int_0^t \|\widetilde{u}_\mu(s) - u(s)\|_X\,ds. \tag{4.26}$$

On the other hand, by estimate (4.6), Lemma 4.5 and Lebesgue's convergence theorem it follows that

$$\int_0^t \|\widetilde{u}_\mu(s) - u(s)\|_X\,ds \to 0 \qquad \text{as} \quad \mu \to 0. \tag{4.27}$$

We use now (4.26), (4.27) and (4.19) to obtain the convergence (4.23), which concludes the proof. □

We end this section with the remark that the points 1) and 2) of Theorem 3.2 correspond to Lemmas 4.1 and 4.6, respectively. Therefore, we conclude from here that the proof of Theorem 3.2 is complete.

5 Existence and uniqueness

We turn now to the variational analysis of problem $\mathcal{Q}$. To this end, we need further notation and preliminaries. First, we use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. Recall that the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \\ &\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

We use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the following spaces:

$$\begin{aligned} &V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : v_i = 0 \text{ on } \Gamma_1 \,\}, \\ &Q = \{\ \boldsymbol{\tau} = (\tau_{ij}) \ : \ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \\ &Q_1 = \{\ \boldsymbol{\tau} \in Q \ : \ \tau_{ij,j} \in L^2(\Omega)\ \}. \end{aligned}$$

These are real Hilbert spaces endowed with the inner products

$$\begin{aligned} &(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx, \\ &(\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_{L^2(\Omega)^d}. \end{aligned}$$

Here and below $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption meas$(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{5.1}$$

Also, for a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential trace, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_\Gamma \boldsymbol{\sigma\nu}\cdot\boldsymbol{v}\,da \quad \forall\,\boldsymbol{v}\in V. \tag{5.2}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl})\ :\ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1\le i,j,k,l\le d\ \},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le \|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q \qquad \forall\,\mathcal{E}\in\mathbf{Q}_\infty,\ \boldsymbol{\tau}\in Q. \tag{5.3}$$

Next, we list the assumptions on the data, derive the variational formulation of the problem $\mathcal{Q}$ and then we state and prove its unique weak solvability. To this end we assume that the elasticity operator $\mathcal{A}$ and the relaxation tensor $\mathcal{B}$ satisfy the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\
\qquad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\
\qquad\quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(c) There exists } m_{\mathcal{A}}>0 \text{ such that}\\
\qquad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\
\qquad\quad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\
\text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\
\text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{5.4}$$

$$\mathcal{B}\in C(\mathbb{R}_+;\mathbf{Q}_\infty). \tag{5.5}$$

The densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d), \quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d). \tag{5.6}$$

Finally, the normal compliance function p and the surface yield function F satisfy

$$\left\{\begin{array}{l}
\text{(a) } p:\mathbb{R}\to\mathbb{R}_+.\\
\text{(b) There exists } L_p>0 \text{ such that}\\
\qquad |p(r_1)-p(r_2)|\le L_p\,|r_1-r_2| \quad \forall\,r_1,\,r_2\in\mathbb{R}.\\
\text{(c) } (p(r_1)-p(r_2))(r_1-r_2)\ge 0 \quad \forall\,r_1,\,r_2\in\mathbb{R}.\\
\text{(d) } p(r)=0 \ \text{ iff } \ r\le 0.
\end{array}\right. \tag{5.7}$$

$$F \in L^2(\Gamma_3), \quad F(\boldsymbol{x}) \geq 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{5.8}$$

In what follows we consider the set of admissible displacements defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \leq g \text{ on } \Gamma_3 \,\}. \tag{5.9}$$

Moreover, we define the operator $P : V \to V$ and the functions $j : V \to \mathbb{R}_+$, $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{5.10}$$

$$j(\boldsymbol{v}) = \int_{\Gamma_3} F v_\nu^+ \, da \quad \forall\, \boldsymbol{v} \in V, \tag{5.11}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \quad \forall\, \boldsymbol{v} \in V,\ t \in [0, T]. \tag{5.12}$$

Here and below, for $r \in \mathbb{R}$ we denote by r^+ its positive part, i.e. $r^+ = \max\{r, 0\}$. Note that assumptions (5.6)–(5.8) imply that the integrals in (5.10)–(5.12) are well-defined.

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (2.1)–(2.6) and let $\boldsymbol{v} \in U$ and $t > 0$ be given. First, we use Green's formula (5.2) and the equilibrium equation (2.2) to see that

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \, dx = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, da.$$

We split the surface integral over Γ_1, Γ_2 and Γ_3 and, since $\boldsymbol{v} - \boldsymbol{u}(t) = \boldsymbol{0}$ a.e. on Γ_1, $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on Γ_2, we deduce that

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \, dx = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, dx$$
$$+ \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, da + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, da.$$

Moreover, since

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_3,$$

taking into account the frictionless condition (2.5) we obtain

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \, dx = \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, dx \tag{5.13}$$
$$+ \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) \, da + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t)) \, da.$$

We write now

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) = (\sigma_\nu(t) + p(u_\nu(t)) + \xi(t))(v_\nu - g)$$
$$+(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t))(g - u_\nu(t))$$
$$-(p(u_\nu(t)) + \xi(t))(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3,$$

then we use the contact conditions (2.6) and the definition (5.9) of the set U to see that

$$\sigma_\nu(t)(v_\nu - u_\nu(t)) \geq -(p(u_\nu(t)) + \xi(t))(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{5.14}$$

We use (2.6), again, and the hypothesis (5.8) on function F to deduce that

$$F\,(v_\nu^+ - u_\nu^+(t)) \geq \xi(t)(v_\nu - u_\nu(t)) \quad \text{on } \Gamma_3. \tag{5.15}$$

Then we add the inequalities (5.14) and (5.15) and integrate the result on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\,da \tag{5.16}$$
$$\geq -\int_{\Gamma_3} p(u_\nu(t))(v_\nu - u_\nu(t))\,da - \int_{\Gamma_3} F\,(v_\nu^+ - u_\nu^+(t))\,da.$$

Finally, we combine (5.13) and (5.16) and use the definitions (5.10)–(5.12) to deduce that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\boldsymbol{v}) - j(\boldsymbol{u}(t)) \tag{5.17}$$
$$\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

We now substitute the constitutive law (2.1) in (5.17) to obtain the following variational formulation of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to U$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\Big)_Q \tag{5.18}$$
$$+(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + j(\boldsymbol{v}) - j(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

In the study of the problem $\mathcal{Q}^V$ we have the following existence and uniqueness result.

Theorem 5.1 *Assume that* (5.4)–(5.8) *hold. Then, Problem* $\mathcal{Q}^V$ *has a unique solution which satisfies* $\boldsymbol{u} \in C(\mathbb{R}_+; U)$.

Proof. To solve the variational inequality (5.18) we use Theorem 3.1 with $X = V$ and $K = U$. To this end we consider the operators $A : V \to V$ and $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ defined by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{5.19}$$

$$(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V = \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \Big)_Q \tag{5.20}$$
$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \ \boldsymbol{v} \in V.$$

It is easy to see that condition (3.2) holds. Next, we use (5.4), (5.7) and (5.1) to see that the operator A satisfies conditions (3.3) with $M = L_{\mathcal{A}} + c_0^2 L_p$ and $m = m_{\mathcal{A}}$. Let $n \in \mathbb{N}^*$. Then, a simple calculation based on assumption (5.5) and inequality (5.3) shows that

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_V \le \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \tag{5.21}$$
$$\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \ \forall\, t \in [0, n].$$

This inequality shows that the operator $\mathcal{S}$, defined by (5.20), satisfies condition (3.4) with

$$d_n = \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty}.$$

Next, we use condition (5.8) to see that the functional j defined by (5.11) is a seminorm on V and, moreover, it satisfies

$$j(\boldsymbol{v}) \le c_0 \|F\|_{L^2(\Gamma_3)} \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.22}$$

Inequality (5.22) shows that the seminorm j is continuous on V and, therefore, (3.5) holds. Finally, using assumption (5.6) and definition (5.12) we deduce that $\boldsymbol{f} \in C(\mathbb{R}_+; V)$ which shows that (3.6) holds, too.

It follows now from Theorem 3.1 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which satisfies the inequality

$$\boldsymbol{u}(t) \in U, \quad (A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \tag{5.23}$$
$$+ j(\boldsymbol{v}) - j(\boldsymbol{u}(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,$$

for all $t \in \mathbb{R}_+$. And, using (5.19) and (5.20) we deduce that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ such that (5.18) holds for all $t \in \mathbb{R}_+$, which concludes the proof. □

Let $\boldsymbol{\sigma}$ be the function defined by (2.1). Then, it follows from (5.4) and (5.5) that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. Moreover, it is easy to see that (5.17) holds for all $t \in \mathbb{R}_+$ and, using standard arguments, it results from here that

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \qquad \forall\, t \in \mathbb{R}_+. \tag{5.24}$$

Therefore, using the regularity $\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$ in (5.6) we deduce that $\mathrm{Div}\,\boldsymbol{\sigma} \in C(\mathbb{R}_+; L^2(\Omega)^d)$ which implies that $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$. A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (2.1), (5.18) for all $t \in \mathbb{R}_+$ is called a *weak solution* to the contact problem $\mathcal{Q}$. We conclude that Theorem 5.1 provides the unique weak solvability of Problem $\mathcal{Q}$. Moreover, the regularity of the weak solution is $\boldsymbol{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q_1)$.

6 Penalization

In this section we show how the abstract result in Theorem 3.2 can be used in the study of the contact problem $\mathcal{Q}$. To this end, for each $\mu > 0$ we consider the following contact problem.

Problem $\mathcal{Q}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\,ds \quad \text{in} \quad \Omega, \tag{6.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}_\mu(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{6.2}$$

$$\boldsymbol{u}_\mu(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{6.4}$$

$$\boldsymbol{\sigma}_{\mu\tau}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{6.5}$$

for all $t \in \mathbb{R}_+$, *and there exists* $\xi_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{array}{l} \sigma_{\mu\nu}(t) + p(u_{\mu\nu}(t)) + \frac{1}{\mu}\,p(u_{\mu\nu} - g) + \xi_\mu(t) = 0, \\ 0 \le \xi_\mu(t) \le F, \\ \xi_\mu(t) = 0 \ \text{ if } \ u_{\mu\nu}(t) < 0, \\ \xi_\mu(t) = F \ \text{ if } \ u_{\mu\nu}(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{6.6}$$

for all $t \in \mathbb{R}_+$.

Here and below $u_{\mu\nu}$ and $\boldsymbol{\sigma}_{\mu\tau}$ represent the normal and the tangential components of the functions $\boldsymbol{u}_\mu$ and $\boldsymbol{\sigma}_\mu$, respectively. Note that the contact condition (6.6) can be obtained from the contact condition (2.6) in the limit when $g \to \infty$. For this reason, its mechanical interpretation is similar to that of condition (2.6) and could be summarised as follows: when there is separation between the body's surface and the foundation then the normal stress vanishes; the penetration arises only if the normal stress reaches the critical value F; when there is penetration the contact follows a normal compliance condition of the form (2.13). For this reason we refer to this condition as to a *multivalued normal compliance contact condition.* It models the case when the foundation is assumed to have a rigid-elastic behaviour. Arguments similar to those used in [9, 20] show that μ can be interpreted as a deformability coefficient of the hard layer of the foundation.

Using notation (5.10)–(5.12) by similar arguments as those used in the case of Problem $\mathcal{Q}$ we obtain the following variational formulation of Problem $\mathcal{Q}_\mu$.

Problem $\mathcal{Q}_\mu^V$. *Find a displacement field* $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)))_Q + \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) \Big)_Q \tag{6.7}$$
$$+ (P\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V + \frac{1}{\mu} \int_{\Gamma_3} p(u_{\mu\nu}(t) - g)(v_\nu - u_{\mu\nu}(t))da$$
$$+ j(\boldsymbol{v}) - j(\boldsymbol{u}_\mu(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \qquad \forall\, \boldsymbol{v} \in V.$$

We have the following existence, uniqueness and convergence result, which states the unique solvability of Problem $\mathcal{Q}_\mu^V$ and describes the behaviour of its solution as $\mu \to 0$.

Theorem 6.1 *Assume that* (5.4)–(5.8) *hold. Then:*

1) *For each* $\mu > 0$ *Problem* $\mathcal{Q}_\mu^V$ *has a unique solution which satisfies* $\boldsymbol{u}_\mu \in C(\mathbb{R}_+; V)$.

2) *The solution* $\boldsymbol{u}_\mu$ *of Problem* $\mathcal{Q}_\mu^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{Q}^V$, *that is*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0 \qquad \text{as} \quad \mu \to 0, \tag{6.8}$$

for all $t \in \mathbb{R}_+$.

Proof. We use Theorem 3.2 with $X = V$ and $K = U$. To this end we define the operator $G : V \to V$ by equality

$$(G\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu - g)v_\nu da \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{6.9}$$

We use (5.1) and (5.7) to show that G is a monotone Lipschitz continuous operator with Lipschitz constant $M = c_0^2 L_p$, i.e. it satisfies condition (3.8)(a) and (b).

Assume now that $\boldsymbol{u} \in V$ and $\boldsymbol{v} \in U$. Then, using (5.9) and (5.7) it is easy to see that

$$p(u_\nu - g)(v_\nu - g) \leq 0 \quad \text{a.e. on } \Gamma_3,$$
$$p(u_\nu - g)(g - u_\nu) \leq 0 \quad \text{a.e. on } \Gamma_3$$

and, therefore

$$
\begin{aligned}
(G\boldsymbol{u}, \boldsymbol{v}-\boldsymbol{u})_V &= \int_{\Gamma_3} p(u_\nu - g)(v_\nu - u_\nu)da \\
&= \int_{\Gamma_3} p(u_\nu - g)(v_\nu - g)da + \int_{\Gamma_3} p(u_\nu - g)(g - u_\nu)da \le 0
\end{aligned}
$$

which shows that (3.8)(c) holds, too.

Finally, assume that $G\boldsymbol{u} = \boldsymbol{0}_V$. Then, $(G\boldsymbol{u}, \boldsymbol{u})_V = 0$ and, therefore,

$$
\int_{\Gamma_3} p(u_\nu - g)u_\nu da = 0. \tag{6.10}
$$

We use (5.7) to obtain the inequality

$$
p(u_\nu - g)u_\nu \ge p(u_\nu - g)g \ge 0 \quad \text{ a.e. on } \Gamma_3.
$$

Therefore, since the integrand in (6.10) is positive, we deduce from (6.10) that

$$
p(u_\nu - g)u_\nu = 0 \quad \text{ a.e. on } \Gamma_3.
$$

This equality combined with assumption (5.7)(d) implies that $u_\nu \le g$ a.e. on Γ_3 and, therefore, we deduce that $\boldsymbol{u} \in U$. Conversely, if $\boldsymbol{u} \in U$ it follows that $u_\nu \le g$ a.e. on Γ_3 and using assumption (5.7) (d) we deduce that $p(u_\nu - g) = 0$ a.e. on Γ_3. From the definition (6.9) of the operator G we deduce that $(G\boldsymbol{u}, \boldsymbol{v})_V = 0$ for all $\boldsymbol{v} \in V$, which implies that $G\boldsymbol{u} = 0_V$. It follows from above that G satisfies the condition (3.8)(d).

We now turn back to (5.19) and (5.20). Thus, it is easy to see that $\boldsymbol{u}_\mu$ is a solution to Problem $\mathcal{Q}_\mu^V$ iff

$$
\begin{aligned}
&(A\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V + \frac{1}{\mu}(G\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \\
&\quad + j(\boldsymbol{v}) - j(\boldsymbol{u}_\mu(t)) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \quad \forall\, \boldsymbol{v} \in V,
\end{aligned} \tag{6.11}
$$

for all $t \in \mathbb{R}_+$. Moreover, $\boldsymbol{u}$ is a solution to Problem $\mathcal{Q}^V$ iff $\boldsymbol{u}$ satisfies inequality (5.23) for all $t \in \mathbb{R}_+$. Recall also that the operator G satisfies condition (3.8). Theorem 6.1 is now a consequence of Theorem 3.2. □

Note that the convergence result (6.8) can be easily extended to the weak solutions of the problems $\mathcal{Q}_\mu$ and $\mathcal{Q}$. Indeed, let $\boldsymbol{\sigma}_\mu$ and $\boldsymbol{\sigma}$ be the functions defined by (6.1) and (2.1), respectively, and let $t \in \mathbb{R}_+$, $n \in \mathbb{N}^*$ be such that $t \in [0, n]$. Then, following the arguments presented in Section 5, it follows that $\boldsymbol{\sigma}_\mu$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$ and, moreover,

$$
\operatorname{Div} \boldsymbol{\sigma}_\mu(t) = \operatorname{Div} \boldsymbol{\sigma}(t) = -\boldsymbol{f}_0(t). \tag{6.12}
$$

Therefore, using (2.1), (6.1) and (6.12) as well as the properties of the operators $\mathcal{A}$ and $\mathcal{B}$ we deduce that

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_{Q_1} = \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \le L_{\mathcal{A}} \|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \\
&\quad + d \max_{r \in [0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^n \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds.
\end{aligned} \tag{6.13}
$$

Next, we take $\boldsymbol{v} = \mathbf{0}_V$ in (6.11), then we use the properties of the operators A, G combined with those of the functional j. As a result we obtain

$$m_{\mathcal{A}} \|\boldsymbol{u}_\mu(t)\|_V \leq \|A\mathbf{0}_V\|_V + \|\mathcal{S}\boldsymbol{u}_\mu(t)\|_V + \|\boldsymbol{f}(t)\|_V.$$

We now use the property (5.21) of the operator $\mathcal{S}$ and a Gronwall argument to see that

$$\|\boldsymbol{u}_\mu(t)\|_V \leq c_n \tag{6.14}$$

where c_n represents a constant which depends on n but is independent on μ. Then, we use the inequality (6.13), the convergence (6.8), the estimate (6.14) and Lebesque's theorem to deduce that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_{Q_1} \to 0 \qquad \text{as} \quad \mu \to 0. \tag{6.15}$$

In addition to the mathematical interest in the convergence result (6.8), (6.15), it is important from the mechanical point of view, since it shows that the weak solution of the viscoelastic contact problem with multivalued normal compliance and unilateral constraint may be approached as closely as one wishes by the solution of the viscoelastic contact problem with multivalued normal compliance, with a sufficiently small deformability coefficient.

7 Conclusion

We presented a penalization method for a class of history-dependent variational inequalities in Hilbert spaces. It contains the existence and the uniqueness of the solution for the penalized problems as well as its convergence to the solution of the original problem. The proofs were based on arguments of compactness and monotonicity. The method can be applied in the study of a large class of nonlinear boundary value problems with unilateral constraints. To provide an example, we presented a new model of quasistatic frictionless contact with viscoelastic materials which, in the variational formulation, leads to a history-dependent variational inequality for the displacement field. We applied the abstract penalization method in the study of this contact problem and we presented the mechanical interpretation of the corresponding results. A numerical validation of the convergence result included in this method will be provided in a forthcoming paper.

References

[1] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems*, John Wiley, Chichester, 1984.

[2] H. Brézis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier* **18** (1968), 115–175.

[3] C. Corduneanu, Problèmes globaux dans la théorie des équations intégrales de Volterra, *Ann. Math. Pure Appl.* **67** (1965), 349–363.

[4] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[5] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.

[6] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[7] W. Han and B.D. Reddy, Computational plasticity: the variational basis and numerical analysis, *Computational Mechanics Advances* **2** (1995), 283–400.

[8] W. Han and B.D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Verlag, New York, 1999.

[9] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society–International Press, 2002.

[10] I. Hlaváček, J. Haslinger, J. Necăs and J. Lovíšek, *Solution of Variational Inequalities in Mechanics*, Springer-Verlag, New York, 1988.

[11] N. Kikuchi and J.T. Oden, Theory of variational inequalities with applications to problems of flow through porous media, *Int. J. Engng. Sci.* **18** (1980), 1173–1284.

[12] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[13] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Classics in Applied Mathematics **31**, SIAM, Philadelphia, 2000.

[14] J.A.C. Martins and M.D.P. Monteiro Marques, eds., *Contact Mechanics*, Kluwer, Dordrecht, 2002.

[15] J. J. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York-London, 1966.

[16] S. Migórski, A. Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.

[17] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[18] M. Raous, M. Jean and J.J. Moreau, eds., *Contact Mechanics*, Plenum Press, New York, 1995.

[19] M. Shillor, ed., Recent advances in contact mechanics, Special issue of *Math. Comput. Modelling* **28** (4–8) (1998).

[20] M. Shillor, M. Sofonea and J.J. Telega, *Models and Analysis of Quasistatic Contact. Variational Methods*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.

[21] M. Sofonea and A. Matei, History-dependent quasivariational inequalities arising in Contact Mechanics, *European Journal of Applied Mathematics* **22** (2011), 471–491.

[22] M. Sofonea and A. Matei, *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Notes, Cambridge University Press **398**, Cambridge, 2012.

[23] M. Sofonea and F. Pătrulescu, Analysis of a history-dependent frictionless contact problem, *Mathematics and Mechanics of Solids* **18** (2013), 409–430.

[24] P. Wriggers and U. Nackenhorst, eds., *Analysis and Simulation of Contact Problems*, Lecture Notes in Applied and Computational Mechanics **27**, Springer, Berlin, 2006.