History-dependent Contact Models for Viscoplastic Materials

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Abstract

We consider two quasistatic contact problems which describe the contact between a viscoplastic body and an obstacle, the so-called foundation. The contact is frictionless and is modelled with normal compliance and memory term of such a type that the penetration is not restricted in the first problem, but is restricted with unilateral constraint, in the second one. For each problem we derive a variational formulation, then we prove its unique solvability. Next, we prove the convergence of the weak solution of the first problem to the weak solution of the second problem, as the stiffness coefficient of the foundation converges to infinity. And, finally, we provide numerical simulations which illustrate the convergence result.

1 Introduction

Phenomena of contact between deformable bodies or between deformable and rigid bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming, metal extrusion, involve contact evolutions and, for this reason, a considerable effort has been developed in their modelling, mathematical analysis and numerical solution. Owing to their inherent complexity, contact phenomena lead to nonlinear and nonsmooth mathematical problems.

The aim of this paper is to study two frictionless contact problems for rate-type viscoplastic materials, within the framework of the Mathematical Theory of Contact Mechanics. We model the behavior of the material with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor. Here $\mathcal{E}$ is a fourth order tensor which describes the elastic properties of the material and $\mathcal{G}$ is a constitutive function which describes its viscoplastic behavior. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t.

Various results, examples and mechanical interpretations in the study of viscoplastic materials of the form (1.1) can be found in [3, 5] and references therein. Displacement-traction boundary value problems with such materials were considered in [5], both in the dynamic and quasistatic case. Quasistatic frictionless and frictional contact problems for materials of the form (1.1) were studied in various papers, see [4] and [13] for a survey. There, various models of contact were stated and they variational analysis, including existence and uniqueness results, was provided. The numerical analysis of the corresponding models can be found in [4] and the references therein. In all the above papers the process of contact was studied in a finite interval of time.

In [14] a quasistatic frictionless contact problem for viscoplastic materials of the form (1.1) was considered. The process was assumed to be quasistatic and the contact was modelled by using the normal compliance condition with infinite penetration. The unique solvability of the solution was obtained by using a fixed point argument. In contrast, in [2] was considered a problem with normal compliance and finite penetration and was proved the unique solvability of the models using new arguments on history-dependent variational inequalities presented in [15]. The present paper represents a continuation of [2] and we consider the problem with normal compliance, finite penetration and memory term. The same contact condition for viscoelastic materials was used in [16]. Also, we state and prove the convergence of the solution of the problem with infinite penetration to the solution of the problem with finite penetration as the stiffness coefficient converges to infinity. And, finally, we provide numerical simulations which illustrate this convergence.

The rest of the paper is structured as follows. In Section 2 we introduce the notations we shall use as well as some preliminary material. In Section 3 we present the classical formulation of the two contact problems. In Section 4 we list the assumptions on the data and derive the variational formulation of the problems. Then we state and prove the unique weak solvability of each model. In Section 5 we state and prove a converge result, Theorem 5.1. Next, in Section 6 we present the numerical solution of the contact problem with normal compliance restricted by unilateral constraints and memory term. And, finally, in Section 7 some numerical simulations are presented including a numerical validation of the convergence result.

2 Notations and preliminaries

Everywhere in this paper we use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of non negative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \\ &\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that meas$\,(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. We consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}, \qquad Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption meas$(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.1}$$

Also, for a regular stress function $\boldsymbol{\sigma}$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j})$ and, finally, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{2.2}$$

For a normed space $(X, \|\cdot\|_X)$ we use the notation $C(\mathbb{R}_+; X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values on X, and $C^1(\mathbb{R}_+; X)$ for the space of continuous differentiable functions defined on $\mathbb{R}_+$ with values on X. For a subset $K \subset X$ we still use the symbols $C(\mathbb{R}_+; K)$ and $C^1(\mathbb{R}_+; K)$ for the set of continuous and continuously differentiable functions defined on $\mathbb{R}_+$ with values on K, respectively.

Let X be a real Hilbert space with inner product $(\cdot, \cdot)_X$ and associated norm $\|\cdot\|_X$. Assume given a set $K \subset X$, the operators $A : K \to X$, $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ and a function $f : \mathbb{R}_+ \to X$ such that:

$$K \text{ is a closed, convex, nonempty subset of } X. \tag{2.3}$$

$$\left\{\begin{array}{l} \text{(a) There exists } m > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geq m \, \|u_1 - u_2\|_X^2 \quad \forall \, u_1, u_2 \in K. \\ \text{(b) There exists } L > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leq L \, \|u_1 - u_2\|_X \quad \forall \, u_1, u_2 \in K. \end{array}\right. \tag{2.4}$$

$$\left\{\begin{array}{l} \text{For every } n \in \mathbb{N} \text{ there exists } r_n > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \leq r_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\ \quad \forall \, u_1, \, u_2 \in C(\mathbb{R}_+; X), \ \forall \, t \in [0, n]. \end{array}\right. \tag{2.5}$$

$$f \in C(\mathbb{R}_+; X). \tag{2.6}$$

We proceed with the following existence and uniqueness result in the study of nonlinear equations involving monotone operators.

Theorem 2.1 *Let X be a Hilbert space and let $A : X \to X$ be a strongly monotone Lipschitz continuous operator. Then, for each $f \in X$ there exists a unique element $u \in X$ such that $Au = f$.*

The following result, proved in [15], will be used in Section 4 of this paper.

Theorem 2.2 *Assume that* (2.3)–(2.6) *hold. Then there exists a unique function $u \in C(\mathbb{R}_+; K)$ such that for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$\begin{aligned} &(Au(t), v - u(t))_X + (\mathcal{S}u(t), v)_X - (\mathcal{S}u(t), u(t))_X \\ &\qquad \geq (f(t), v - u(t))_X \qquad \forall \, v \in K. \end{aligned} \tag{2.7}$$

We have the following consequence of Theorem 2.2.

Corollary 2.3 *Let X be a Hilbert space and assume that* (2.3)–(2.6) *hold. Then there exists a unique function $u \in C(\mathbb{R}_+; X)$ such that*

$$(Au(t), v)_X + (\mathcal{S}u(t), v)_X = (f(t), v)_X \quad \forall \, v \in X, \quad \forall \, t \in \mathbb{R}_+. \tag{2.8}$$

Following the terminology introduced in [15] we refer to (2.7) as a history-dependent quasivariational inequality. To avoid any confusion, we note that here and below the notation $Au(t)$ and $\mathcal{S}u(t)$ are short hand notation for $A(u(t))$ and $(\mathcal{S}u)(t)$, i.e. $Au(t) = A(u(t))$ and $\mathcal{S}u(t) = (\mathcal{S}u)(t)$, for all $t \in \mathbb{R}_+$.

3 The models

In this section we present the two problems which describe the frictionless contact process and present the assumption on the data. The physical setting is as follows. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the problem is quasistatic, and we study the contact process in the interval of time $\mathbb{R}_+ = [0, \infty)$.

In the first problem, unlike in [2], the contact is modelled with normal compliance and memory term in such a way that the penetration is not limited. Under these conditions, the classical formulation of the problem is the following.

Problem $\mathcal{P}_1$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{in} \quad \Omega \times (0, \infty), \tag{3.1}$$

$$\text{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega \times (0, \infty), \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2 \times (0, \infty), \tag{3.4}$$

$$-\sigma_\nu = p(u_\nu) + \int_0^t b(t - s) u_\nu^+(s) ds \qquad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{3.7}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}$ or t. Equation (3.1) represents the viscoplastic constitutive law of the material introduced in Section 1 and equation (3.2) is the equilibrium equation. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, and condition (3.6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (3.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

The function p is Lipschitz continuous, increasing and vanishes for a negative argument, i.e.

$$\begin{cases} \text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \quad \forall\, r_1, r_2 \in \mathbb{R}. \\ \text{(d) } p(r) = 0 \ \text{ for all } r < 0. \end{cases} \tag{3.8}$$

In the second problem the contact is again modelled with normal compliance and memory term but in such a way that the penetration is limited and associated to a unilateral constraint. The classical formulation of the problem is the following.

Problem $\mathcal{P}_2$. *Find a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{in} \quad \Omega \times (0, \infty), \tag{3.9}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega \times (0, \infty), \tag{3.10}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1 \times (0, \infty), \tag{3.11}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2 \times (0, \infty), \tag{3.12}$$

$$\left.\begin{aligned} u_\nu \le g,\ \sigma_\nu + p(u_\nu) + \int_0^t b(t-s) u_\nu^+(s)\, ds \le 0 \\ (u_\nu - g)(\sigma_\nu + p(u_\nu) + \int_0^t b(t-s) u_\nu^+(s)\, ds) = 0 \end{aligned}\right\} \qquad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{3.13}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3 \times (0, \infty), \tag{3.14}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{3.15}$$

Here $g \ge 0$ is given and p is a function which satisfies (3.8). Conditions (3.9)–(3.12) and (3.14)– (3.15) have the same interpretation as in the contact problem $\mathcal{P}_1$.

We now present the new contact condition (3.13), condition (3.5) can be presented using similar arguments. It can be derived in the following way. First, we assume that the penetration is limited by the bound g and, therefore, at each time moment $t \in \mathbb{R}_+$, the normal displacement satisfies the inequality

$$u_\nu(t) \le g \quad \text{on } \Gamma_3. \tag{3.16}$$

Next, we assume that the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^D(t) + \sigma_\nu^R(t) + \sigma_\nu^M(t) \quad \text{on } \Gamma_3, \tag{3.17}$$

in which the functions σ_ν^D, σ_ν^R and σ_ν^M describe the deformability, the rigidity and the memory properties of the foundation, at each $t \in \mathbb{R}_+$. Also, we assume that the function σ_ν^D satisfies the normal compliance contact condition

$$- \sigma_\nu^D(t) = p(u_\nu(t)) \quad \text{on } \Gamma_3. \tag{3.18}$$

Condition (3.18) combined with assumption (3.8) shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); also, when there is penetration (i.e. when $u_\nu \geq 0$), then the reaction of the foundation is towards the body (since $\sigma_\nu \leq 0$) and it is increasing with the penetration (since p is an increasing function). Finally, we note that in this condition the penetration is not restricted and the normal stress is uniquely determined by the normal displacement.

Condition (3.18) was first introduced in [11, 12] in the study of dynamic contact problems with elastic and viscoelastic materials. The term *normal compliance* for this condition was first used in [8, 9]. A first example of normal compliance function p which satisfies condition (3.8) is

$$p(r) = c_\nu r^+ \tag{3.19}$$

where $r^+ = \max\{r, 0\}$ and c_ν is a positive constant. In this case condition (3.18) shows that the reaction of the foundation is proportional to the penetration and, therefore, (3.8), (3.18) model the contact with a linearly elastic foundation. A second example of normal compliance function p which satisfies condition (3.8) is given by

$$p(r) = \begin{cases} c_\nu r^+ & \text{if} \quad r \leq \alpha, \\ c_\nu \alpha & \text{if} \quad r > \alpha, \end{cases}$$

where α is a positive coefficient related to the wear and hardness of the surface and, again, $c_\nu > 0$. In this case the contact condition (3.18) means that when the penetration is too large, i.e. when it exceeds α, the obstacle backs off and offers no additional resistance to the penetration. We conclude that in this case the foundation has an elastic-perfectly plastic behavior.

The part σ_ν^R of the normal stress satisfies the Signorini condition in the form with a gap function, i.e.

$$\sigma_\nu^R(t) \leq 0, \quad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \quad \text{on } \Gamma_3. \tag{3.20}$$

And, finally, the function σ_ν^M satisfies the memory condition

$$- \sigma_\nu^M(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \quad \text{on } \Gamma_3, \tag{3.21}$$

in which b represents a given function, the so-called surface memory function. Contact conditions of the form (3.21) have a simple physical interpretation if there are no cycles of contact and separation during the time interval of interest. For instance, assume in what follows that b is a positive function. Moreover, assume that in the time interval $[0, t]$ there is only penetration (i.e. $u_\nu(s) \geq 0$ for all $s \in [0, t]$). Then (3.21) shows that the reaction of the foundation at t is towards the body (since $\sigma_\nu(t) \leq 0$). Also, if in the time interval $[0, t]$ there is separation (i.e. $u_\nu(s) < 0$ for all $s \in [0, t]$) then there is no reaction at the moment t (since $\sigma_\nu(t) = 0$).

Now, assume a situation in which u_ν is positive in time interval $[0, t_0]$ and negative on the time interval $[t_0, t]$. Then, following (3.21) we have

$$-\sigma_\nu(t) = \int_0^{t_0} b(t-s)\, u_\nu^+(s)\, ds,$$

since the integral on the remaining interval $[t_0, t]$ vanishes. Assume, in addition, that the support of the function b is included in the interval $[0, \delta]$ with $\delta > 0$. Two possibilities arise. First, if $t - t_0 > \delta$ it follows that $b(t-s) = 0$ for all $s \in [0, t_0]$ and (3.21) shows the normal stress $\sigma_\nu(t)$ vanishes. Second, if $t - t_0 \leq \delta$ (3.21) implies that $\sigma_\nu(t) \leq 0$ i.e. a residual pression exists at the moment t on the body's surface. We interpret this as a memory effect in which the foundation prevents the separation, moves towards the body and exerts a pression on a short interval of time of length δ. Various other mechanical interpretation of the condition (3.21) could be obtained if b is assumed to be a negative function.

We combine equalities (3.17), (3.18) and (3.21) to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\, ds \quad \text{on } \Gamma_3. \tag{3.22}$$

Then we substitute equality (3.22) in (3.20) and use inequality (3.16) to obtain the contact condition (3.13).

4 Existence and uniqueness results

In this section we list the assumptions on the data, derive the variational formulations of the problems $\mathcal{P}_1$ and $\mathcal{P}_2$ and then we state and prove their unique weak solvability. To this end we assume that the elasticity tensor $\mathcal{E}$ and the constitutive function $\mathcal{G}$ satisfy the following conditions.

$$\left\{\begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \leq i,j,k,l \leq d. \\ \text{(c) There exists } m_\mathcal{E} > 0 \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_\mathcal{E} \|\boldsymbol{\tau}\|^2\ \ \forall \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. in } \Omega. \end{array}\right. \tag{4.1}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad\quad \le L_{\mathcal{G}} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right) \\ \qquad\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\ \qquad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } Q. \end{array}\right. \tag{4.2}$$

The surface memory function satisfies

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)). \tag{4.3}$$

We also assume that the body forces and the surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{4.4}$$

In the study of Problem $\mathcal{P}_1$ we assume that the initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \tag{4.5}$$

and, finally, in the study of Problem $\mathcal{P}_2$ we assume that

$$\boldsymbol{u}_0 \in U, \quad \boldsymbol{\sigma}_0 \in Q, \tag{4.6}$$

where U denotes the set of admissible displacements defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}. \tag{4.7}$$

In the rest of the section we denote by c a positive generic constant that may depend on time and whose value may change from line to line. Also, we use the symbol " $\rightharpoonup$" to denote the weak convergence in the Hilbert space V.

We turn now to the variational formulation of the problems $\mathcal{P}_1$ and $\mathcal{P}_2$. To this end, we use Riesz's representation Theorem to define the operator $P : V \to V$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{4.8}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{4.9}$$

Assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.7) and let $t > 0$ be given. We integrate equation (3.1) with the initial conditions (3.7) to obtain

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{4.10}$$

Then we use the Green formula (2.2), the equilibrium equation (3.2), the boundary conditions (3.3)–(3.6) and notation (4.8)–(4.9) to see that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \Big(\int_0^t b(t-s) u_\nu^+(s) ds, v_\nu \Big)_{L^2(\Gamma_3)} \\ + (P\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V. \tag{4.11}$$

We present the following existence and uniqueness result proved in [2].

Lemma 4.1 *Assume that* (4.2) *and* (4.5) *hold. Then, for each function* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\mathcal{S}_1 \boldsymbol{u} \in C(\mathbb{R}_+; Q)$ *such that*

$$\mathcal{S}_1 \boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_1 \boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{4.12}$$

Moreover, the operator $\mathcal{S}_1 : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ *satisfies the following property: for every* $n \in \mathbb{N}$ *there exists* $k_n > 0$ *such that*

$$\|\mathcal{S}_1 \boldsymbol{u}_1(t) - \mathcal{S}_1 \boldsymbol{u}_2(t)\|_Q \le k_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \tag{4.13}$$

$$\forall\, \boldsymbol{u}_1,\ \boldsymbol{u}_2 \in C(\mathbb{R}_+; V),\ \forall\, t \in [0, n].$$

Using the operator $\mathcal{S}_1 : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ defined in Lemma 4.1 we deduce that (4.10) and (4.11) are equivalent with

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1 \boldsymbol{u}(t),$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{S}_1 \boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \Big(\int_0^t b(t-s) u_\nu^+(s) ds, v_\nu \Big)_{L^2(\Gamma_3)} \\ + (P\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V.$$

We use again Riesz's representation Theorem to define the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by equality

$$(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V = (\mathcal{S}_1 \boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \\ + \Big(\int_0^t b(t-s) u_\nu^+(s)\, ds, v_\nu \Big)_{L^2(\Gamma_3)} \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \boldsymbol{v} \in V, \tag{4.14}$$

and we obtain the following variational formulation of the Problem $\mathcal{P}_1$.

Problem $\mathcal{P}_1^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to V$ *and a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1 \boldsymbol{u}(t), \tag{4.15}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V + (P\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V \tag{4.16}$$

hold, for all $t \in \mathbb{R}_+$.

In the study of the problem $\mathcal{P}_1^V$ we have the following existence and uniqueness result.

Theorem 4.2 *Assume that* (3.8) *and* (4.1)–(4.5) *hold. Then, Problem* $\mathcal{P}_1^V$ *has a unique solution, which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; V), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q). \tag{4.17}$$

Proof. We define the operator $A : V \to V$ by equality

$$(A\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + (P\boldsymbol{v}, \boldsymbol{w})_V \quad \forall\, \boldsymbol{v},\, \boldsymbol{w} \in V. \tag{4.18}$$

With this notation we consider the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$, the following equality holds

$$(A\boldsymbol{u}(t), \boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V. \tag{4.19}$$

To solve (4.19) we employ Corollary 2.3 with $X = V$. We use (4.1), (3.8) and (2.1) to see that the operator A verifies condition (2.4), i.e. it is strongly monotone and Lipschitz continuous. In addition, using (4.3) we note that the operator $\mathcal{S}$ satisfies condition (2.5) with

$$r_n = k_n e^{nk_n} + c^2 \max_{r\in[0,n]} \|b(r)\|_{L^\infty(\Gamma_3)}, \tag{4.20}$$

for every $n \in \mathbb{N}$.

Finally, using (4.4) and (4.9) we deduce that $\boldsymbol{f}$ has the regularity expressed in (2.6). It follows now from Corollary 2.3 that there exists a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which solves the equality (4.19), for any $t \in \mathbb{R}_+$.

Based on the results above we deduce the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ which satisfies (4.16) for any $t \in \mathbb{R}_+$. Let $\boldsymbol{\sigma}$ be defined by (4.15). Then it follows that the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ is the unique couple of functions with regularity (4.17) which satisfies (4.15)–(4.16). □

Assume now that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.9)–(3.15) and, again, let $t > 0$ be given. Then, using similar arguments as above we obtain the following variational formulation of Problem $\mathcal{P}_2$.

Problem $\mathcal{P}_2^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to V$ *and a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1\boldsymbol{u}(t), \tag{4.21}$$

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &+(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \ \ \forall\, \boldsymbol{v} \in U \end{aligned} \tag{4.22}$$

hold, for all $t \in \mathbb{R}_+$.

In the study of the problem $\mathcal{P}_2^V$ we have the following existence and uniqueness result.

Theorem 4.3 *Assume that* (3.8), (4.1)–(4.4) *and* (4.6) *hold. Then, Problem* $\mathcal{P}_2^V$ *has a unique solution, which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; U), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q). \tag{4.23}$$

Proof. We use the Theorem 2.2 with $X = V$ and $K = U$ and arguments similar to those used in the proof of Theorem 4.2. □

5 A convergence result

Everywhere in this section we assume that the function p satisfies condition (3.8) and let q be a function which satisfies

$$\begin{cases} \text{(a) } q : \ [g, +\infty[\to \mathbb{R}_+. \\ \text{(b) There exists } L_q > 0 \text{ such that} \\ \qquad |q(r_1) - q(r_2)| \le L_q |r_1 - r_2| \quad \forall\, r_1,\, r_2 \ge g. \\ \text{(c) } (q(r_1) - q(r_2))(r_1 - r_2) > 0 \quad \forall\, r_1,\, r_2 \ge g,\, r_1 \ne r_2. \\ \text{(d) } q(g) = 0. \end{cases} \tag{5.1}$$

Let $\mu > 0$ and consider the function p_μ defined by

$$p_\mu(r) = \begin{cases} p(r) & \text{if} \quad r \le g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if} \quad r > g. \end{cases} \tag{5.2}$$

Using assumption (5.1) it follows that the function p_μ satisfies condition (3.8), i.e.

$$\begin{cases} \text{(a) } p_\mu : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_{p_\mu} > 0 \text{ such that} \\ \qquad |p_\mu(r_1) - p_\mu(r_2)| \le L_{p_\mu} |r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p_\mu(r) = 0 \ \text{ for all } r < 0. \end{cases} \tag{5.3}$$

This allows us to consider the operator $P_\mu : V \to V$ defined by

$$(P_\mu \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\mu(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V \tag{5.4}$$

and, moreover, we note that P_μ is a monotone, Lipschitz continuous operator.

With these notation, we consider the following contact problem.

Problem $\mathcal{P}_{1\mu}$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}_\mu = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\mu) + \mathcal{G}(\boldsymbol{\sigma}_\mu, \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu)) \qquad \text{in} \quad \Omega \times (0,\infty), \tag{5.5}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}_\mu + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega \times (0,\infty), \tag{5.6}$$

$$\boldsymbol{u}_\mu = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1 \times (0,\infty), \tag{5.7}$$

$$\boldsymbol{\sigma}_\mu \boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2 \times (0,\infty), \tag{5.8}$$

$$-\sigma_{\mu\nu} = p_\mu(u_{\mu\nu}) + \int_0^t b(t-s)u_{\mu\nu}^+(s)ds \qquad \text{on} \quad \Gamma_3 \times (0,\infty), \tag{5.9}$$

$$\boldsymbol{\sigma}_{\mu\tau} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_3 \times (0,\infty), \tag{5.10}$$

$$\boldsymbol{u}_\mu(0) = \boldsymbol{u}_0,\ \ \boldsymbol{\sigma}_\mu(0) = \boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{5.11}$$

The equations and boundary conditions in problem (5.5)–(5.11) have a similar interpretations as those in problem (3.9)–(3.15). The difference arises in the fact that here we replace the contact condition with normal compliance, unilateral constraint and memory term (3.13) with the contact condition with normal compliance and memory term (5.9). In this condition μ represents a penalization parameter which may be interpreted as a deformability of the foundation, and then $\frac{1}{\mu}$ is the surface stiffness coefficient. Indeed, when μ is smaller the reaction force of the foundation to penetration is larger and so the same force will result in a smaller penetration, which means that the foundation is less deformable. When μ is larger the reaction force of the foundation to penetration is smaller, and so the foundation is less stiff and more deformable.

Note that here and below $u_{\mu\nu}$ is the normal component of the displacement field $\boldsymbol{u}_\mu$ and $\sigma_{\mu\nu}$, $\boldsymbol{\sigma}_{\mu\tau}$ represent the normal and tangential components of the stress tensor $\boldsymbol{\sigma}_\mu$, respectively.

Assume now that (4.1)–(4.4) and (4.5) hold. Using arguments similar as in Section 4 for contact problem $\mathcal{P}_1$ we obtain the following variational formulation for Problem $\mathcal{P}_{1\mu}$.

Problem $\mathcal{P}_{1\mu}^V$**.** *Find a displacement field* $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ *and a stress field* $\boldsymbol{\sigma}_\mu : \mathbb{R}_+ \to Q$*, such that*

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \mathcal{S}_1\boldsymbol{u}_\mu(t), \tag{5.12}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{v})_V + (P_\mu \boldsymbol{u}_\mu(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V \tag{5.13}$$

hold, for all $t \in \mathbb{R}_+$.

It follows from Theorem 4.2 that Problem $\mathcal{P}_{1\mu}^V$ has a unique solution $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ which satisfies (4.17). Finally, it follows from Theorem 4.3 that Problem $\mathcal{P}_2^V$ has a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (4.23). The behavior of the solution $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ as $\mu \to 0$ is given in the following result.

Theorem 5.1 *Assume that* (3.8), (4.1)–(4.4), (4.6) *and* (5.1) *hold. Then, the solution* $(\boldsymbol{u}_\mu, \boldsymbol{\sigma}_\mu)$ *of Problem* $\mathcal{P}_{1\mu}^V$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *of Problem* $\mathcal{P}_2^V$*, that is*

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \tag{5.14}$$

as $\mu \to 0$*, for all* $t \in \mathbb{R}_+$.

In addition to the mathematical interest in the result above, this result is important from the mechanical point of view, since it shows that the weak solution of the viscoplastic contact problem with normal compliance, memory term and finite penetration may be approached as closely as we wish by the solution of the viscoplastic contact problem with normal compliance, memory term and infinite penetration, with a sufficiently small deformability coefficient.

The proof of Theorem 5.1 is carried out in several steps.

Let $\mu > 0$. In the first step we consider the auxiliary problem of finding a displacement field $\widetilde{\boldsymbol{u}}_\mu : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V + (P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \boldsymbol{v})_V \\ = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{5.15}$$

This problem is an intermediate problem between (5.13) and (4.22), since here $\mathcal{S}\boldsymbol{u}$ is known, taken from the problem with finite penetration $\mathcal{P}_2^V$.

We have the following existence and uniqueness result.

Lemma 5.2 *There exists a unique function* $\widetilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ *which satisfies* (5.15)*, for all* $t \in \mathbb{R}_+$.

Proof. We define the operator $A_\mu : V \to V$ and the function $\widetilde{\boldsymbol{f}} : \mathbb{R}_+ \to V$ by equalities

$$(A_\mu \boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \boldsymbol{u}, \boldsymbol{v})_V \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{5.16}$$

$$(\widetilde{\boldsymbol{f}}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V - (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+ \tag{5.17}$$

and note that (4.3), (4.4), (4.9), (4.12) and (4.14) yield

$$\widetilde{\boldsymbol{f}} \in C(\mathbb{R}_+; V). \tag{5.18}$$

Let $t \in \mathbb{R}_+$. Based on (5.16)–(5.17), it is easy to see that the variational equation (5.15) is equivalent with the nonlinear equation

$$A_\mu \widetilde{\boldsymbol{u}}_\mu(t) = \widetilde{\boldsymbol{f}}(t). \tag{5.19}$$

Next, as in the proof of Theorem 4.2 by (4.1) and the properties of operator P_μ it follows that A_μ is a strongly monotone and Lipschitz continuous operator. And, Theorem 2.1 implies the existence of a unique solution $\widetilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ for the nonlinear equation (5.19), which concludes the proof. □

We proceed with the following convergence result.

Lemma 5.3 *As $\mu \to 0$,*

$$\widetilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \boldsymbol{u}(t) \quad in \;\; V,$$

for all $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$. We take $\boldsymbol{v} = \widetilde{\boldsymbol{u}}_\mu(t)$ in (5.15) to obtain

$$(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V \\ +(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V. \tag{5.20}$$

On the other hand, the properties (5.3) of the function p_μ yield

$$(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V \geq 0. \tag{5.21}$$

We combine (5.20), (5.21) and use (4.1) (c) to obtain that

$$\|\widetilde{\boldsymbol{u}}_\mu(t)\|_V \leq c\,(\|\boldsymbol{f}(t)\|_V + \|\mathcal{S}\boldsymbol{u}(t)\|_V). \tag{5.22}$$

This inequality shows that the sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu \subset V$ is bounded. Hence, there exists a subsequence of the sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$, still denoted $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$, and an element $\widetilde{\boldsymbol{u}}(t) \in V$ such that

$$\widetilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \widetilde{\boldsymbol{u}}(t) \;\text{ in } V, \quad \text{as } \mu \to 0. \tag{5.23}$$

It follows from (5.20) that

$$(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V - (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{S}\boldsymbol{u}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V$$

and, since $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is a bounded sequence in V, we deduce that

$$(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V \leq c.$$

This implies that

$$\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t))\widetilde{u}_{\mu\nu}(t)\, da \leq c$$

and, since $\displaystyle\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t)) g\, da \geq 0$, it follows that

$$\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \leq c. \tag{5.24}$$

We consider now the measurable subsets of Γ_3 defined by

$$\Gamma_{31} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \widetilde{u}_{\mu\nu}(t)(\boldsymbol{x}) \le g \,\}, \qquad \Gamma_{32} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \widetilde{u}_{\mu\nu}(t)(\boldsymbol{x}) > g \,\}. \tag{5.25}$$

Clearly, both Γ_{31} and Γ_{32} depend on t and μ but, for simplicity, we do not indicate explicitly this dependence. We use (5.24) to write

$$\int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da + \int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \le c$$

and, since

$$\int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t))\widetilde{u}_{\mu\nu}(t)\, da \ge 0,$$

we obtain

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \le \int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t)) g\, da + c.$$

Thus, taking into account that $p_\mu(r) = p(r)$ for $r \le g$, by the monotonicity of the function p we can write

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \le \int_{\Gamma_{31}} p(\widetilde{u}_{\mu\nu}(t)) g\, da + c \le \int_{\Gamma_3} p(g) g\, da + c.$$

Therefore, we deduce that

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \le c. \tag{5.26}$$

We use now the definitions (5.2) and (5.25) to see that, a.e. on Γ_{32}, we have

$$p_\mu(\widetilde{u}_{\mu\nu}(t)) = \frac{1}{\mu}\, q(\widetilde{u}_{\mu\nu}(t)) + p(g), \quad p(g)(\widetilde{u}_{\mu\nu}(t) - g) > 0.$$

Consequently, the inequality (5.26) yields

$$\int_{\Gamma_{32}} q(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\, da \le c\mu. \tag{5.27}$$

Next, we consider the function defined by

$$\widetilde{p} : \mathbb{R} \to \mathbb{R}_+ \quad \widetilde{p}(r) = \begin{cases} 0 & \text{if} \quad r \le g, \\ q(r) & \text{if} \quad r > g \end{cases}$$

and we note that by (5.1) it follows that $\widetilde{p}$ is a continuous increasing function and, moreover,

$$\widetilde{p}(r) = 0 \qquad \text{iff} \qquad r \le g. \tag{5.28}$$

We use (5.27), equality $q(\widetilde{u}_{\mu\nu}(t)) = \widetilde{p}(\widetilde{u}_{\mu\nu}(t))$ a.e. on Γ_{32} and (5.25) to deduce that

$$\int_{\Gamma_3} \widetilde{p}(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)^+ \le c\mu,$$

where $(\widetilde{u}_{\mu\nu}(t)-g)^+$ denotes the positive part of $\widetilde{u}_{\mu\nu}(t)-g$. Therefore, passing to the limit as $\mu\to 0$, by using (5.23) as well as compactness of the trace operator we find that

$$\int_{\Gamma_3} \widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+\,da \le 0.$$

Since the integrand $\widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+$ is positive a.e. on Γ_3, the last inequality yields

$$\widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+ = 0 \quad \text{a.e. on } \Gamma_3$$

and, using (5.28) and definition (4.7) we conclude that

$$\widetilde{\boldsymbol{u}}(t)\in U. \tag{5.29}$$

Next, we test in (5.15) with $\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t)$, where $\boldsymbol{v}\in U$, to obtain

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\\ +(P_\mu\widetilde{\boldsymbol{u}}_\mu(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V=(\boldsymbol{f}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V.\end{aligned} \tag{5.30}$$

Since $\boldsymbol{v}\in U$ we have $p_\mu(v_\nu)=p(v_\nu)$ a.e. on Γ_3. Taking into account this equality and the monotonicity of the function p_μ we have

$$p(v_\nu)(v_\nu-\widetilde{u}_{\mu\nu}(t))\ge p_\mu(\widetilde{u}_{\mu\nu}(t))(v_\nu-\widetilde{u}_{\mu\nu}(t)) \quad \text{a.e. on } \Gamma_3$$

and, therefore, by using (5.4) we obtain

$$(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\ge(P_\mu\widetilde{\boldsymbol{u}}_\mu(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V. \tag{5.31}$$

Then, using (5.31) and (5.30) we find that

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\\ +(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\ge(\boldsymbol{f}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\quad\forall\,\boldsymbol{v}\in U.\end{aligned} \tag{5.32}$$

We pass to the lower limit in (5.32) and use (5.23) to obtain

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}(t))_V\\ +(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}(t))_V\ge(\boldsymbol{f}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}(t))_V\quad\forall\,\boldsymbol{v}\in U.\end{aligned} \tag{5.33}$$

Next, we take $\boldsymbol{v}=\widetilde{\boldsymbol{u}}(t)$ in (4.22) and $\boldsymbol{v}=\boldsymbol{u}(t)$ in (5.33). Then, adding the resulting inequalities we find that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t))-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\le 0.$$

This inequality combined with (4.1) implies that

$$\widetilde{\boldsymbol{u}}(t)=\boldsymbol{u}(t).$$

It follows from here that the whole sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is weakly convergent to the element $\boldsymbol{u}(t)\in V$, which concludes the proof. □

We proceed with the following strong convergence result.

Lemma 5.4 *As* $\mu \to 0$,

$$\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0,$$

for all $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$. Using (4.1) we write

$$\begin{aligned} m_{\mathcal{E}}\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V^2 &\le (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &= (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q. \end{aligned}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (5.32) to obtain

$$\begin{aligned} -(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q &\le (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \\ &+ (P\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V - (\boldsymbol{f}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \end{aligned}$$

and, therefore, combining the above inequalities we find that

$$\begin{aligned} &m_{\mathcal{E}}\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V^2 \le (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &+ (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V + (P\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V - (\boldsymbol{f}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V. \end{aligned}$$

We pass to the upper limit in this inequality and use Lemma 5.3 to conclude the proof of the lemma. □

We are now in position to provide the proof of Theorem 5.1.

Proof. Let $t \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ be such that $t \in [0, n]$. Let also $\mu > 0$. Then, testing with $\boldsymbol{v} = \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t)$ in (5.15) and (5.13), we have

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \\ &\quad + (P_\mu\widetilde{\boldsymbol{u}}_\mu(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V, \end{aligned}$$

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \\ &\quad + (P_\mu\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V = (\boldsymbol{f}(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V. \end{aligned}$$

We subtract the previous equalities and use the monotonicity of the operator P_μ to deduce that

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &\quad \le (\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \end{aligned}$$

and, therefore,

$$\|\boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{1}{m_{\mathcal{E}}}\,\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t)\|_V. \tag{5.34}$$

We use (5.34) to find that

$$\|\boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{r_n}{m_{\mathcal{E}}}\int_0^t \|\boldsymbol{u}(s) - \boldsymbol{u}_\mu(s)\|_V\, ds$$

where r_n is given by (4.20). It follows from here that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_\mathcal{E}} \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V ds$$

and, using a Gronwall argument, we obtain

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_\mathcal{E}} \int_0^t e^{\frac{r_n}{m_\mathcal{E}}(t-s)} \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds. \quad (5.35)$$

Note that $e^{\frac{r_n}{m_\mathcal{E}}(t-s)} \le e^{\frac{r_n}{m_\mathcal{E}}t} \le e^{\frac{nr_n}{m_\mathcal{E}}}$ for all $s \in [0,t]$ and, therefore, (5.35) yields

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_\mathcal{E}} e^{\frac{nr_n}{m_\mathcal{E}}} \int_0^t \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds. \quad (5.36)$$

On the other hand, by estimate (5.22), Lemma 5.4 and Lebesgue's convergence Theorem it follows that

$$\int_0^t \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds \to 0 \qquad \text{as} \quad \mu \to 0. \quad (5.37)$$

We use now (5.36), (5.37) and Lemma 5.4 to see that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0 \qquad \text{as} \quad \mu \to 0. \quad (5.38)$$

Next, by (4.21) and (5.12) we obtain

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \le \|\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))\|_Q + \|\mathcal{S}\boldsymbol{u}_\mu(t) - \mathcal{S}\boldsymbol{u}(t)\|_V$$

and, using (4.1), (4.13), (4.3) and (4.20) it follows that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \le c\, \|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t))\|_V + r_n \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds.$$

We use again the convergence (5.38) and Lebesque's Theorem to find that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \qquad \text{as} \quad \mu \to 0. \quad (5.39)$$

Theorem 5.1 is now a consequence of the convergences (5.38) and (5.39). □

6 Numerical solutions

This section is devoted to the numerical solution of the contact problems presented in Section 3, including the numerical validation of the convergence result in Theorem 5.1. In order to avoid repetitions, we restrict ourselves to present details only on the numerical approach of Problem $\mathcal{P}_2$, which is based on penalization and the augmented Lagrangean method. To this end we introduce a new variational formulation of Problem $\mathcal{P}_2$, more convenient for the numerical solution.

An adapted variational formulation. We consider the space $X_\nu = \{\, v_{\nu|\Gamma_3} \ : \ \boldsymbol{v} \in V \,\}$ with its usual norm and denote by X'_ν and $\langle \cdot, \cdot \rangle_{X'_\nu, X_\nu}$ the dual of X_ν and the duality pairing mapping, respectively. We also consider the function $\varphi : X_\nu \to (-\infty, +\infty]$ and the operators L, $H : X_\nu \to X'_\nu$ defined by

$$\varphi(u_\nu) = \int_{\Gamma_3} I_{\mathbb{R}_-}(u_\nu - g)\, da, \quad \forall\, u_\nu \in X_\nu,$$

$$\langle L u_\nu, v_\nu \rangle_{X'_\nu, X_\nu} = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, u_\nu,\, v_\nu \in X_\nu,$$

$$\langle H u_\nu, v_\nu \rangle_{X'_\nu, X_\nu} = \int_{\Gamma_3} \Big(\int_0^t b(t-s) u_\nu^+(s) ds \Big) v_\nu \, da \quad \forall\, u_\nu,\, v_\nu \in X_\nu, \qquad (6.1)$$

where $I_{\mathbb{R}_-}$ represents the indicator function of the set $\mathbb{R}_- = (-\infty, 0]$.

We note that, for all $t \in \mathbb{R}_+$, condition (3.13) is equivalent to the subdifferential inclusion

$$-\, \sigma_\nu(t) \in \partial \varphi(u_\nu(t)) + L u_{\nu|\Gamma_3}(t) + H u_{\nu|\Gamma_3}(t) \quad \text{in} \quad X'_\nu, \qquad (6.2)$$

where $\partial\varphi$ denotes the subdifferential of φ. This inclusion suggests to introduce a new unknown of the problem, the Lagrange multiplier, which represents the normal stress on the contact surface. Thus, proceeding in a standard way and using the inclusion (6.2) we obtain the following variational formulation of Problem $\mathcal{P}_2$, in terms of three unknown fields.

Problem $\widetilde{\mathcal{P}}_2^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to V$*, a stress field* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ *and a Lagrange multiplier* $\lambda : \mathbb{R}_+ \to X'_\nu$ *such that, for all* $t \in \mathbb{R}_+$*,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \qquad (6.3)$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q - \langle \lambda(t), v_{\nu|\Gamma_3} \rangle_{X'_\nu, X_\nu} = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V, \qquad (6.4)$$

$$-\lambda(t) \in \partial\varphi(u_\nu(t)) + L u_{\nu|\Gamma_3}(t) + H u_{\nu|\Gamma_3}(t). \qquad (6.5)$$

Fully-discrete approximation. Let $k > 0$ be the time step size and define

$$t_n = n\, k, \quad 0 \le n \le N,$$

where N is a sufficiently large integer. Below, for a continuous function $v(t)$ with values in a function space, we use the notation $v_j = v(t_j)$, for $0 \le j \le N$. Assume that Ω is a polyhedral domain. Moreover, let $\{\mathcal{T}^h\}$ be a regular family of triangular finite element partitions of $\overline{\Omega}$ that are compatible with the boundary decomposition

$\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, i.e. if one side of an element $T \in \mathcal{T}^h$ has more than one point on Γ, then the side lies entirely in $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ or $\overline{\Gamma}_3$. The space V is approximated by the finite dimensional space $V^h \subset V$ of continuous and piecewise affine functions, that is,

$$\begin{aligned} V^h = \{\, \boldsymbol{v}^h \in [C(\overline{\Omega})]^d \ :\ \boldsymbol{v}^h|_T \in [P_1(T)]^d \ \ \forall T \in \mathcal{T}^h, \\ \boldsymbol{v}^h = \boldsymbol{0} \ \text{ at the nodes on } \ \Gamma_1 \,\}, \end{aligned} \tag{6.6}$$

where $P_1(T)$ represents the space of polynomials of degree less or equal to one in T. The space Q is approximated by the finite element space of piecewise constants, denoted Q^h. For any $\boldsymbol{\tau} \in Q$, we denote by $\Pi_{Q^h}\boldsymbol{\tau}$ its finite element projection onto Q^h, that is

$$(\Pi_{Q^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q = (\boldsymbol{\tau}, \boldsymbol{\tau}^h)_Q \quad \forall \boldsymbol{\tau}^h \in Q^h.$$

We also consider the discrete space $Y_\nu^h \subset X_\nu' \cap L^2(\Gamma_3)$ related to the discretization of the Lagrange multiplier λ. see [6, 7] for considerations about the discretization step.

Let $\boldsymbol{u}_0^h \in V^h$ and $\boldsymbol{\sigma}_0^h \in Q^h$ be the finite element approximations of $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$, respectively. Then, we consider the following fully discrete numerical approximation of Problem $\widetilde{\mathcal{P}}_2^V$.

Problem $\mathcal{P}_V^{hk}$. *Find a discrete displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=1}^N \subset V^h$, *a discrete stress field* $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}_n^{hk}\}_{n=1}^N \subset Q^h$ *and a discrete Lagrange multiplier* $\lambda^{hk} = \{\lambda_n^{hk}\}_{n=1}^N \subset Y_\nu^h$ *such that, for all* $n = 1, \ldots, N$,

$$\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{Q^h}\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}) + \sum_{j=0}^{n-1} k\mathcal{P}_{Q^h}\mathcal{G}_j^{hk}, \tag{6.7}$$

$$(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_Q - \langle \lambda_n^{hk}, v^h_{\nu|_{\Gamma_3}} \rangle_{X_\nu', X_\nu} = (\boldsymbol{f}_n, \boldsymbol{v}^h)_V \quad \forall \boldsymbol{v}^h \in V^h, \tag{6.8}$$

$$-\lambda_n^{hk} \in \partial\varphi(u_{\nu|_{\Gamma_3}}{}_n^{hk}) + L u_{\nu|_{\Gamma_3}}{}_n^{hk} + \widetilde{H} u_{\nu|_{\Gamma_3}}{}_n^{hk}. \tag{6.9}$$

Note that the sum in (6.7) corresponds to the approximation of the integral in (6.3) by using a rectangle method (Top-left corner approximation) for the time integration.

Furthermore, in (6.9) we propose to approximate the operator H by using a trapezoidale rule for the time integral which appears in (6.1). The approximated operator $\widetilde{H}$ is defined as follows:

$$\widetilde{H} u_\nu = \sum_{i=0}^{n} \frac{\Delta t}{2} [b(t_n - t_{i-1}) u_{\nu i-1} + b(t_n - t_i) u_{\nu i}] \quad \forall\, u_\nu \in X_\nu. \tag{6.10}$$

Here and below we use the short-hand notation $\mathcal{G}_j^{hk} = \mathcal{G}(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}))$.

Numerical method. In the rest of this subsection, to simplify the notation, we skip the dependence of various variables with respect to the discretization parameters n, k and h, i.e., for example, we write $\boldsymbol{u}$ instead of $\boldsymbol{u}_n^{hk}$.

For the numerical treatment of the condition (6.9), we use the penalized method for the compliance contact combined with the augmented Lagrangean approach for the unilateral condition. To this end, we consider additional fictitious nodes for the Lagrange multiplier in the initial mesh. The construction of these nodes depends on the contact element used for the geometrical discretization of the interface Γ_3. In the case of the numerical example presented below, the discretization is based on "node-to-rigid" contact element, which is composed by one node of Γ_3 and one Lagrange multiplier node. This contact interface discretization is characterized by a finite dimensional subspace $H^h_{\Gamma_3} \subset Y^h_\nu$. Let N^h_{tot} be the total number of nodes and denote by α^i the basis functions used to define the space V^h for $i = 1, \ldots, N^h_{tot}$. Moreover, let $N^h_{\Gamma_3}$ represent the number of nodes on the interface Γ_3 and let μ^i be the shape functions of the finite element space $H^h_{\Gamma_3}$, for $i = 1, \ldots, N^h_{\Gamma_3}$, i.e.

$$H^h_{\Gamma_3} = \{\gamma^h \in Y^h_\nu \ : \ \gamma^h = \sum_{i=1}^{N^h_{\Gamma_3}} \gamma^i \mu^i\}.$$

Usually, if a P_1 finite element method is used for the displacement, then a P_0 finite element method is considered for the multipliers. The expression of functions $\boldsymbol{v}^h \in V^h$ and $\gamma^h \in H^h_{\Gamma_3}$ is given by

$$\boldsymbol{v}^h = \sum_{i=1}^{N^h_{tot}} \boldsymbol{v}^i \alpha^i \quad \text{and} \quad \gamma^h = \sum_{i=1}^{N^h_{\Gamma_3}} \gamma^i \mu^i, \tag{6.11}$$

where $\boldsymbol{v}^i$ represents the value of the corresponding functions $\boldsymbol{v}^h$ at the i-th node of $\mathcal{T}^h$. Also, γ^i denotes the value of the function γ^h at the i-th node of the contact element of the discretized contact interface. More details about this discretization step can be found in [1, 6, 7, 17].

It can be shown that the numerical approach of Problem $\mathcal{P}^{hk}_V$ is governed at each time step n by a system of nonlinear equations of the form

$$\mathbf{R}(\boldsymbol{u}, \boldsymbol{\lambda}) = \widetilde{\mathbf{G}}(\boldsymbol{u}) + \mathcal{F}(\boldsymbol{u}, \boldsymbol{\lambda}) = \mathbf{0}, \tag{6.12}$$

where the functions $\widetilde{\mathbf{G}}$ and $\mathcal{F}$ are defined below. Here the unknowns are the discrete displacement field $\boldsymbol{u} \in \mathbb{R}^{d \cdot N^h_{tot}}$ and the Lagrange multiplier generalized vector $\boldsymbol{\lambda} \in \mathbb{R}^{N^h_{\Gamma_3}}$, defined by

$$\boldsymbol{u} = \{\boldsymbol{u}^i\}_{i=1}^{N^h_{tot}}, \ \boldsymbol{\lambda} = \{\lambda^i\}_{i=1}^{N^h_{\Gamma_3}}, \tag{6.13}$$

where $\boldsymbol{u}^i$ represents the value of the corresponding function $\boldsymbol{u}^{hk}_n$ at the i-th node of $\mathcal{T}^h$. Also, λ^i denotes the value of the corresponding function λ^{hk}_n at the i-th node of the contact element of the discretized contact interface. The generalized elastic term $\widetilde{\mathbf{G}}(\boldsymbol{u}) \in \mathbb{R}^{d \cdot N^h_{tot}} \times \mathbb{R}^{N^h_{\Gamma_3}}$ is defined by $\widetilde{\mathbf{G}}(\boldsymbol{u}) = (\mathbf{G}(\boldsymbol{u}), \mathbf{0}_{N^h_{\Gamma_3}})$, where $\mathbf{0}_{N^h_{\Gamma_3}}$ is the zero element of $\mathbb{R}^{N^h_{\Gamma_3}}$, $\mathbf{G}(\boldsymbol{u}) \in \mathbb{R}^{d \cdot N^h_{tot}}$ denotes the term given by

$$(\mathbf{G}(\boldsymbol{u}) \cdot \boldsymbol{v})_{\mathbb{R}^{d \times N^h_{tot}}} = (\boldsymbol{\sigma}^{hk}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}^h))_Q - (\boldsymbol{f}, \boldsymbol{v}^h)_V \quad \forall \boldsymbol{v} = \{\boldsymbol{v}^i\}_{i=1}^{N^h_{tot}},$$

$\boldsymbol{v}^h$ is defined by (6.11) and $\boldsymbol{\sigma}_n^{hk}$ is related to $\boldsymbol{u}_n^{hk}$ by the discrete constitutive law (6.7). The contact operator $\mathcal{F}(\boldsymbol{u}, \boldsymbol{\lambda})$, which allows to deal with the contact condition (6.9) is defined by

$$
\begin{aligned}
(\mathcal{F}(\boldsymbol{u}, \boldsymbol{\lambda}) \cdot (\boldsymbol{v}, \boldsymbol{\gamma}))_{\mathbb{R}^{d\cdot N^h_{tot}} \times \mathbb{R}^{N^h_{\Gamma_3}}} &= \int_{\Gamma_3} \nabla \boldsymbol{u} l^r_\nu(\boldsymbol{u}_n^{hk}, \lambda_n^{hk}) \cdot \boldsymbol{v}^h d\,\Gamma \qquad (6.14)\\
&+ \int_{\Gamma_3} \nabla_\lambda l^r_\nu(\boldsymbol{u}_n^{hk}, \lambda_n^{hk}) \cdot \gamma^h d\,\Gamma + \int_{\Gamma_3} \nabla \boldsymbol{u} \mathcal{P}_c([(u_\nu)_n^{hk}]_g) \cdot \boldsymbol{v}^h d\,\Gamma \\
&+ \int_{\Gamma_3} \nabla \boldsymbol{u} \mathcal{P}_H([(u_\nu)_n^{hk}]_g) \cdot \boldsymbol{v}^h d\,\Gamma \\
&\forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{d\cdot N^h_{tot}},\ \forall\, \boldsymbol{\lambda}, \boldsymbol{\gamma} \in \mathbb{R}^{N^h_{\Gamma_3}},\ \boldsymbol{u}_n^{hk}, \boldsymbol{v}^h \in V^h,\ \forall\, \lambda_n^{hk}, \gamma^h \in H^h_{\Gamma_3}.
\end{aligned}
$$

Here $\mathcal{P}_c$, $\mathcal{P}_H : \mathbb{R} \to \mathbb{R}$ are derivable functions such that $\nabla \boldsymbol{u} \mathcal{P}_c = p$ on $(-\infty, g]$ and $\nabla \boldsymbol{u} \mathcal{P}_H = \widetilde{H}$, $[\,\cdot\,]_g : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$
[s]_g = \begin{cases} s & \text{if} \quad s \le g, \\ 0 & \text{if} \quad s > g, \end{cases}
$$

and $\nabla \boldsymbol{u}$ represents the gradient operator with respect the variable $\boldsymbol{u}$; also, l^r_ν denotes the augmented Lagrangean functional given by

$$
l^r_\nu(\boldsymbol{u}_n^{hk}, \lambda_n^{hk}) = -\frac{1}{2r}\,(\lambda_n^{hk})^2 + \frac{1}{2r}\,\left[(\lambda_n^{hk} + r((u_\nu)_n^{hk} - g))^+\right]^2, \qquad (6.15)
$$

where r is a positive penalty coefficient.

The solution of the nonlinear system (6.12) is based on a generalized Newton method, which permits to treat simultaneously the two unknowns $\boldsymbol{u}$ and $\boldsymbol{\lambda}$. Nevertheless, to keep this paper in a reasonable length, we skip the presentation of the numerical algorithm and we pass in what follows to description of the numerical example. Details on this kind of algorithms can be found in [1, 10, 17].

7 Numerical simulations

Physical setting of the numerical example. For the numerical simulations we consider the physical setting depicted in Figure 1. There, $\Omega = [0, 2] \times [0, 1] \subset \mathbb{R}^2$ with $\Gamma_1 = (\{0\} \times [0.5, 1]) \cup (\{2\} \times [0.5, 1])$, $\Gamma_2 = ([0, 2] \times \{1\}) \cup (\{0\} \times [0, 0.5]) \cup (\{2\} \times [0, 0.5])$, $\Gamma_3 = [0, 2] \times \{0\}$. The domain Ω represents the cross section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part Γ_1 the body is clamped and, therefore, the displacement field vanishes there. Vertical tractions act on the part $[0, 2] \times \{1\}$ of the boundary Γ_2 and the part $(\{0\} \times [0, 0.5]) \cup (\{2\} \times [0, 0.5])$ is traction free. No body forces are assumed to act on the body during the process. The body is in frictionless contact with an obstacle on the part $\Gamma_3 = [0, 2] \times \{0\}$ of the boundary. For the discretization we use 7935 elastic finite elements and 129 contact elements.

The total number of degrees of freedom is equal to 8326 and we take a time step k equal to $0.01s$.

We model the material's behavior with a constitutive law of the form (1.1) in which elasticity tensor $\mathcal{E}$ satisfies

$$(\mathcal{E}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2, \tag{7.1}$$

where E is the Young modulus, κ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

Moreover, in order to facilitate the numerical implementation, we assume that $\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$, where the tensor $\mathcal{C}$ satisfies

$$(\mathcal{C}\boldsymbol{\tau})_{\alpha\beta} = \gamma_1(\tau_{11}+\tau_{22})\delta_{\alpha\beta} + \gamma_2\tau_{\alpha\beta}, \qquad 1 \le \alpha, \beta \le 2. \tag{7.2}$$

For the computation below we use the following data:

$$\begin{aligned}
&t \in [0,T], \quad T = 1s, \quad k = 0.01s, \quad N = 100,\\
&E = 10000 N/m^2, \quad \kappa = 0.3, \quad \gamma_1 = 1N/m^2, \quad \gamma_2 = 2N/m^2,\\
&\boldsymbol{f}_0 = (0,0)N/m^2,\\
&\boldsymbol{f}_2 = \begin{cases} (0,0)\, N/m & \text{on } (\{0\}\times[0,0.5]) \cup (\{2\}\times[0,0.5]),\\ (-2000,0)\, N/m & \text{on } [0,2]\times\{1\}, \end{cases}\\
&p(r) = c_\nu r^+, \quad c_\nu = 200, \quad g = 0.05\, m,\\
&b(r) = c_\nu, \quad q(r) = r^+, \quad p_\mu(r) = \tfrac{1}{\mu}q(r) + p(g), \quad \tfrac{1}{\mu} = 200,
\end{aligned}$$

Numerical results. The main purpose of this part consists to present a numerical validation of the theoretical convergence result obtained in Theorem 5.1. Our results are presented in Figures 2–6 and are described in what follows.

First, the deformed configuration as well as the contact interface forces at $t = 1s$ are plotted in Figure 2, which corresponds to the numerical solution of problem $\mathcal{P}_2^V$. In order to compare the deformed mesh related to Problem $\mathcal{P}_2^V$ with those obtained for the numerical solution of problem $\mathcal{P}_{1\mu}^V$, we plotted in Figures 3 and 4, respectively, the deformed configurations for the numerical solution of problems $\mathcal{P}_{1\mu}^V$ with memory term (in which the function $b = c_\nu$) and without memory term ($b = 0$), respectively. Then, in Figures 3 and 4, we note that the penetration of the contact nodes is no longer restricted by unilateral constraint and exceed the limit g. Moreover, the absence of the memory term leads to larger penetrations in the foundation.

In Figure 5 we present the evolution of the convergence of the discrete solution of the problem $\mathcal{P}_{1\mu}^V$ to the discrete solution of the problem $\mathcal{P}_2^V$ as the deformability of the foundation μ tends to zero. More precisely, we plot 4 deformed meshes and the associated contact forces for 4 values of $1/\mu$ which represents here the stiffness

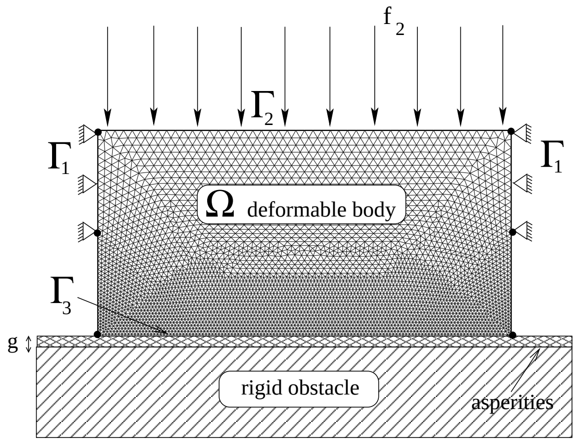

Figure 1: Initial configuration of the two-dimensional example.

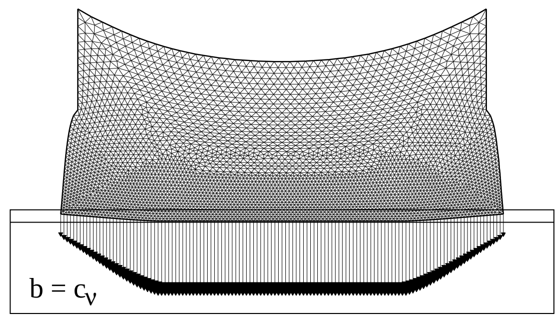

Figure 2: Deformed mesh and contact interface forces related to Problem $\mathcal{P}_2^V$.

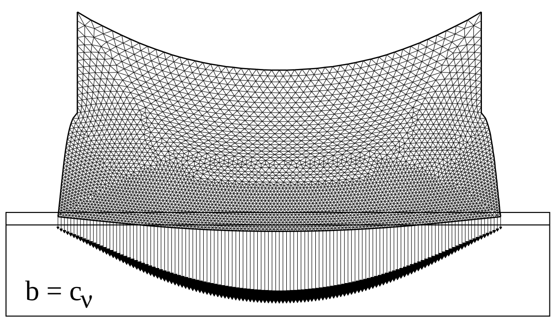

Figure 3: Deformed mesh and contact interface forces related to Problem $\mathcal{P}_{1\mu}^V$.

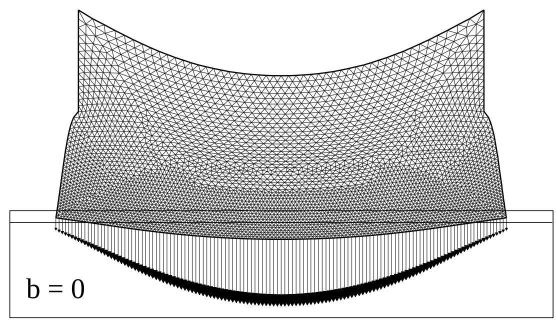

Figure 4: Deformed mesh and contact interface forces related to Problem $\mathcal{P}_{1\mu}^V$ without memory term (b=0).

of the foundation after the limit g is reached. One can see that for $1/\mu = 10$ all the contact nodes are in strong penetration, whereas for $1/\mu = 10000$ two-third of the nodes slightly exceed the limit $g = 0.05\,m$ and will come into a unilateral contact.

For the convergence result, we denote by $\boldsymbol{u}_\mu^{hk}$ and $\boldsymbol{u}^{hk}$ the discrete solution of the contact problems $\mathcal{P}_{1\mu}^V$ and $\mathcal{P}_2^V$, respectively, for a given $\mu > 0$. The numerical estimations of the difference

$$\|\boldsymbol{u}_\mu^{hk} - \boldsymbol{u}^{hk}\|_V + \|\boldsymbol{\sigma}_\mu^{hk} - \boldsymbol{\sigma}^{hk}\|_Q$$

at the time $t = 1\,s$, for various values of the coefficient μ, are presented in Figure 6. It results from here that this difference converges to zero as $1/\mu$ tends towards infinity. We conclude that our results in Figure 6 represent a numerical validation of the theoretical convergence result obtained in Theorem 5.1.

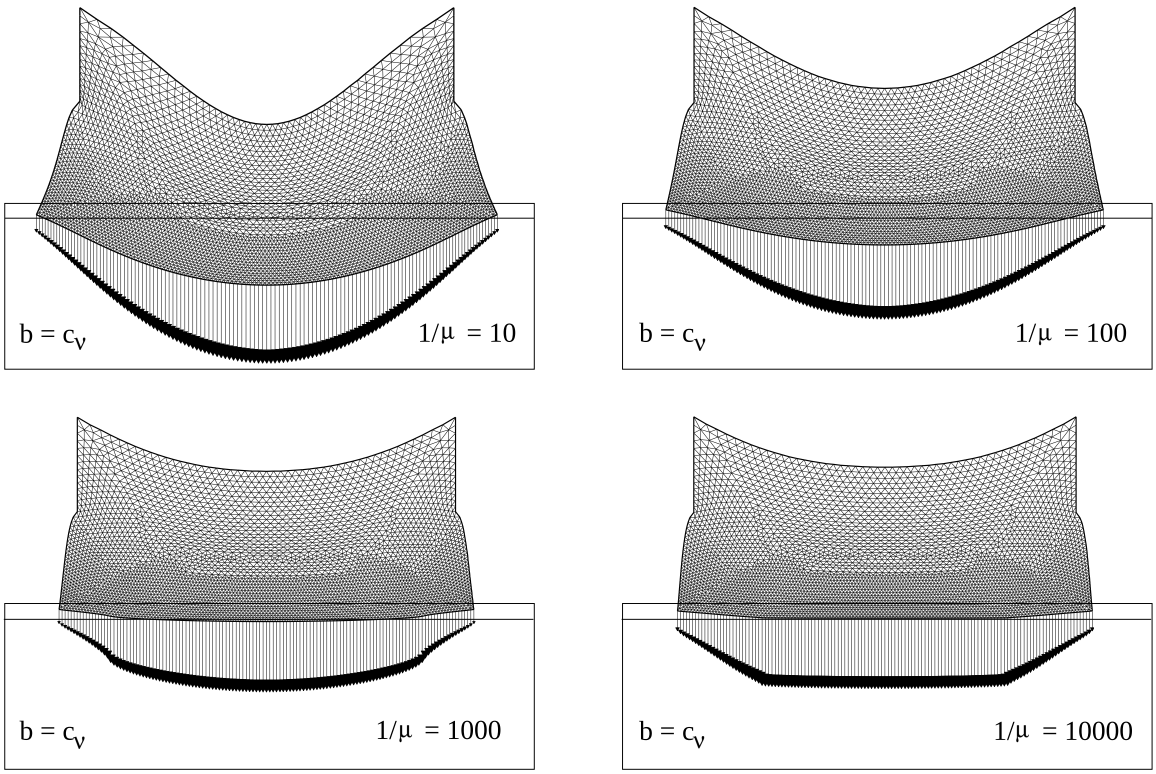

Figure 5: Deformed meshes and contact interface forces for $1/\mu = 10$, $1/\mu = 100$, $1/\mu = 1000$ and $1/\mu = 10000$ related to Problem $\mathcal{P}_{1\mu}^{V}$.

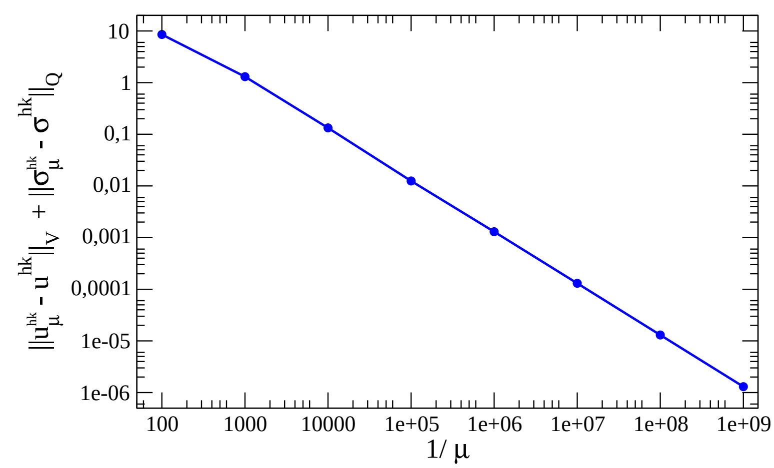

Figure 6: Numerical validation of the convergence result in Theorem 5.1.

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