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Dedicated to Maria S. Pop on her 60^{th} anniversary

ON THE CHEBYSHEV METHOD FOR APPROXIMATING THE SOLUTIONS OF POLYNOMIAL OPERATOR EQUATIONS OF DEGREE 2

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ABSTRACT. In this paper, the Chebyshev method for approximating the solutions of polynomial operator equations of degree 2 is presented. The convergence of the Chebyshev method is studied.
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1. Introduction

The polynomial operator equations represent an important class of operator equations [1]. Among them, the polynomial operator equations of degree 2 have a special importance because the convergence hypotheses for the usual methods (Newton method, chord method, Steffensen method, Chebyshev method, etc.) are much simplified compared to the general case [6]. In this note we shall study the convergence of the Chebyshev method for the mentioned equations.

Let X be a Banach space and consider a mapping $f : X \to X$. We remind that the mapping f is a polynomial operator of degree two if

a) f is three times differentiable;
b) $f'''(x) = \theta_3, \forall x \in X$, where θ_3 is the trilinear null operator.

2. The convergence of the Chebyshev method

Consider the equation

$$f(x) = \theta, \tag{2.1}$$

where $\theta \in X$ is the zero element. Given an initial approximation $u_0 \in X$ of a solution $\bar{u}$ of the above equation, the Chebyshev method generates the sequence $(u_n)_{n\geq 0}$ by

$$u_{n+1} = u_n - \Gamma_n f(u_n) - \tfrac{1}{2}\Gamma_n f''(u_n)(\Gamma_n f(u_n))^2, \; n = 0,1,2,\ldots, \; u_0 \in X$$

where $\Gamma_n = f'(u_n)^{-1}$.

Consider $r > 0$ and denote $S = \{u \in X : \|u - u_0\| \leq r\}$. Since $f'''(x) = \theta_3, \forall x \in X$, it is clear that $f''(x)$ does not depend on x, so we may take $m_2 = \|f''(x)\|$.

From the Taylor formula we obtain

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \tfrac{1}{2}f''(u_0)(u - u_0)^2 \tag{2.2}$$

$$f'(u) = f'(u_0) + f''(u_0)(u - u_0). \tag{2.3}$$

By (2.3) it follows

$$\|f'(u)\| \leq \|f'(u_0)\| + m_2 r, \; \forall u \in S,$$

which implies

$$\sup_{u\in S}\|f'(u)\| \leq \|f'(u_0)\| + m_2 r. \tag{2.4}$$

Similarly (2.2), leads to

$$\sup_{u\in S}\|f(u)\| \leq \|f(u_0)\| + r\|f'(u_0)\| + \tfrac{1}{2}m_2 r^2. \tag{2.5}$$

We shall make the following notations:

$$m_0 = \|f(u_0)\| + r\|f'(u_0)\| + \tfrac{1}{2}m_2 r^2 \tag{2.6}$$

$$\mu = \tfrac{1}{2}m_2^2 b^4\left(1 + \tfrac{1}{4}m_2 m_0 b^2\right), \tag{2.7}$$

$$v = b\left(1 + \tfrac{1}{2}m_2 m_0 b^2\right), \tag{2.8}$$

where

$$b = \tfrac{b_0}{1 - m_2 b_0 r} \text{ and } b_0 = \left\|f'(u_0)^{-1}\right\|. \tag{2.9}$$

With the above notations, the following result holds:

Theorem 1. *If for some* $u_0 \in X$ *and* $r > 0$, *the mapping* f *satisfies*

i. $\exists f'(u_0)^{-1}$;
ii. $m_2 b_0 r < 1$;
iii. *the numbers* μ *and* v *given by* (2.7) *and* (2.8) *verify*

$$\rho_0 = \sqrt{\mu} \cdot \|f(u_0)\| < 1 \text{ and}$$
$$\frac{v \cdot \rho_0}{\sqrt{\mu(1-\rho_0)}} \le r,$$

then the following relations hold:

j. *the sequence* $(u_n)_{n\ge 0}$ *generated by the Chebyshev method converges;*
jj. *denoting* $\bar{u} = \lim_{n\to\infty} u_n$, *then* $\bar{u} \in S$ *and* $f(\bar{u}) = \theta$;
jjj. $\|u_{n+1} - u_n\| \le \frac{v\rho_0^{3^n}}{\sqrt{\mu}}, \quad n = 0, 1, \dots;$
jv. $\|\bar{u} - u_n\| \le \frac{v\rho_0^{3^n}}{\sqrt{\mu}\left(1-\rho_0^{3^n}\right)}, \quad n = 0, 1, \dots$

Proof. First we shall show that hypothesis *ii*) implies the existence of the application $f'(u)^{-1}$ for all $u \in S$ and, moreover, $\left\|f'(u)^{-1}\right\| \le b$. Indeed, one has

$$\left\|f'(u_0)^{-1}\left(f'(u_0) - f'(u)\right)\right\| \le m_2 b_0 r, \ \forall u \in S.$$

Applying the Banach Lemma and taking into account relation *ii*) it follows the existence of $f'(u)^{-1}$ for all $u \in S$ and, moreover,

$$\left\|f'(u)^{-1}\right\| \le \frac{b_0}{1-m_2 b_0 r} = b. \tag{2.10}$$

Denote by g the mapping $g : S \to X$ given by

$$g(u) = -\Gamma(u) f(u) - \tfrac{1}{2}\Gamma(u) f''(u) \left(\Gamma(u) f(u)\right)^2 \tag{2.11}$$

where $\Gamma(u) = f'(u)^{-1}$.

It can be easily seen that for all $u \in S$, the following identity holds:

$$\begin{aligned} & f(u) + f'(u) g(u) + \tfrac{1}{2} f''(u) g^2(u) = \\ & = \tfrac{1}{2} f''(u) \left(f'(u)^{-1} f(u), f'(u)^{-1} f''(u) \left(f'(u)^{-1} f(u)\right)^2\right) \\ & \quad + \tfrac{1}{8} f''(u) \left(f'(u)^{-1} f''(u) \left(f'(u)^{-1} f(u)\right)^2\right)^2, \end{aligned}$$

whence

$$\left\|f(u) + f'(u) g(u) + \tfrac{1}{2} f''(u) g^2(u)\right\| \le \mu \|f(u)\|^3, \qquad \forall u \in S. \tag{2.12}$$

Since $u_{n+1} = u_n + g(u_n)$, from the Taylor formula we get

$$f(u_{n+1}) = f(u_n) + f'(u_n)(u_{n+1} - u_n) + \tfrac{1}{2} f''(u_n)(u_{n+1} - u_n)^2$$
$$= f(u_n) + f'(u_n) g(u_n) + \tfrac{1}{2} f''(u_n) g^2(u_n),$$

and by (2.2),

$$\|f(u_{n+1})\| \leq \mu \|f(u_n)\|^3, \tag{2.13}$$

provided that $u_n \in S$. Since $u_0 \in S$ one obtains

$$\|u_1 - u_0\| = \|g(u_0)\| \leq v \|f(u_0)\| \leq \tfrac{v\sqrt{\mu}\|f(u_0)\|}{\sqrt{\mu}(1-\rho_0)} = \tfrac{v\rho_0}{\sqrt{\mu}(1-\rho_0)} \leq r,$$

i.e. $u_1 \in S$.

Suppose now that the following relations hold:

α) $u_i \in S,\ i = \overline{0,k}$;

β) $\|f(u_i)\| \leq \mu \|(u_{i-1})\|^3,\ i = \overline{1,k}$.

From $u_k \in S$ and (2.2) it results

$$\|f(u_{k+1})\| \leq \mu \|f(u_k)\|^3 \tag{2.14}$$

and

$$\|u_{k+1} - u_k\| \leq v \|f(u_k)\|. \tag{2.15}$$

Inequality (2.14) leads to

$$\|f(u_i)\| \leq \tfrac{1}{\sqrt{\mu}} \left(\sqrt{\mu} \|f(u_0)\|\right)^{3^i}, \qquad i = \overline{1, k+1}.$$

By (2.13) one gets

$$\|u_{k+1} - u_0\| \leq \sum_{i=1}^{k+1} \|u_1 - u_{i-1}\| \leq \sum_{i=1}^{k+1} v \|f(u_{i-1})\|$$
$$\leq \tfrac{v}{\sqrt{\mu}} \sum_{i=1}^{k+1} \rho_0^{3^{i-1}} \leq \tfrac{v\rho_0}{\sqrt{\mu}(1-\rho_0)},$$

i.e., $u_{k+1} \in S$.

It is easy to show that

$$\|u_{n+m} - u_n\| \leq \tfrac{v\rho_0^{3^n}}{\sqrt{\mu}\left(1-\rho_0^{3^n}\right)}, \qquad n = 0, 1, \ldots, m \in \mathbb{N} \tag{2.16}$$

and, since $\rho_0 < 1$, it follows that the sequence $(u_n)_{n\geq 0}$ is Cauchy, so it converges. Denoting $\bar{u} = \lim u_n$, it is clear that $f(\bar{u}) = \theta$. Letting $m \to \infty$ in (2.16) leads us to jv. $\square$

The Chebyshev method may be applied with the aid of the following algorithm:

Let u_n be an arbitrary approximation of the solution of (2.1), and which satisfies the hypotheses of Theorem 1. The next approximation u_{n+1} may be obtained by

1. Solve the linear operator equation

$$f'(u_n)\,p_n = f(u_n)\,,$$

2. Solve the linear operator equation

$$f'(u_n)\,q_n = f''(u_n)\,p_n^2$$

3. Compute

$$u_{n+1} = u_n - p_n - \tfrac{1}{2}q_n.$$

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