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ON SOME AITKEN-STEFFENSEN-HALLEY-TYPE METHODS FOR APPROXIMATING THE ROOTS OF SCALAR EQUATIONS

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Abstract. In this note we extend the Aitken-Steffensen method to the Halley transformation. Under some rather simple assumptions we obtain error bounds for each iteration step; moreover, the convergence order of the iterates is 3, i.e. higher than for the Aitken-Steffensen case.

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1. INTRODUCTION

Let $f : [a,b] \to \mathbb{R}$, $a,b \in \mathbb{R}$, $a < b$ and suppose that $f \in C^4[a,b]$, and $f'(x) > 0$, $\forall x \in [a,b]$. Consider the function $h : [a,b] \to \mathbb{R}$ given by

$$h(x) = \frac{f(x)}{\sqrt{f'(x)}}.$$

As it was shown in [2], the Halley method for solving:

$$f(x) = 0, \tag{1.1}$$

is given by

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}, \qquad n = 0,1,\dots,\ x_0 \in [a,b]. \tag{1.2}$$

This sequence is in fact generated by the Newton method for solving $h(x) = 0$.

The first and second order derivatives of h are given by

$$h'(x) = \frac{2(f'(x))^2 - f''(x)\cdot f(x)}{2(f'(x))^{3/2}} \tag{1.3}$$

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$$h''(x) = \frac{3(f''(x))^2 - 2f'''(x)f'(x)}{4(f'(x))^{5/2}} \cdot f(x) \tag{1.4}$$

which yield the following equalities for a solution $\bar{x}$ of (1.1):

$$h'(\bar{x}) = \left(f'(\bar{x})\right)^{1/2}, \qquad \text{and} \tag{1.5}$$

$$h''(\bar{x}) = 0. \tag{1.6}$$

Relation (1.6) ensures the convergence order 3 for the sequence $(x_k)_{k\geq 0}$.

In the papers [2]–[8] and [12] there are studied the convergence and the convergence order of some sequences generated by some interpolatory methods applied to equation $h(x) = 0$.

We shall consider other two equations equivalent to (1.1) of the form:

$$x - \varphi_1(x) = 0, \qquad \text{and} \tag{1.7}$$

$$x - \varphi_2(x) = 0 \tag{1.8}$$

The Aitken method for solving $h(x) = 0$ is given by the iteration

$$x_{n+1} = \varphi_1(x_n) - \frac{h(\varphi_1(x_n))}{[\varphi_1(x_n), \varphi_2(x_n); h]}, \qquad n = 0, 1, \ldots,\ x_0 \in [a, b]. \tag{1.9}$$

In this note we shall study the convergence of these iterates. We shall show that the functions φ_1 and φ_2 may be chosen in order to obtain bilateral approximations at each iteration step; this fact allows the control of the errors. On the other hand, the convergence order of $(x_n)_{n\geq 0}$ given by (1.9) is at least equal to 3.

Hypotheses $f \in C^4[a, b]$ and $f'(x) > 0$, $\forall x \in [a, b]$ imply, taking into account (1.5), that there exist $\alpha, \beta \in \mathbb{R}$, $a \leq \alpha < \bar{x} < \beta \leq b$ such that $h'(x) > 0, \forall x \in [\alpha, \beta]$.

2. ERROR EVALUATION AND LOCAL CONVERGENCE

Consider the interval $[\alpha, \beta]$ given above. We shall make the following assumptions on φ_1 and φ_2:

i. the function $f \in C^4[a, b]$;
ii. equation (1.1) has the solution $\bar{x} \in [a, b]$;
iii. the inequality $f'(x) > 0$ holds for $x \in [\alpha, \beta]$;

iv. the function $\varphi_1$verifies the relation $0 < [x, y; \varphi_1] < 1$ for all $x, y \in [\alpha, \beta]$, where $[x, y; \varphi]$ denotes the first order divided difference of φ on x and y;
v. the function φ_2 verifies the relations $-1 < [x, y; \varphi_2] < 0$ for all $x, y \in [\alpha, \beta]$.

We can state the following result:

THEOREM 2.1. *Assume that* i–v *hold, and for some* $x_0 \in [\alpha, \beta]$ *sufficiently close to* $\bar{x}$ *we have* $\varphi_1(x_0), \varphi_2(x_0) \in [\alpha, \beta]$. *Then the following relations hold:*

j. *the sequences* $(x_n)_{n\geq 0}, (\varphi_1(x_n))_{n\geq 0}$ *and* $(\varphi_2(x_n))_{n\geq 0}$ *converge to* $\bar{x}$*;*
jj. *for any* $n = 0, 1, \ldots,$ *one has*

$$|\bar{x} - x_{n+1}| \leq \max\left\{ |x_{n+1} - \varphi_1(x_n)|, |x_{n+1} - \varphi_2(x_n)| \right\};$$

jjj. *there exists* $k \in \mathbb{R}$, $k > 0$,*which does not depend on* $n \in \mathbb{N}$ *such that*

$$|x_{n+1} - \bar{x}| \leq k\, |x_n - \bar{x}|^3, \qquad n = 0, 1, \ldots$$

Proof. By $\varphi_1(x_0), \varphi_2(x_0) \in [\alpha, \beta]$ it obviously follows that $h'(\varphi_1(x_0)) > 0$ and $h'(\varphi_2(x_0)) > 0$. Denote by I_0 the interval having the extremities $\varphi_1(x_0)$ and $\varphi_2(x_0)$. We notice, taking into account the mean formula, that

$$[\varphi_1(x_0), \varphi_2(x_0); h] > 0.$$

When $x_0 < \bar{x}$, by iv. and $\bar{x} = \varphi_1(\bar{x})$ it follows $\varphi_1(x_0) < \bar{x}$. Analogously, $\varphi_1(x_0) > \bar{x}$ for $x_0 > \bar{x}$. Taking into account v. and $\bar{x} = \varphi_2(\bar{x})$ we get $\varphi_2(x_0) > \bar{x}$ for $x_0 < \bar{x}$ and $\varphi_2(x_0) < \bar{x}$ for $x_0 > \bar{x}$. It is obvious that in both situations $\bar{x} \in I$. It can be easily seen that for all $n = 0, 1, \ldots$ we have

$$\varphi_1(x_n) - \tfrac{h(\varphi_1(x_n))}{[\varphi_1(x_n), \varphi_2(x_n); h]} = \varphi_2(x_n) - \tfrac{h(\varphi_2(x_n))}{[\varphi_1(x_n), \varphi_2(x_n); h]},$$

which, for $n = 0$ imply $x_1 > \varphi_1(x_0)$ and $x_1 < \varphi_2(x_0)$ if $x_0 < \bar{x}$, respectively $x_1 < \varphi_1(x_0)$ and $x_1 > \varphi_2(x_0)$ if $x_0 > \bar{x}$, i.e., $x_1 \in \operatorname{int} I_0$. It is clear now that, analogously, $x_1 < \varphi_1(x_1) < \bar{x} < \varphi_2(x_1)$ if $x_1 < \bar{x}$ or $x_1 > \varphi_1(x_1) > \bar{x} > \varphi_2(x_1)$ if $x_1 > \bar{x}$. Denoting by I_1 the interval determined by $\varphi_1(x_1)$ and $\varphi_2(x_1)$ then

$$I_1 \subset I_0,$$

and the element x_2 constructed by (1.9) satisfies $x_2, \bar{x} \in I_1$.

Repeating the above reason we get $x_{n+1}, \bar{x} \in I_n$, the interval being determined by $\varphi_1(x_n), \varphi_2(x_n)$, and also that

$$I_{n+1} \subset I_n.$$

and $\bar{x} \in I_{n+1}$. It is clear now that jj. holds. In order to obtain jjj. we shall use the identity

$$\begin{aligned} h(\bar{x}) =& h(\varphi_1(x_n)) + [\varphi_1(x_n), \varphi_2(x_n); h](\bar{x} - \varphi_1(x_n)) \\ &+ [\bar{x}, \varphi_1(x_n), \varphi_2(x_n); h](\bar{x} - \varphi_1(x_n))(\bar{x} - \varphi_2(x_n)), \end{aligned}$$

which, together with (1.9) and $h(\bar{x}) = 0$, imply

$$\bar{x} - x_{n+1} = -\tfrac{[\bar{x}, \varphi_1(x_n), \varphi_2(x_n); h]}{[\varphi_1(x_n), \varphi_2(x_n); h]}(\bar{x} - \varphi_1(x_n))(\bar{x} - \varphi_2(x_n)).$$

For the difference $\bar{x} - \varphi_1(x_n)$, by iv. one gets

$$\bar{x} - \varphi_1(x_n) = [\bar{x}, x_n; \varphi_1](\bar{x} - x_n),$$

i.e.,

$$\bar{x} - \varphi_1(x_n) < |\bar{x} - x_n|.$$

Analogously, by v. we get

$$|\bar{x} - \varphi_2(x_n)| < |\bar{x} - x_n|.$$

The mean formula for divided differences implies

$$\begin{aligned} [\bar{x}, \varphi_1(x_n), \varphi_2(x_n); h] &= \tfrac{1}{2} h''(\xi_n), \qquad \text{with } \xi_n \in I_n, \text{ and} \\ [\varphi_1(x_n), \varphi_2(x_n); h] &= h'(\eta_n), \qquad \eta_n \in I_n. \end{aligned}$$

For $h''(\xi_n)$ we have

$$\left|h''(\xi_n)\right| = \left|h''(\xi_n) - h''(\bar{x})\right| = \left|h'''(\theta_n)\right| |\bar{x} - \xi_n|.$$

Since $\xi_n \in I_n$ it follows

$$|\bar{x} - \xi_n| < |\bar{x} - x_n|.$$

Denoting $m_1 = \inf_{x \in [\alpha, \beta]} |h'(x)|$, $M_3 = \sup_{x \in [\alpha, \beta]} |h'''(x)|$, the above relations lead to

$$|\bar{x} - x_{n+1}| \le \tfrac{M_3}{2m_1} |\bar{x} - x_n|^3, \qquad n = 0, 1, \ldots,$$

i.e., jjj. for $k = \frac{M_3}{2m_1}$.

Since the initial approximation x_0 was supposed sufficiently close to the solution $\bar{x}$, then

$$\sqrt{\tfrac{M_3}{2m_1}}\,|\bar{x}-x_0|<1$$

implies, together with jjj., statement j. □

REMARK 2.2. Supposing that $f''(x)\geq 0$ for all $x\in[a,b]$, and if instead of iii. we assume that $f'(x)>0$, then obviously $f(x)<0$ for $a\leq x<\bar{x}$, and so in Theorem 2.1 we may take $\alpha=a$. From the above conditions it follows that $h'(x)>0$ for $x\in[a,\bar{x}]$, and since $f(\bar{x})=0$, one gets $\beta>\bar{x}$. □

3. DETERMINING THE FUNCTIONS φ_1 AND φ_2

Under reasonable hypotheses on f, we shall show that there exist two classes of functions among we can choose the functions φ_1 and φ_2 such that hypotheses iv. and v. to be satisfied.

Besides assumptions i.–iii. on f, we shall suppose that f is strictly convex, i.e., $f''(x)>0$, $\forall x\in[a,b]$. This condition implies that f' is increasing on $[a,b]$. If, moreover, $f'(x)<2\lambda$, with $0<\lambda\leq f'_r(a)$, then we may consider the functions

$$\varphi_1(x)=x-\frac{f(x)}{\mu} \quad \text{and} \tag{3.1}$$

$$\varphi_2(x)=x-\frac{f(x)}{\lambda} \tag{3.2}$$

where μ may be taken as any real number greater than the left derivative of f at b, $f'_l(b)$.

In the following we shall show that the functions $\varphi_1(x)$ and $\varphi_2(x)$ chosen above obey iv. and v. The derivatives of these functions are given by

$$\varphi_1'(x)=1-\frac{f'(x)}{\mu}, \qquad \text{resp. } \varphi_1'(x)=1-\frac{f'(x)}{\lambda}.$$

Obviously, $0\leq\varphi_1'(x)<1$. Also, the monotonicity of f' implies $\varphi_2'(x)<0$, while $f'(x)<2\lambda$ implies $-1<\varphi_2'(x)$.

Taking into account Remark 2.2, under the above hypotheses it is obvious that if $x_0<\bar{x}$, then condition $\varphi_1(x_0)\in[\alpha,\beta]$ from Theorem 2.1 is obviously satisfied. Indeed, this fact follows from $\varphi_1'(x)<1$, since $\varphi_1(x_0)-\bar{x}=\varphi_1(x_0)-\varphi_1(\bar{x})=\varphi_1'(\xi)(x_0-\bar{x})<0$, $x_0<\theta<\bar{x}$, i.e., $\varphi_1(x_0)<\bar{x}$.

On the other hand, $|\varphi_1(x_0)-\bar{x}|<|x_0-\bar{x}|$, and so the relations $x_0<\varphi_2(x_0)<\bar{x}$ hold. The hypothesis $\varphi_2(x_0)\in[\alpha,\beta]$ must be kept.

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