ON A STEFFENSEN TYPE METHOD

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Abstract

We study a general Steffensen type method based on the inverse interpolation Lagrange polynomial of second degree.

We show how the auxiliary functions may be constructed and we analyze some conditions on them which lead to monotone approximations. We obtain some local convergence results, which are illustrated by some numerical examples.

1 Introduction

As it is well known, the Steffensen method for approximating the solutions of equations is an interpolatory type method with controlled nodes [3], [4], [6], [7], [8].

More precisely, if we generate in the Lagrange polynomial of inverse interpolation of degree 1, the nodes of interpolation, in a particular way, we obtain one of the known variants of the Steffensen's method [1], [5], [11], [8], [13], [14].

Consider the equation

$$f(x) = 0 \tag{1}$$

where $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$. In a sufficiently general case, in order to obtain approximations of a root $\overline{x} \in [a, b]$ of equation (1), we shall consider another equation, equivalent to (1), of the form

$$x = g(x),\ g : [a, b] \to [a, b]. \tag{2}$$

Let $F = f([a, b])$ be the set of values of the function f for $x \in [a, b]$. We suppose that f is one to one, i.e. there exists $f^{-1} : F \to [a, b]$. We consider the interpolation nodes $a_i \in [a, b]$, $i = 1, 2$, and $b_i = f(a_i)$, $i = 1, 2$ the values of function f at these nodes. The Lagrange interpolation polynomial of first degree for the function f^{-1}, on the nodes b_i, $i = 1, 2$, has the form:

$$L_1(y) = a_1 + [b_1, b_2; f^{-1}](y - b_1). \tag{3}$$

Taking into account relation $\overline{x} = f^{-1}(0)$, for $y = 0$ from (3), we obtain an approximation of root $\overline{x}$, given by relation

$$\overline{x} \cong a_1 - [b_1, b_2; f^{-1}]b_1$$

from which, if we take into account equality

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]}, \tag{4}$$

we obtain:

$$\overline{x} \cong a_1 - \frac{f(a_1)}{[a_1, a_2; f]} \tag{5}$$

i.e. the regula falsi, which leads us to the chord method.

Let $x_n \in [a, b]$ $n \in \mathbb{N}^*$ be an approximation of root $\overline{x}$ of equation (1). If in (5) we take $a_1 = x_n$ and $a_2 = g(x_n)$, we obtain the Steffensen's method, which is written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]},\ n = 0, 1, ...,\ x_0 \in [a, b]. \tag{6}$$

In the present work we shall study a Steffensen type method, more general than (6), which will rely on the Lagrange polynomial of inverse interpolation of second degree. Accordingly, let $a_i \in [a, b],\ i = 1, 2, 3$ be three nodes of interpolation, and $b_i = f(a_i),\ i = 1, 2, 3$. The Lagrange interpolation polynomial of second degree, for the inverse function f^{-1}, has the form:

$$L_2(y) = a_1 + [b_1, b_2; f^{-1}](y - b_1) + [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2) \tag{7}$$

with equality

$$f^{-1}(y) = L_2(y) + [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3) \tag{8}$$

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for every $y \in F$.

For $y = 0$ from (7) and (8) we obtain:

$$\overline{x} = a_1 - [b_1, b_2; f^{-1}]b_1 + [b_1, b_2, b_3; f^{-1}]b_1 b_2 - [0, b_1, b_2, b_3; f^{-1}]b_1 b_2 b_3. \quad (9)$$

It is easy to see that the following equality takes place:

$$\left[b_1, b_2, b_3; f^{-1}\right] = -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}. \quad (10)$$

From (4), (9) and (10) we obtain for $\overline{x}$ the approximation

$$\overline{x} \simeq a_4 = a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f]f(a_1)f(a_2)}{[a_1, a_2; f][a_2, a_3; f][a_1, a_3; f]}, \quad (11)$$

with the error given by relation

$$\overline{x} - a_4 = -[0, b_1, b_2, b_3; f^{-1}]f(a_1)f(a_2)f(a_3). \quad (12)$$

Further on, we shall suppose that $f \in C^3[a, b]$ and $f'(x) \neq 0$ for every $x \in [a, b]$. Hence $f^{-1} \in C^3(F)$, and the following relation takes place

$$\left[f^{-1}(y)\right]''' = \frac{3[f''(x)]^2 - f'(x)f'''(x)}{[f'(x)]^5}, \quad (13)$$

where $y = f(x)$, [6], [7], [12], [15]. From the mean value theorem for divided differences, it results that there exists $\eta \in \operatorname{int}(F)$ so that

$$[0, b_1, b_2, b_3; f^{-1}] = \frac{[f^{-1}(y)]'''_{y=\eta}}{6}. \quad (14)$$

If we take into account that f is one to one and onto, it results that for $\eta \in \operatorname{int}(F)$, there exists $\xi \in]a, b[$, so that $\eta = f(\xi)$ and from (14) and (13) we have:

$$[0, b_1, b_2, b_3; f^{-1}] = \frac{3[f''(\xi)]^2 - f'(\xi)f'''(\xi)}{6[f'(\xi)]^5}. \quad (15)$$

Using as in (6) function g for the control of interpolation nodes, we shall obtain, from (11), a generalized method of Steffensen type.

Thus, let $x_n \in [a, b]$, $n \in \mathbb{N}^*$, an approximation of solution $\overline{x}$ of equation (1); then, we shall obtain approximation x_{n+1} from (11) considering $a_1 = x_n$, $a_2 = g(x_n)$, $a_3 = g(g(x_n))$, i.e.

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - \frac{[x_n, g(x_n), g(g(x_n)); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}, \quad n = 0, 1, ... \quad (16)$$

We shall name the method (16), the Steffensen's method of degree three.

For the study of convergence of sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ generated by (16), we shall analyze first the conditions on functions f and g, as well as on initial value $x_0 \in [a, b]$, so that the two considered sequences to be monotonically. This fact will give us the possibility to control the error at each iteration step. We also study the local convergence, and we shall show that, under certain assumptions, function g can be chosen to assure assumptions regarding monotonous convergence [2], [9], [10], [11].

2 The Convergence of Steffensen Method of Order Three

In the following we shall study the convergence of sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ given by (16).

We shall consider the following assumptions on the functions f and g:

(α_1) *equation* (1) *has a unique root* $\overline{x} \in]a, b[$;

(α_2) *equation* (2) *is equivalent to* (1);

(α_3) *function g is decreasing on* $[a, b]$;

(α_4) *there exists* $\ell \in \mathbb{R}$, $0 < \ell \leq 1$ *so that, for every* $x \in [a, b]$ *the following relation takes place:*

$$|g(x) - g(\overline{x})| \leq \ell\, |x - \overline{x}|\,; \quad (17)$$

(α_5) $f \in C^3([a, b])$;

(α_6) *function* $E_f : [a, b] \to \mathbb{R}$, *given by relation*:

$$E_f(x) = 3(f''(x))^2 - f'(x)f'''(x) \quad (18)$$

verifies condition $E_f(x) \leq 0$ *for every* $x \in [a, b]$.

The following theorems take place, depending on the properties of monotonicity and convexity of function f:

Theorem 1 *If functions* f, g *and initial value* $x_0 \in [a, b]$ *verify conditions:*

i$_1$. $g(x_0) \in [a, b]$;

ii$_1$. $f'(x) > 0$, *for every* $x \in [a, b]$;

iii$_1$. $f''(x) \geq 0$, *for every* $x \in [a, b]$;

iv$_1$. *function* f *and* g *verify assumptions* $\alpha_1) - \alpha_6)$,

then, the elements of sequences $(x_n)_{n\geq 0,}$ $(g(x_n))_{n\geq 0}$ *and* $(g(g(x_n)))_{n\geq 0}$ *generated by (16) remain in the interval* $[a, b]$ *and, moreover, following properties hold:*

j$_1$. *if* $f(x_0) < 0$, *then, for every* $n = 0, 1, ...$, *the following relations are verified:*

$$x_n \leq x_{n+1} \leq \overline{x} \leq g(x_{n+1}) \leq g(x_n); \quad (19)$$

jj$_1$. *if* $f(x_0) > 0$, *then, for every* $n = 0, 1, ...,$ *the following relations are verified:*

$$x_n \geq x_{n+1} \geq \overline{x} \geq g(x_{n+1}) \geq g(x_n); \qquad (20)$$

jjj$_1$. $\lim x_n = \lim g(x_n) = \overline{x}$;

jv$_1$. $|x_{n+1} - \overline{x}| \leq |x_{n+1} - g(x_n)|\,,\ n = 0, 1, ...,$

Theorem 2 *If* $x_0 \in [a, b]$ *and functions* f, g *verify conditions:*

i$_2$. $g(x_0) \in [a, b]$;

ii$_2$. $f'(x) < 0$, *for every* $x \in [a, b]$;

iii$_2$. $f''(x) < 0$, *for every* $x \in [a, b]$;

iv$_2$. *functions* f *and* g *verify conditions* $\alpha_1) - \alpha_6)$,

then, the elements of sequences $(x_n)_{n\geq 0}$, $(g(x_n))_{n\geq 0}$ *and* $(g(g)x_n)))_{n\geq 0}$ *generate by (16) remain in the interval* $[a, b]$ *and, moreover, the following properties hold:*

j$_2$. *if* $f(x_0) > 0$ *then, for every* $n = 0, 1, ...,$*relations (19) hold;*

jj$_2$. *if* $f(x_0) \leq 0$, *then for every* $n > 0, 1, ...,$*relations (20) hold;*

jjj$_2$. *relations from* **jjj**$_1$ *and* **jv**$_1$ *from Theorem 1 are verified.*

Theorem 3 *If* $x_0 \in [a, b]$ *and functions* f, g *verify conditions:*

i$_3$. $g(x_0) \in [a, b]$;

ii$_3$. $f'(x) < 0$, *for every* $x \in [a, b]$;

iii$_3$. $f''(x) \geq 0$, *for every* $x \in [a, b]$;

iv$_3$. *functions* f *and* g *verify assumptions* $\alpha_1) - \alpha_6)$,

then, the elements of sequences the $(x_n)_{n\geq 0}$, $(g(x_n))_{n\geq 0}$ *and* $(g(g)x_n)))_{n\geq 0}$ *generated by (16), remain in the interval* $[a, b]$ *and the following properties hold:*

j$_3$. *if* $f(x_0) > 0$, *then for every* $n = 0, 1, ...,$ *relations (19) hold;*

jj$_3$. *if* $f(x_0) < 0$, *the for every* $n = 0, 1, ...,$ *relations (20) hold.*

jjj$_3$. *statements* **jjj**$_1$ *and* **jv**$_1$ *of Theorem 1 hold.*

Theorem 4 *If* $x_0 \in [a, b]$ *and functions* f, g *verify conditions:*

i$_4$. $g(x_0) \in [a, b]$;

ii$_4$. $f'(x) > 0$, *for every* $x \in [a, b]$;

iii$_4$. $f''(x_0) \leq 0$, *for every* $x \in [a, b]$;

iv$_4$. *functions* f *and* g *verify assumptions* $\alpha_1) - \alpha_6)$,

then, the elements of sequences $(x_n)_{n\geq 0,}$ $(g(x_n))_{n\geq 0}$ *and* $(g(g(x_n)))_{n\geq 0}$, *generated by (16), remain in the interval* $[a, b]$, *and moreover, the following properties hold:*

j$_4$. *if* $f(x_0) < 0$ *then relations (19) hold;*

jj$_4$. *if* $f(x_0) > 0$ *then relations (20) hold;*

jjj$_4$. *statements* **jjj**$_1$ *and* **jv**$_1$ *of Theorem 1 hold.*

Proof. (Theorem 1). Let $x_n \in [a, b]$, $n \in \mathbb{N}^*$, for which $g(x_n) \in [a, b]$. Suppose that $f(x_n) < 0$, then, from **ii**$_1$, it results $x_n < \overline{x}$. Assumption $\alpha_3)$ leads us to relation $g(x_n) > \overline{x}$ and $g(g(x_n)) < \overline{x}$. From (17) we have:

$$|g(g(x_n)) - \overline{x}| \leq \ell\, |g(x_n) - \overline{x}| \leq \ell^2\, |x_n - \overline{x}|\,,$$

i.e.

$$x_n \leq g(g(x_n)) < \overline{x}.$$

It is obvious that the following relations take place:

$$a \leq x_n \leq g(g(x_n)) < \overline{x} < g(x_n) \leq b. \qquad (21)$$

Further on, in order to simplify writing, we shall denote by $D(x_n)$ the expression

$$D(x_n) = \\ = \frac{[x_n, g(x_n), g(g(x_n)); f]}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}$$

and then (16) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - D(x_n) f(x_n) f(g(x_n)), \qquad (22)$$

$n = 0, 1, ..., .$ From **ii**$_1$, **iii**$_1$ and (21) it follows that results $D(x_n) \geq 0$, $f(x_n) < 0$ and $f(g(x_n)) > 0$ which together with (22) lead us to relation $x_{n+1} > x_n$. By use of Newton's identity (9) we obtain, for the nodes considered in (16)

$$\overline{x} - x_{n+1} = \\ = -\,[0, f(x_n), f(g(x_n)), f(g(g(x_n))); f^{-1}]\cdot \\ \cdot f(x_n) f(g(x_n)) f(g(g(x_n))). \qquad (23)$$

From (15) and (21) it follows that there exists $\xi_n \in\,]x_n, g(x_n)[$ such that (23) can be represented in the form:

$$\overline{x} - x_{n+1} = -\frac{E_f(\xi_n) f(x_n) f(g(x_n)) f(g(g(x_n)))}{6[f'(\xi_n)]^5} \qquad (24)$$

Taking into account **ii**$_1$ and assumption $\alpha_6)$ from (24) we obtain $\overline{x} - x_{n+1} \geq 0$, i.e. $\overline{x} \geq x_{n+1}$. The last relation, together with α_3) lead us to relations $g(x_{n+1}) \geq \overline{x}$, and from $x_{n+1} > x_n$, we have $g(x_{n+1}) < g(x_n)$. From the facts proven above, it is obvious that relations (19) hold. If $f(x_n) > 0$, then we shall use identity

$$\frac{f(x_n)}{[x_n, g(x_n); f]} + D(x_n) f(x_n) f(g(x_n)) = \tag{25}$$
$$= \frac{f(x_n)}{[x_n, g(g(x_n)); f]} + D(x_n) f(x_n) f(g(g(x_n))).$$

As $f(x_n) > 0$ then, from $\mathbf{ii}_1$ and α_3) relations $x_n > \overline{x}$, $g(x_n) < \overline{x}$ imply the stated result.

From (17) it is easy to see that relation $g(g(x_n)) \geq x_n$ takes place, i.e. we have:

$$a \leq g(x_n) \leq \overline{x} \leq g(g(x_n)) \leq x_n \leq b.$$

From the above relation it results that $f(g(x_n)) < 0$; $f(x_n) > 0$ and $f(g(g(x_n))) > 0$ and from (25) and (16) it results $x_{n+1} < x_n$. From (24) for $\xi_n \in]g(x_n), x_n[$ it results $x_{n+1} \geq \overline{x}$ which, together with $g(x_{n+1}) > g(x_n)$ leads us to relations (20).

The consequence $\mathbf{jjj}_1$ is a result of relation (19) or (20). For $\mathbf{jv}_1$, from (19), respectively (20), it results that it exists $u = \lim x_n$. Getting to the limit for $n \to \infty$ in (16) we obtain $f(u) = 0$, and from assumption α_2) it results $u = g(u)$, i.e. $u = \overline{x}$, and $\lim g(x_n) = g(\overline{x}) = \overline{x}$. Thus, Theorem 1 is proven. ■

Proof. (Theorem 2). We notice that if instead of equation (1) we consider equation $-f(x) = 0$, then function $h : [a, b] \to \mathbb{R}$, $h(x) = -f(x)$ verifies all assumptions of Theorem 1 relating to f. If $f \in C^3([a, b])$, then $-f \in C^3([a, b])$ and α_5) hold. Also, if f verifies α_6) from relation $E_f(x) = E_{-f}(x)$ it results that $-f$ verifies too α_6).

Obviously, assumptions $\mathbf{ii}_1$ and $\mathbf{iii}_1$ are verified by $h = -f$. Also, relations (16) do not change if we replace f by $-f$. Taking into account Theorem 1, it is obvious that the consequences from Theorem 2 take place. ■

Proof. (Theorem 3). Let $x_n \in [a, b]$, $n \in \mathbb{N}^*$, for which $g(x_n) \in [a, b]$, an approximation of $\overline{x}$. Suppose $f(x_n) < 0$, then, obviously, form $\mathbf{ii}_3$ it results $x_n > \overline{x}$, $g(x_n) < \overline{x}$, and $g(g(x_n)) > \overline{x}$. From $\mathbf{ii}_3$, $\mathbf{iii}_3$, taking into account the last relations, and from (16) it results $x_{n+1} < x_n$. From (24), α_6), $\mathbf{ii}_3$ and $\mathbf{iii}_3$ it results $\overline{x} - x_{n+1} < 0$, i.e. $x_{n+1} > \overline{x}$. From $x_{n+1} < x_n$ and from α_3) it results $g(x_{n+1}) > g(x_n)$. Relation (20) results from the facts proven above. If $f(x_n) > 0$ then, obviously $x_n < \overline{x}$, $g(x_n) > \overline{x}$ and $g(g(x_n)) < \overline{x}$. By use of relation (25) from $\mathbf{ii}_3$, $\mathbf{iii}_3$ and (16) it results $x_{n+1} > x_n$ and $g(x_{n+1}) < g(x_n)$. From (24), α_6) and $\mathbf{iii}_3$ taking into account the last relations, it results $\overline{x} - x_{n+1} > 0$, i.e. $x_{n+1} < \overline{x}$, and thus relations (19) hold. Consequence $\mathbf{jjj}_3$ results from (19) and (20). ■

Proof. (Theorem 4.) We shall use the same reasoning as in the case of Theorem 2, i.e. we shall notice that if f verifies the assumptions of Theorem 4, then $h_1 : [a, b] \to \mathbb{R}$, $h_1(x) = -f(x)$ verifies the assumptions of Theorem 3. We notice that $E_f(x) = E_{-f}(x)$ for every $x \in [a, b]$ and thus α_6) is also verified for $-f$. Also, function h_1 verifies assumptions $\mathbf{ii}_3$ and $\mathbf{iii}_3$. It thus follows that the consequences of Theorem 3 take place, which imply the proof of Theorem 4. As one can notice, in every case, relation $\mathbf{jv}_1$. offers us a control of error at every step of iteration. ■

3 The Local Convergence of Steffensen's Method of Order Three

In the following, in order to point out the convergence order of method (16), we shall provide a result concerning the local convergence of this method. Accordingly, we shall admit that functions f and g verify the following assumptions:

β_1) *there exists* $m, M \in \mathbb{R}$, $m > 0$, $M > 0$, *so that* $m \leq |f'(x)| \leq M$, *for every* $x \in [a, b]$;

β_2) *there exists* $E > 0$, $E \in \mathbb{R}$ *so that* $|E_f(x)| \leq E$ *for every* $x \in [a, b]$.

The following theorem holds:

Theorem 5 *If functions f and g verify assumptions β_1), β_2), α_1), α_2), α_4) and for some $x_0 \in [a, b]$ the following relations are verified:*

$$\rho_0 = p\,|\overline{x} - x_0| < 1, \quad p = \tfrac{M\ell}{m^2}\left(\tfrac{EM\ell}{6m}\right)^{\frac{1}{2}}; \tag{26}$$

$$\delta = \left[\overline{x} - \tfrac{1}{p}, \overline{x} + \tfrac{1}{p}\right] \subseteq [a, b], \tag{27}$$

then, the elements of sequences $(x_n)_{n\geq 0}$, $(g(x_n))_n$ and $(g(g(x_n)))_{n\geq 0}$ remain in the interval $[a, b]$, and for every $n = 0, 1, ...$, the following relations hold:

$$|\overline{x} - x_{n+1}| \leq p^2\,|\overline{x} - x_n|^3;$$
$$|x_n - \overline{x}| \leq \tfrac{1}{p}\rho_0^{3^{n+1}}, \; n = 1, 2, ... \tag{28}$$

i.e. $\lim x_n = \lim g(x_n) = \lim g(g(x_n)) = \overline{x}$.

Proof. From α_4), α_2) and (26) it results $g(x_0) \in \delta$ and, more, $g(g(x_0)) \in \delta$. If we take account of (24), for $n = 0$, and of assumptions β_1), β_2), and α_4) the following relation results:

$$|\overline{x} - x_1| \leq \tfrac{EM^3\ell^3}{6m^5}\,|\overline{x} - x_0|^3 \leq p^2\,|\overline{x} - x_0|^3. \tag{29}$$

From (26), (27) and (29) it results $x_1 \in \delta$ and relation (28) for $n = 1$ holds. If we suppose that for $n \geq 1$, $x_n \in \delta$, then, from (24), proceeding as above, we deduce:

$$|\overline{x} - x_{n+1}| \leq p^2\,|\overline{x} - x_n|^3 \tag{30}$$

or

$$p\,|\overline{x}-x_{n+1}| \le (p\,|\overline{x}-x_n|)^3. \tag{31}$$

From (30) results $x_{n+1} \in \delta$, and from (31) results (28). Relations (26) and (28) imply $\lim x_n = \lim g(x_n) = \lim g(g(x_n)) = \overline{x}$. ■

Remark 6 *Relations (28) show that the q-convergence order of the method given by (16) is at least 3.*

4 Construction of Auxiliary Function g

Further on, we shall show that, within supplementary assumptions upon f, depending on its monotony and convexity, we can construct the functions g, which fulfill, respectively, the assumptions of Theorems 1–4. More precisely, the essential assumptions upon function g, are given by α_2), α_3), α_4) and $\mathbf{i}_1$ or, analogously, $\mathbf{i}_2$, $\mathbf{i}_3$, $\mathbf{i}_4$.

The following theorems take place:

Theorem 7 *If f verifies assumptions $\mathbf{ii}_1$ and $\mathbf{iii}_1$ of Theorem 1 and moreover, if it exists $\ell \in \mathbb{R}$, $0 < \ell \le 1$, so that $f'(b) \le (1+\ell) f'(a)$, then, function g given by relation*

$$g(x) = x - \lambda f(x) \tag{32}$$

where $\lambda \in \left[\frac{1}{f'(a)}, \frac{1+\ell}{f'(b)}\right]$ fulfills the conditions given by assumptions α_2), α_3) and α_4).

Proof. From $\mathbf{ii}_1$ and $\mathbf{iii}_1$ it results $f'(x) > 0$ and $f''(x) \ge 0$ for every $x \in [a, b]$. ■

It is obvious that function g given by (32) verifies α_2).

For α_3) and α_4) it is sufficient that function $g'(x)$ should verify relations

$$-\ell \le 1 - \lambda f'(x) < 0 \tag{33}$$

where $0 < \ell \le 1$.

From (33) $\mathbf{ii}_1$ and $\mathbf{iii}_1$ it results:

$$\frac{1}{f'(a)} \le \lambda < \frac{1+\ell}{f'(b)}. \tag{34}$$

It is not difficult to show that if λ verifies (34), then $g'(x)$ verifies

$$-\ell \le g'(x) < 0,$$

i.e. assumptions α_3) and α_4) are verified.

Theorem 8 *If assumptions $\mathbf{ii}_2$ and $\mathbf{iii}_2$ of Theorem 2 are fulfilled, and, moreover, if for an $\ell \in \mathbb{R}$, $0 < \ell \le 1$ relation $(1+\ell) f'(a) < f'(b)$, takes place, then, function g given by relation (32), where $\lambda \in]\frac{1+\ell}{f'(b)}, \frac{1\ell}{f'(a)}[$ verifies the conditions given by assumptions α_2), α_3) and α_4).*

Proof. From $\mathbf{ii}_2$ and $\mathbf{iii}_2$ it results $f'(x) < 0$ and $f''(x) \leqq 0$ for every $x \in [a, b]$. Let $h : [a, b] \to \mathbb{R}$ given by relation $h(x) = -f(x)$; then g has the form

$$g(x) = x + \lambda h(x).$$

From relation $h''(x) \ge 0$ it results that function h' is increasing, and moreover $h'(x) > 0$ for every $x \in [a, b]$. It is thus obvious that the following relations are valid $h'(a) \le h'(x) \le h'(b)$ i.e.

$$\frac{1}{h'(a)} \ge \frac{1}{h'(x)} \ge \frac{1}{h'(b)}. \tag{35}$$

Function g given by (32) verifies assumption α_2). For α_3)and α_4) the following relations are sufficient:

$$-\ell \le 1 + \lambda h'(x) < 0, \text{ for every } x \in [a, b], \tag{36}$$

where $0 < \ell \le 1$.

From (36) follows that

$$-1 - \ell \le \lambda h(x) < -1,$$

hence

$$\tfrac{1+\ell}{h'(x)} \ge -\lambda > \tfrac{1}{h'(x)} \text{ for every } x \in [a, b]. \tag{37}$$

From assumption $\lambda \in \left[\frac{1+\ell}{f'(b)}, \frac{1}{f'(a)}\right]$ follows

$$\tfrac{1}{h'(a)} < -\lambda < \tfrac{1+\ell}{h'(b)}. \tag{38}$$

From (35), (37) and (38) we deduce relations

$$\tfrac{1}{h'(x)} \le \tfrac{1}{h'(a)} < -\lambda < \tfrac{1+\ell}{h'(b)} \le \tfrac{1+\ell}{h'(x)}. \tag{39}$$

It is obvious that if λ verifies (38), from (39) it results that λ verifies (37), i.e. (36) takes place, and thus function g verifies α_2) and α_3). ■

Theorem 9 *If assumptions $\mathbf{ii}_3$ and $\mathbf{iii}_3$ of Theorem 3 are fulfilled, and, moreover, if for an $\ell \in \mathbb{R}$, $0 < \ell \le 1$, relation $(1+\ell) f'(b) < f'(a)$, takes place, then, function g, given by relation (32), where $\lambda \in \left[\frac{1+\ell}{f'(a)}, \frac{1}{f'(b)}\right]$, verifies assumptions α_2), α_3) and α_4).*

Proof. We consider again function $h : [a, b] + \mathbb{R}$, $h(x) = -f(x)$. From $\mathbf{ii}_3$ and $\mathbf{iii}_3$ results $h'(x) > 0$ and $h''(x) \le 0$ i.e. function $h'(x)$ is decreasing, and the following relations hold:

$$\tfrac{1}{h'(a)} \le \tfrac{1}{h'(x)} \le \tfrac{1}{h'(b)}, \tag{40}$$

for every $x \in [a, b]$. In order that g verifies α_3) and α_4), the following relations are sufficient:

$$\tfrac{1}{h'(x)} < -\lambda \le \tfrac{1+\ell}{h'(x)}. \tag{41}$$

If we take into account the substitution considered, and the assumption upon parameter λ from (40), we deduce that λ verifies (41). ■

Assumption $(1+\ell) f'(b) < f'(a)$ assures us that the set of values which λ could take is not a void one.

Theorem 10 *If function f verifies assumptions $\mathbf{ii}_4$ and $\mathbf{iii}_4$ of Theorem 4, and if, moreover, for an $\ell \in \mathbb{R}$, $0 < \ell \leq 1$, relation $f'(a) < (1+\ell)f'(b)$, takes place, then function g given by (32) where $\lambda \in]\frac{1}{f'(b)}, \frac{1+\ell}{f'(a)}[$, verifies assumptions $\alpha_2)$, $\alpha_3)$ and $\alpha_4)$.*

Proof. From $\mathbf{iii}_4$ it results that function f' is decreasing, i.e. the following relations take place:

$$f'(a) \geq f'(x) \geq f'(b), \tag{42}$$

for every $x \in [a, b]$.

From relation $-\ell \leq g'(x) < 0$, the following relations result, for λ :

$$\frac{1+\ell}{f'(x)} \geq \lambda > \frac{1}{f'(x)}. \tag{43}$$

From relations (42), (43) it results

$$\frac{1}{f'(x)} \leq \frac{1}{f'(b)} < \lambda \leq \frac{1+\ell}{f'(a)} \leq \frac{1+\ell}{f'(x)},$$

Which shows us that if $\lambda \in]\frac{1}{f'(b)}, \frac{1+\ell}{f'(a)}[$, then, relation (43) is verified, which assures us that assumptions $\alpha_3)$ and $\alpha_4)$ are verified. Relation $f'(a) < (1+\ell)f'(b)$ assures us that the set of values of λ is not empty. ■

5 Numerical Examples

Further on, we shall present two numerical examples, which illustrate some of the obtained results.

Example 1 *Let*

$$f(x) = e^x + 6x - 4 = 0 \tag{44}$$

for $x \in [0, 1]$. Because $f'(x) = e^x + 6 > 0$ and $f''(x) = e^x > 0$ for $x \in [0, 1]$, we construct function g in such a manner that Theorem 1 can be applied to this example.

Function E_f is given by relation

$$E_f(x) = 2e^x(e^x - 3).$$

It is clear that $E_f(x) < 0$ for every $x \in [0, 1]$. It is shown at once that if we take function g given by relation

$$g(x) = x - \tfrac{1}{6}f(x), \tag{45}$$

then assumptions $\mathbf{i}_1$, $\alpha_3)$, $\alpha_4)$ and α_5 upon function g are verified for $x_0 = 0$, and $\ell = \frac{e}{6}$ and thus $\lambda = \frac{1}{6}$ is an acceptable value.

If in (16) we consider functions f and g given by (44), respectively (45), then we obtain, for the root $\overline{x} \in (0, 1)$ of equation (44) the approximations given in Table 1.

Obviously, sequence $(x_n)_{n\geq 0}$, generated from (16) in the conditions of Theorem 1, verifies its conclusions, i.e. sequences $(x_n)_{n\geq 0}$ and $(g(g(x_n)))_{n\geq 0}$ are increasing, and sequence $(g(x_n))_{n\geq 0}$ is decreasing. From Table 1, by use of $\mathbf{jv}_1$ the following relation clearly results:

$$|x_2 - \overline{x}| < 10^{-14},$$

where $\overline{x}$ is the root of the given equation.

n	x_n	$g(x_n)$	$g(g(x_n))$
0	0	0.5	0.39187978821665
1	0.41440725449098	0.41442110496351	0.41441761121909
2	0.41441831498704	0.41441831498704	0.41441831498704

Table 1. Numerical results for $f(x) = e^x + 6x - 4$.

Example 2 *We consider equation:*

$$f(x) = xe^x + 4x + 4 = 0 \tag{46}$$

for $x \in [-1, 0]$. For the derivatives of order 1 and 2 of f, we have relations

$$f'(x) = (x+1)e^x + 4 > 0,\ x \in [-1, 0];$$
$$f''(x) = (x+2)e^x > 0,\ x \in [-1, 0].$$

Once more, we shall show that the Theorem 1 can be applied. It is easy to see that function $E_f(x)$ may be put in the form:

$$E_f(x) = e^x(x+3)\left[\tfrac{2x^2+8x+9}{x+3}e^x - 4\right].$$

An elementary reasoning leads us to conclusion $E_f(x) < 0$ for every $x \in [-1, 0]$.

We consider function g given by relation

$$g(x) = x - \tfrac{1}{5}f(x). \tag{47}$$

We conclude that all the assumptions of Theorem 1 are verified. By use of relations (16) we obtain the results from Table 2. In this case we notice that sequences $(x_n)_{n\geq 0}$ and $(g(g(x_n)))_{n\geq 0}$ are decreasing, and sequence $(g(x_n))_{n\geq 0}$ is increasing. For the error, we have relation:

$$|\overline{x} - x_2| < 10^{-14}.$$

n	x_n	$g(x_n)$	$g(g(x_n))$
0	0	-0.8	-0.8881073657412
1	-0.90850552567187	-0.90845262256514	-0.90844243232071
2	-0.90844000122266	-0.90844000122266	-0.90844000122266

Table 2. Numerical results for $f(x) = xe^x + 4x + 4$.

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