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ON SPLINE INTERPOLATION AT ALL INTEGER POINTS OF THE REAL AXIS

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Let (y_ν) $(-\infty < \nu < \infty,$ ν rational integer) be a doubly-infinite sequence of real or complex numbers. By a *cardinal interpolation problem* we mean the problem of constructing a function $F(x)$ $(x \in \mathbf{R})$ satisfying the relations

$$F(\nu) = y_\nu \text{ for all integer } \nu, \tag{1}$$

while F is to meet appropriate additional conditions specified beforehand. There are many cardinal interpolation problems depending on the additional conditions which are imposed. We refer to (1) as a cardinal interpolation problem because (1) is solved formally by the so-called *cardinal series*

$$F(x) = \sum_{-\infty}^{\infty} y_\nu \frac{\sin \pi(x-\nu)}{\pi(x-\nu)} \qquad \text{(See [8; Chap. 11]).} \tag{2}$$

The paper is divided into three parts. Our main results are described in Part 3 and concern certain cardinal interpolation problems. These results are based on those of a recent joint paper with M. GOLOMB [4]. This paper not being yet in print, it seemed indispensable to describe in Part 2 its main contents.

For motivation and background I discuss in Part 1 the formal solutions (by spline functions) of the problem (1) which were given in my old paper [5]. These were found useful during the war for numerical purposes. In Part 3 these formal solutions are characterized by certain extremum properties and their connection with the theory of entire functions of exponential type is uncovered. This connection may also be interpreted as a new summation method for the series (2) which is more powerful than existing methods. Being based on spline functions, we propose to call it the *spline summation* of the cardinal series. Part 3 is expository in the sense that no proofs are given; these will appear elsewhere. The paper concludes with a number of open problems and conjectures.

1. The spline solutions of the cardinal interpolation problem

In the present first part we discuss the interpolation problem (1) from the formal computational point of view of the paper [5]. The solutions there given will now be described, postponing to Part 3 a discussion of their analytic characterizations.

Let us assume for the moment that $y_\nu = 0$ if $|\nu| > N$, where N is very large. It follows that the series (2) is a finite sum which represents an entire function $F(x)$ satisfying (1). However, the series (2) is not convenient for numerical purposes because of the slow decay of the function

$$\frac{\sin \pi x}{\pi x} = O\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty. \tag{1.1}$$

This implies that the sum (2) will contain very many terms which can not be neglected. Moreover, an error in the value of y_ν will affect $F(x)$ even if the distance $|x - \nu|$ is large.

The interpolation method used in [5] proceeds as follows. We select a natural number m and denote by $S_m(x)$ $(x \in \mathbf{R})$ a function satisfying the following conditions:

$$S_m(x) \in \pi_{2m-1} \text{ in each interval } (\nu, \nu+1), \tag{1.2}$$

$$S_m(x) \in C^{2m-2}(\mathbf{R}), \tag{1.3}$$

$$S_n(\nu) = y_\nu \text{ for all } \nu. \tag{1.4}$$

Here and below π_k denotes the class of polynomials of degrees not exceeding k. In words: We interpolate the points (ν, y_ν) by a spline function $S_m(x)$ of degree $2m - 1$ having knots at all integer points of the real axis.

Thus, if $m = 1$, $S_1(x)$ is the piecewise linear function obtained by linear interpolation between consecutive points. Notice that $S_1(x)$ is uniquely defined. Matters are different if $m > 1$. Indeed, let us choose $P(x) \in \pi_{2m-1}$ such that $P(0) = y_0$ and $P(1) = y_1$, but otherwise arbitrary. $P(x)$ still depends on $2m - 2$ free parameters. We now define

$$S_m(x) = P(x) \text{ in the interval } (0, 1),$$

and extend its definition to all real x by setting

$$S_m(x) = P(x) + \sum_{i=1}^{\infty} a_i (x-i)_+^{2m-1} + \sum_{-\infty}^{j=0} b_j (j-x)_+^{2m-1},$$

where we use the function

$$x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

I claim that the coefficients a_i and b_j are uniquely defined by the interpolation requirements (1.4). For a_1 is uniquely defined by asking that $S_m(2) = y_2$, then a_2 by $S_m(3) = y_3$ etc. Likewise b_0 is given by $S_m(-1) = y_{-1}$, b_{-1} by $S_m(-2) = y_{-2}$ etc. This makes it abundently clear that the spline function $S_m(x)$ satisfying the conditions (1.2), (1.3) and (1.4) still depends on $2m - 2$ linear parameters.

Nevertheless, a useful spline interpolant $S_m(x)$ was constructed in [5; § 4.2] as follows: We start from the rectangular frequency function

$$M_1(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ 0 & \text{elsewhere} \end{cases}$$

and let

$$M(x) = \overbrace{M_1 * M_1 * \ldots * M_1}^{2m}(x) \tag{1.5}$$

be the frequency function obtained by convoluting $2m$ factors all of which are identical with $M_1(x)$. The Fourier transform of $M_1(x)$ being

$$\int_{-\infty}^{\infty} M_1(x) e^{-iux}\, dx = \frac{\sin (u/2)}{u/2},$$

we conclude that

$$\int_{-\infty}^{\infty} M(x)e^{-iux}\,dx = \psi(u), \tag{1.6}$$

where

$$\psi(u) = \left(\frac{\sin(u/2)}{u/2}\right)^{2m}. \tag{1.7}$$

Inverting (1.6) we obtain

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(u)e^{iux}\,du, \quad (-\infty < x < \infty). \tag{1.8}$$

It is easily shown in various ways that $M(x)$ satisfies the conditions (1.2) and (1.3). Moreover, $M(x) > 0$ in the interval $(-m, m)$, and $M(x) = 0$ in its complement. It was also shown in [5; Theorem 5 on page 72] that any $S_m(x)$ satisfying (1.2). and (1.3) may be represented uniquely in the form

$$S_m(x) = \sum_{-\infty}^{\infty} c_\nu M(x-\nu) \tag{1.9}$$

for appropriate values of the c_ν. Conversely, it is clear that the series (1.9) represents a function satisfying (1.2) and (1.3) whatever the values of the coefficients c_ν may be.

Let us now consider the ,,unit data''

$$y_\nu = \delta_\nu = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \neq 0, \end{cases}$$

and let us find a spline solution $L_m(x)$ of the ''unit'' interpolation problem

$$L_m(\nu) = \delta_\nu \text{ for all } \nu. \tag{1.10}$$

Such a spline function was given in [5; formula (9) on page 79, for $k = 2m$ and $t = 0$]. It is defined by the Fourier integral

$$L_m(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi(u)}{\Phi(u)}e^{iux}\,du, \quad (-\infty < x < \infty), \tag{1.11}$$

where $\psi(u)$ is defined by (1.7) and

$$\Phi(u) = \sum_{j=-\infty}^{\infty} \psi(u + 2\pi j). \tag{1.12}$$

By (1.12) and (1.7) $\Phi(u)$ is evidently a periodic function of period 2π which is positive for all real u.

We can readily see that $L_m(x)$ satisfies (1.2) and (1.3) as follows: We consider the Fourier expansion of the reciprocal of $\Phi(u)$

$$\frac{1}{\Phi(u)} = \sum_\nu c_\nu e^{-i\nu u}$$

and introduce it into (1.11). Interchanging the integration and summation symbols we obtain by (1.8)

$$L_m(x) = \sum_\nu c_\nu \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(u)e^{iu(x-\nu)}\,du = \sum_\nu c_\nu M(x-\nu)$$

which is a spline function of our class in view of the representation (1.9). That also (1.10) are satisfied is seen as follows: For integer $x = \nu$ (1.11) gives

$$L_m(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi(u)}{\Phi(u)}e^{i\nu u}\,du = \frac{1}{2\pi}\sum_j \int_{2\pi j}^{2\pi j+2\pi} \frac{\psi(u)}{\Phi(u)}e^{i\nu u}\,du$$

$$= \frac{1}{2\pi}\sum_j \int_0^{2\pi} \frac{\psi(u+2\pi j)}{\Phi(u)}e^{i\nu u}\,du = \frac{1}{2\pi}\int_0^{2\pi} \frac{\Phi(u)}{\Phi(u)}e^{i\nu u}\,du$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\nu u}\,du = \delta_\nu.$$

Finally, $L_m(x)$ being a solution of the ,,unit'' interpolation problem (1.10), it is clear that the series

$$S_m(x) = \sum_{-\infty}^{\infty} y_\nu L_m(x-\nu), \tag{1.13}$$

if convergent, will represent a function satisfying the conditions (1.2), (1.3) and (1.4)

One advantage of the interpolation formula (1.13) over the cardinal series (2) is due to the exponential decay of $L_m(x)$ as $|x| \to \infty$ (compare with (1.1) !). Another advantage of (1.13) is this: If

$$P(x) \in \pi_{2m-1} \quad \text{and} \quad y_\nu = P(\nu) \text{ for all } \nu,$$

then the series (1.13) converges and

$$S_m(x) = P(x) \text{ for all real } x.$$

For these reasons (1.13) was found useful for numerical applications.

Nevertheless, various pertinent questions are as yet unanswered. Here is one: We have seen above that the conditions (1.2), (1.3) and (1.4) do not determine the interpolating spline function $S_m(x)$ uniquely. *What additional properties characterize the particular interpolating spline function* $S_m(x)$, *defined by* (1.13), *among all other interpolating spline functions of degree* $2m-1$?

This, and other questions will be answered in Part 3.

2. The extension of functions and spline interpolation

Let A be a closed set of reals and let f be a function, or mapping, from A into the complex field $\mathbf{C}$. A function F from $\mathbf{R}$ into $\mathbf{C}$ is said to be an extension of f, provided that

$$F(x) = f(x) \text{ if } x \in A. \tag{2.1}$$

Worthwhile problems arise if we ask for conditions for the existence of extensions F belonging to some specified space of functions. In [4] the authors discuss the extension problem which requires that

$$F \in \mathcal{H}^m, \tag{2.2}$$

where

$$\mathcal{H}^m = \{F;\ F^{(m)} \in L_2(\mathbf{R})\},\ (m \text{ prescribed, } m \geq 1). \tag{2.3}$$

Alternatively, we may describe $\mathcal{H}^m$ as the class of functions F obtained as m-fold integrals of functions in $L_2(\mathbf{R})$.

The extension problem described by (2.1), (2.2), will be denoted by the symbol

$$\text{Ext. Prob. } (A, f, m). \tag{2.4}$$

Concerning it, Golomb and Schoenberg proposed the following three problems.

PROBLEMS. I. *To describe conditions which insure that* (2.4) *admits solutions.*

II. *If* (2.4) *admits solutions, to inquire into the existence and uniqueness of solutions* S, *of* (2.4), *such that*

$$\int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx \leqq \int_{-\infty}^{\infty} (F^{(m)}(x))^2 dx$$

for all solutions F *of* (2.4). *Such functions* S *are called optimal extensions of* f, *or optimal solutions of* (2.4).

III. *To give an intrinsic, or structural, characterization of the optimal solutions.*

The case when the set A *is finite.* We assume that

$$A = \{x_1, x_2, \ldots, x_n\},\quad x_1 < x_2 < \ldots x < x_n,\ n \geqq m, \tag{2.5}$$

and wish to point out that all three Problems I, II, III are for this case completely solved by known results concerning *spline interpolation*. We write as usual

$$f(x_i) = y_i,\ (i = 1, \ldots, n). \tag{2.6}$$

It is known that the optimal solution S of the extension (or interpolation) problem (2.4) is unique and uniquely characterized by the following properties (See e.g. [3; Theorems 1 and 2 on p. 158])

$$\begin{cases} 1'. & S \in \pi_{m-1} \text{ in the intervals } (-\infty, x_1) \text{ and } (x_n, +\infty), \\ 2'. & S \in \pi_{2m-1} \text{ in the intervals } (x_1,x_2), (x_2,x_3), \ldots, (x_{n-1},x_n), \\ 3'. & S \in C^{2m-2}(\mathbf{R}), \end{cases} \tag{2.7}$$

and

$$S(x_i) = y_i \quad (i = 1, \ldots, n). \tag{2.8}$$

A function S enjoying the properties (2.7) 1′, 2′, 3′ is called a *natural spline function* of degree $2m-1$ with knots x_i. We denote their class by the symbol

$$NS_m(A). \tag{2.9}$$

A moment's reflexion will show that the conditions (2.7) 1′, 2′, 3′ are equivalent with the conditions

$$(2.10)\quad \begin{cases} 1. & S \in \mathcal{H}^m \\ 2. & S \in \pi_{2m-1} \text{ in each of the intervals } (-\infty, x_1), (x_1, x_2), \ldots, (x_n, \infty), \\ 3. & S \in C^{2m-2}(\mathbf{R}). \end{cases}$$

This is so because the two simultaneous conditions $S \in \pi_{2m-1}$ and $S^{(m)} \in L_2$ $(-\infty, x_1)$ are equivalent with the condition $S \in \pi_{m-1}$ and similarly for the interval (x_n, ∞).

For the case of a *finite* set A we therefore conclude the following:

I. The problem (2.4) has always solutions.

II. The optimal solution S always exists and is unique.

III. The optimal solution S is characterized, besides the interpolatory conditions (2.8), by the structural properties (2.10) 1, 2, and 3.

For the interpolating natural spline function S (i.e. the optimal extension) the integral

$$(2.11)\qquad \int_{-\infty}^{\infty} (S^{(m)}(x))^2 dx$$

can be evaluated; it is represented by a Hermitian form in $n-m$ variables, whose coefficients depend on the set (2.5) and the number m, while the variables are the $n-m$ (consecutive) divided differences of order m of the n ordinates y_i (See [7, §2]). Thus, for $m = 1$, we find

$$\int_{-\infty}^{\infty} (S'(x))^2 dx = \sum_{1}^{n} \frac{|f(x_i) - f(x_{i-1})|^2}{x_i - x_{i-1}},$$

an expression which already appears in some early work of F. Riesz.

The case of a finite set A being disposed of, we shall now describe the solutions of Problems I, II, and III, as given in [4], for the case when

$$(2.12)\qquad A \text{ is an infinite closed set of reals.}$$

THEOREM I (Golomb-Schoenberg). *Assuming* (2.12), *the problem* (2.4) *has solutions if and only if the following condition is satisfied: Let*

$$\Delta = \{x_1, x_2, \ldots, x_n\} \subset A, \ (x_i \textit{ distinct}, \ n \geq m),$$

and let $S_\Delta(x)$ *denote the natural spline function of degree* $2m-1$ *which interpolates* f *at the* n *points of* Δ. *Then there should exist a constant* $K = K_f$, *independent of* Δ, *such that*

$$(2.13)\qquad \int_{-\infty}^{\infty} (S_\Delta^{(m)}(x))^2 dx \leq K^2.$$

THEOREM II. (Golomb-Schoenberg). *If the condition* (2.13) *is satisfied then the problem* (2.4) *admits a unique optimal extension* S.

The solution of Problem III requires two preliminary definitions. The first definition describes, for a fixed set A and all possible (or admissible) f, the class of optimal extensions which is to be characterized.

Definition 1. *Let* A *be fixed and such that* (2.12) *holds. For an arbitrary* $F \in \mathcal{H}^m$ *we define its restriction to* A

$$f(x) = f_F(x) = F(x) \textit{ if } x \in A.$$

Evidently this f *admits extensions in* $\mathcal{H}^m$, *e.g.* F. *By Theorem* II *it has a unique optimal extension* $S = S_F$, *and we consider the class of all these extensions which we denote by the symbol*

$$(2.14)\qquad \mathcal{S}_m(A) = \{S_F;\ \text{for all } F \in \mathcal{H}^m\}.$$

Problem III *asks for a characterization of this class.*

Definition 2. *Let* A *be fixed and such that* (2.12) *holds. A function* $S(x)$ $(x \in \mathbf{R})$ *is called a natural spline function of degree* $2m-1$ *knotted on the set* A, *provided that it satisfies the following conditions:*

$$(2.15)\quad \begin{cases} 1. & S \in \mathcal{H}^m, \\ 2. & S \in \pi_{2m-1} \text{ in every open interval } I \text{ such that } A \cap I = \emptyset, \\ 3. & S \in C^{2m-2}(J) \text{ in every open interval } J \text{ such that } A' \cap J = \emptyset, \end{cases}$$

where A' *is the derived set of* A.

We denote by the symbol $NS_m(A)$ *the entire class of functions satisfying the conditions* (2.15), 1, 2, *and* 3.

A solution of Problem III is given by the following

THEOREM III (Golomb-Schoenberg).

$$(2.16)\qquad \mathcal{S}_m(A) = NS_m(A).$$

In words: A solution S of the problem (2.4) *is an optimal extension if and only if it is a natural spline function of degree $2m - 1$ knotted on the set A.*

In Definitions 1 and 2 and Theorem III we have assumed that the set A is infinite. However, if A is a finite set of n points, $n \geq m$, then the results remain valid, because Definition 2 is then easily seen to define the class of ordinary natural spline functions of degree $2m - 1$ having as knots the n points of A. This follows from the fact that $A' = \emptyset$.

3. The case when A is the set of all rational integers

For the remainder of this paper we discuss the problem (2.4) for the special case when

$$A = Z = \{\nu ;\ \nu \text{ rational integer}\}. \tag{3.1}$$

As in (2.6), we change notation by writing $f(\nu) = y_\nu$ so that our "data" is a sequence of numbers

$$(y_\nu),\ (-\infty < \nu < \infty). \tag{3.2}$$

The problem (2.4) now becomes

$$\text{Ext. Prob. } (Z,\ (y_\nu),\ m). \tag{3.3}$$

This is precisely the interpolation problem (1) of our Introduction, *with the added restriction that the interpolating functions, or extensions, should belong to $\mathcal{H}^m$*. We may therefore apply all results of the general theory of Part 2 to this special case.

In the present case the general existence Theorem I simplifies considerably. From the explicit expression of the integral (2.11) as a Hermitian form it is now easy to derive

THEOREM 1. *The problem* (3.3) *has solutions in $\mathcal{H}^m$ if and only if*

$$\sum_{\nu=-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty. \tag{3.4}$$

Let us assume that the series (3.4) converges. By Theorem II we are assured of the existence of a unique optimal extension S. Moreover, Definition 2 and Theorem III allow to characterize S by structural properties. The characteristic properties (2.15) are fully used in our case (3.1) if in Condition (2.15) 2 we select

$$I = (\nu,\ \nu + 1) \text{ for all integers } \nu.$$

Likewise, observing that $A' = Z' = \emptyset$, we may select in Condition (2.15) 3 the single open interval $J = \mathbf{R}$. This establishes

THEOREM 2. *Let* (3.4) *hold. Among all spline functions of degree $2m-1$, with knots at all integers, and which interpolate the sequence (y_ν), there is exactly one, which we call S_m, which is in $\mathcal{H}^m$, i.e.*

$$S_m^{(m)} \in L_2(\mathbf{R}).$$

This particular interpolating spline function S_m is the optimal solution of the problem (3.3).

Theorems 1 and 2 were announced in [6; Theorem 7, page 27].

Let us now return to the function $L_m(x)$ defined by (1.11). We have already shown in the Introdution that $L_m(x)$ is a spline function of degree $2m - 1$ with knots at the integers. On the other hand $\psi(u) = O(u^{-2m})$, by (1.7). Now (1.11) implies that

$$L_m^{(m)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi(u)}{\Phi(u)} (iu)^m e^{iux}\, du \in L_2(\mathbf{R}),$$

as being the Fourier transform of a function in $L_2(\mathbf{R})$. By (1.10) and Theorem 2 we conclude that *$L_m(x)$ is the optimal extension of the sequence (δ_ν)*. From this it is easy to derive the following general result.

THEOREM 3. *We assume* (3.4) *to hold. The optimal extension S_m of Theorem 2 is given by the series*

$$S_m(x) = \sum_{-\infty}^{\infty} y_\nu L_m(x - \nu), \tag{3.5}$$

which converges locally uniformly on the real axis.

These results answer the question raised at the end of Part 1. Assuming (3.4), they also show that the interpolation formula (3.5) furnishes the optimal solution in $\mathcal{H}^m$ for the cardinal interpolation problem (1).

Further problems arise from the following remark. Let (3.4) hold and let p be a positive integer. By Cauchy's inequality we obtain

$$|\Delta^{m+p} y_\nu|^2 = \left| \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^m y_{\nu+j} \right|^2 \leq \sum_{j=0}^{p} \binom{p}{j}^2 \cdot \sum_{j=0}^{p} |\Delta^m y_{\nu+j}|^2,$$

whence

$$|\Delta^{m+p} y_\nu|^2 \leq \binom{2p}{p} \sum_{j=0}^{p} |\Delta^m y_{\nu+j}|^2.$$

Summing these inequalities for all integers ν we obtain

(3.6) $$\sum_{\nu} |\Delta^{m+p} y_\nu|^2 \leq (p+1) \binom{2p}{p} \sum_{\nu} |\Delta^m y_\nu|^2.$$

This inequality shows that if (3.4) holds for a value $m = k$ (≥ 0), then (3.4) also holds for all $m \geq k$.

By Theorems 1 and 2 we obtain the

Corollary 1. *If* (3.4) *holds for a value* $m = k \geq 0$, *then the spline function*

(3.7) $$S_m(x) \in \mathcal{H}^m$$

such that

(3.8) $$S_m(\nu) = y_\nu \text{ for all integers } \nu,$$

exists for all $m \geq \max(k, 1)$.

This raises the following new question:

PROBLEM 1. *What happens to* $S_m(x)$ *as we let* $m \to \infty$?

The remainder of this paper will describe the answer to this question.

We need two definitions.

Definition 3. 1. *For an integer* $k \geq 0$ *we consider the class of sequence*

(3.9) $$l_k^2 = \left\{ (y_\nu)\,;\, \sum_{-\infty}^{\infty} |\Delta^k y_\nu|^2 < \infty \right\}.$$

2. *For an integer* $k \geq 0$ *we consider the class of entire functions of a complex variable*

(3.10) $$\mathcal{PW}_k^\pi = \{F(x)\,;\, F(x) \text{ entire of exponential type} \leq \pi \text{ and such that for real } x,\ F^{(k)}(x) \in L_2(\mathbf{R})\}.$$

The symbol $\mathcal{PW}$ refers to Paley and Wiener, since they discovered the characteristic representation of the elements of the class $\mathcal{PW}_0^\pi$ (see e.g. [1; 103]). Evidently, the inequality (3.6) implies the inclusions

$$l_0^2 \subset l_1^2 \subset \ldots \subset l_k^2 \subset \ldots .$$

Likewise the Paley-Wiener theorem easily shows that

$$\mathcal{PW}_0^\pi \subset \mathcal{PW}_1^\pi \subset \cdots \subset \mathcal{PW}_k^\pi \subset \cdots .$$

The relation between the classes $\mathcal{PW}_k^\pi$ and l_k^2 is described by the following theorem.

THEOREM 4. *If*

(3.11) $$F(x) \in \mathcal{PW}_k^\pi$$

and if we write

(3.12) $$F(\nu) = y_\nu, \quad (\nu \in Z),$$

then

(3.13) $$(y_\nu) \in l_k^2.$$

Conversely, if (y_ν) *is a sequence such that* (3.13) *holds, then there exists a unique function* $F(x)$ *satisfying* (3.11) *and* (3.12).

We may summarize this theorem by saying that there is a one-to-one correspondence between the two classes

$$\mathcal{PW}_k^\pi \text{ and } l_k^2$$

which is defined by the relations (3.12).

The connection of Theorem 4 with spline functions is as follows. Let $(y_\nu) \in l_k^2$. It follows that $(y_\nu) \in l_m^2$ for all values of m such that

(3.14) $$m \geq \max(k, 1).$$

By Corollary 1 we conclude the existence of the spline functions

$$S_m(x) \in \mathcal{H}^m$$

interpolating the sequence (y_ν) for all values of m satisfying (3.14). This sequence of spline functions enjoys the following property.

THEOREM 5. *Let* $F(x)$ *be the unique element of* $\mathcal{PW}_k^\pi$ *satisfying* (3.12). *Then*

(3.15) $$\lim_{m \to \infty} S_m(x) = F(x),$$

locally uniformly for all real x. If $k \geqq 1$ also the relations

$$\lim_{m\to\infty} S_m^{(\nu)}(x) = F^{(\nu)}(x) \quad (\nu = 0, 1, \ldots, k-1)$$

hold locally uniformly for real x, while

$$\lim_{m\to\infty} S_m^{(k)}(x) = F^{(k)}(x)$$

holds uniformly for all real x.

This, then is the answer to Problem 1. Originally, I establish Theorem 5 first and afterwards derived from it Theorem 4. Very recently Richard A. Askey found an elegant direct proof of Theorem 4. Thereby Theorem 4 can be used in establishing Theorem 5 thereby greatly simplifying its proof.

An example. The sequence $(y_\nu) = (\delta_\nu)$ satisfies the condition of the definition (3.9) with $k = 0$, i.e. $(\delta_\nu) \in l_0^2$. The corresponding interpolating function $F(x)$ (Theorem 4) is evidently

$$F(x) = \frac{\sin \pi x}{\pi x} \in \mathcal{PW}_0^\pi.$$

On the other hand we know by Theorem 3 that

$$S_m(x) = L_m(x), \qquad (m \geqq 1)$$

is the spline interpolant of the sequence (δ_ν). By Theorem 5 we now conclude that the relation

$$\lim_{m\to\infty} L_m(x) = \frac{\sin \pi x}{\pi x} \tag{3.16}$$

holds uniformly for all real x.

The relation (3.16) implies that formally (or termwise)

$$\lim_{m\to\infty} \sum_\nu y_\nu L_m(x-\nu) = \sum_\nu y_\nu \frac{\sin \pi(x-\nu)}{\pi(x-\nu)},$$

where the series on the right hand side is usually divergent. However, Theorem 4 and particularly the relation (3.15) of Theorem 5, suggest the following summation method:

Let

$$(y_\nu) \in l_k^2, \text{ for some } k \geqq 0. \tag{3.17}$$

We define the (S) *sum* of the cardinal series by

$$(S) \sum_\nu (y_\nu) \frac{\sin \pi(x-\nu)}{\pi(x-\nu)} = F(x). \tag{3.18}$$

where $F(x)$ is the unique element of $\mathcal{PW}_k^\pi$ (Theorem 4) such that

$$F(\nu) = y_\nu \text{ for all integer } \nu. \tag{3.19}$$

Constructively, we can define $F(x)$, for real x, from Theorem 5 by

$$\lim_{m\to\infty} S_m(x) = F(x), \tag{3.20}$$

where $S_m(x)$ is the spline function of degree $2m-1$ which interpolates the sequence (y_ν).

If we substitute (3.19) into (3.18) we obtain the identity

$$F(x) = (S) \sum_\nu F(\nu) \frac{\sin \pi(x-\nu)}{\pi(x-\nu)}, \tag{3.21}$$

which is valid for any $F(x)$ belonging to the class

$$\mathcal{PW}_*^\pi = \bigcup_{k=0}^\infty \mathcal{PW}_k^\pi,$$

in particular for any polynomial.

This summation method may be called the *spline summation of the cardinal series.* The relationship with previous methods of summing the cardinal series (See [8, §11]) should be discussed, but we shall not do it here.

Open problems and conjectures. All these refer to the subjects of Part 3. Further questions might occur to the reader.

1. In what sense does the relation (3.15) of Theorem 5 hold also for complex values of x? $F(x)$ is an entire function while $S_m(x)$ is only defined on the real axis where it is piecewise polynomial. On the basis of his experience (unpublished) with a somewhat similar situation concerning the approximation by spline function of solutions of analytic differential equations, the author conjectures the following: *Let $P_{m,\nu}(x)$ denote the*

polynomial of degree $2m - 1$ *which represents* $S_m(x)$ *in the interval* $(\nu, \nu + 1)$, *then*

$$\lim_{m \to \infty} P_{m,\nu}(x) = F(x), \tag{3.22}$$

locally uniformly in the complex plane.

2. We may also consider a cardinal interpolation problem when a certain fixed number of derivatives are also preassigned. The simplest such cardinal *Hermite* interpolation problem is

$$F(\nu) = y_\nu,\ F'(\nu) = y'_\nu, \text{ for all integer } \nu, \tag{3.23}$$

which depends on the pair of sequences

$$Y_2 = \{(y_\nu),\ (y'_\nu)\}. \tag{3.24}$$

Connections with the theory of functions are again likely because of an analogue of the cardinal series which is easily found to be

$$F(x) = \sum_{-\infty}^{\infty} y_\nu C_0(x - \nu) + \sum_{-\infty}^{\infty} y'_\nu C_1(x - \nu) \tag{3.25}$$

where

$$C_0(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2, \quad C_1(x) = \frac{(\sin \pi x)^2}{\pi^2 x}. \tag{3.26}$$

Again we may ask the question: *Let* $m \geqq 2$; *under what conditions does the problem* (3.23) *admit solutions* $F \in \mathcal{H}^m$?

These conditions are expected to be as follows: We regard all integer nodes to be *double* nodes. If we write them consecutively in a row we obtain the infinite array

$$\ldots, -1, -1, 0, 0, \ldots, \nu, \nu, \nu + 1, \nu + 1, \ldots. \tag{3.27}$$

We select from this sequence all sets of $m + 1$ consecutive elements and we form the divided difference of order m (with single and double nodes) for each of these sets and computed by means of the data (3.24). Let Σ_m denote the sum of the squares of the moduli of all these divided differences. Thus for $m = 2$ we obtain

$$\Sigma_2 = \sum_{-\infty}^{\infty} (|[y_\nu, y_\nu, y_{\nu+1}]|^2 + |[y_\nu, y_{\nu+1}, y_{\nu+1}]|^2). \tag{3.28}$$

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I expect that (3.23) has a solution $F \in \mathcal{H}^m (m \geqq 2)$ if and only if

$$\Sigma_m < \infty. \tag{3.29}$$

Also that the *optimal* solutions, i.e. those which minimize

$$\int_{-\infty}^{\infty} |F^{(m)}|^2 \, dx,$$

will be spline functions $S_m(x)$ of degree $2m - 1$ having *double* knots at all integers. This means that we are now lowering our continuity requirements by asking that

$$S_m(x) \in C^{2m-3}(\mathbf{R}).$$

Let us look for a moment at the case of the lowest possible value of m, namely $m = 2$. Now $S_2(x)$ is the cubic spline of class $C^1(\mathbf{R})$ which satisfies (3.23). For this case of the lowest value of m, the problem of constructing $S_2(x)$ breaks up into a sequence of elementary interpolation problems: $S_2(x)$ is identical in the interval $(\nu, \nu + 1)$ with the cubic defined by the four data

$$S_2(\nu) = y_\nu, \qquad S_2(\nu + 1) = y_{\nu+1}$$

$$S'_2(\nu) = y'_\nu, \qquad S'_2(\nu + 1) = y'_{\nu+1}.$$

When is this spline function $S_2(x) \in \mathcal{H}^2$? We apply the conjectured condition (3.29): Evaluating the divided differences appearing in (3.28) we obtain the condition

$$\Sigma_2 = \sum_{-\infty}^{\infty} (|y'_\nu - \Delta y_\nu|^2 + |y'_{\nu+1} - \Delta y_\nu|^2) < \infty. \tag{3.30}$$

It is fairly easy to verify directly that the cubic spline $S_2(x)$ is in $\mathcal{H}^2$ if and only if (3.30) holds.

Also the relation between the interpolating spline functions and the cardinal series, as $m \to \infty$, will very likely generalize. As in the case of simple nodes, we observe that if (3.29) holds for a value of m (even the value $m = 1$ is acceptable), then it will hold for all larger values of m.

Let l_k^2 denote the class of pairs of sequences (3.24) such that the condition

$$\Sigma_k < \infty \qquad (k \geqq 1),$$

holds. Furthermore, let $\mathcal{PW}_k^{2\pi}$ be the class of entire functions $F(x)$ of exponential type $\leqq 2\pi$ such that

$$F^{(k)}(x) \in L_2(\mathbf{R}), \qquad (k \geqq 1)$$

Then we expect that there is a one-to-one correspondence between the classes

$$\mathcal{PW}_k^{2\pi} \quad \text{and } l_k^2$$

which is defined by the relations (3.23). Furthermore, that if the pair (3.24) is in l_k^2 and $S_m(x)$ is the interpolating spline function of degree $2m-1$ $(m \geqq \max(k, 2))$, then

$$\lim_{m\to\infty} S_m(x) = F(x), \quad (x \in \mathbf{R}),$$

where F is the corresponding element in $\mathcal{PW}_k^{2\pi}$.

Finally, that the conjectures just stated for (3.23) should generalize to the cardinal Hermite problem

$$F(\nu) = y_\nu,\ F'(\nu) = y'_\nu, \ldots, F^{(r-1)}(\nu) = y_\nu^{(r-1)}, \text{ for all } \nu.$$

The critical exponential type for this case should be $r\pi$.

3. An entirely different cardinal interpolation problem (1) was discussed some ten years ago by B. EPSTEIN, D. S. GREENSTEIN and J. MINKER in [2]. Let $\sigma > 0$ and let H denote the Hilbert space of functions $F(z)$ analytic in the strip $D_\sigma : |I_m z| < \sigma$, and such that

$$\iint_{D_\sigma} |F(x+iy)|^2 \, dx\, dy < \infty. \tag{3.27}$$

They show that the interpolation problem (1) has solutions in H if and only if

$$\sum_{-\infty}^{\infty} |y_\nu|^2 < \infty,$$

and determine the unique solution which minimizes the norm defined by the left side of (3.27).

Our discussion in Part 3 suggests that it might be worthwhile to study the interpolation problem (1) within the class H^m of functions $F(z)$ such that

$$F^{(m)}(z) \in H,$$

and in particular, to seek solutions of (1), within H^m, which minimize the integral

$$\iint_{D_\sigma} |F^{(m)}(z)|^2 \, dx\, dy.$$

The solutions of this problem might even converge to our spline interpolant $S_m(x)$ of Theorem 2 as we let the width $\sigma \to 0+$.

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Added in proof, April 1968: Here are a few important early references concerning spline functions that have more recently come to my attention:

[A] Runge C., *Theorie und Praxis der Reihen,* Sammlung Schubert XXXII, Leipzig, 1904.

[B] Quade W. und Collatz L., *Zur Interpolationstheorie der reellen periodischen Funktionen,* Sitzungsberichte der Preussischen Akad. der Wiss., Phys.-Math. Kl. XXX, 383–429 (1938).

[C] Popoviciu T., *Notes sur les fonctions convexes d'ordre supérieur* (IX), Bull. Math. de la Société Roumaine des Sciences, **43**, 95–141 (1941).

In [A], pages 192–196, Runge discusses the interpolation of periodic functions by periodic spline functions with equidistant knots for the purpose of obtaining improved values for the Fourier coefficients of the given function whose values are known only at the knots. In [B] the authors greatly elaborate Runge's idea and for the same purpose as Runge. In the process they derive and anticipate many results concerning spline interpolation of periodic functions by periodic spline functions with equidistant knots, including an analysis of the order of approximation so obtained. Finally, in [C], pages 96–105, Popoviciu uses spline functions directly for the purpose for which they are so emminently suited: the approximation of functions. He introduces spline functions of degree n with arbitrary knots, which he calls elementary function of order n. In particular he shows that a continuous non-concave function of order n in a finite interval $[a,b]$ is the uniform limit of elementary functions of order n that are also non-concave of order n in $[a, b]$ [C, Théorème 6, 96].

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AN ITERATIVE METHOD OF SOLVING LINEAR EQUATIONS WITH UNBOUNDED INVERTIBLE OPERATORS IN A SEPARABLE HILBERT SPACE

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The author seeks to investigate in this note an approximate solution of the linear equation $Au = f$, where A is an unbounded linear operator whose domain is dense in a separable Hilbert space and which possesses an inverse. This method is essentially an extension of the method of orthogonal projection, studied by the author (SEN, 1965) for the case of a bounded linear operator having inverse, in a separable Hilbert space.

Compared to the well-known methods of solution in Hilbert space like Ritz method, method of least squares (MIKHLIN, 1950), method of Galerkin (MIKHLIN, 1950), method of steepest descent (KANTOROVITCH,, 1948), this method has somewhat wider applicability because it does not presuppose the positive-definite or k.p.d.-property (PETRYSHYN, 1962) of the class of operators in question. This method is iterative in nature and can solve infinite-dimensional matrix equations in l_2.

Let us consider a separable Hilbert space H. Let, $Au = f$ be an equation in it, such that,

i) the domain D_A of A is dense in H;
ii) f is a given element of H;
iii) A is linear (additive and homogeneous);
and iv) A has an inverse.

Let us suppose that there exists a bounded self-adjoint linear operator, $\mathcal{B}$ such that,

i) the domain of $\mathcal{B} \supseteq$ the domain of A;