GENERALIZAREA FORMULEI DE CUADRATURĂ A LUI N. OBRESCHKOFF LA INTEGRALE DUBLE

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Se ştie că dacă funcţia $f(x)$ este de clasa C^{p+q} pe intervalul $[x_1, x_2]$, atunci avem formula de cuadratură

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{C_p^1}{C_{p+q}^1}\frac{x_2-x_1}{1!}f(x_2) - \frac{C_p^2}{C_{p+q}^2}\frac{(x_2-x_1)^2}{2!}f'(x_2) + \dots +$$

$$+ \dots + (-1)^{p-1}\frac{C_p^p}{C_{p+q}^p}\frac{(x_2-x_1)^p}{p!}f^{(p-1)}(x_2) +$$

$$+ \frac{C_q^1}{C_{p+q}^1}\frac{(x_2-x_1)}{2!}f(x_1) + \frac{C_q^2}{C_{p+q}^2}\frac{(x_2-x_1)^2}{2!}f'(x_1) + \dots + \frac{C_q^q}{C_{p+q}^q}\frac{(x_2-x_1)^q}{q!}f^{(q-1)}(x_1) +$$

$$+ \frac{(-1)^{p+q}}{(p+q)!}\int_{x_1}^{x_2}(x-x_1)^p(x-x_2)^q f^{(p+q)}(x)\,dx, \tag{1}$$

numită formula lui N. O b r e s c h k o f f [1]. În cazul particular $p = q$, formula se numeşte formula lui K. Petr [2]. Ea are forma

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{C_p^1}{C_{2p}^1}\frac{x_2-x_1}{1!}[f(x_1)+f(x_2)] + \frac{C_p^2}{C_{2p}^2}\frac{(x_2-x_1)^2}{2!}[f'(x_1)-f'(x_2)] + \dots +$$

$$+ \frac{C_p^p}{C_{2p}^p}\frac{(x_2-x_1)^p}{p!}[f^{(p-1)}(x_1)+(-1)^{p-1}f^{(p-1)}(x_2)] +$$

$$+ \frac{1}{(2p)!}\int_{x_1}^{x_2}(x-x_1)^p(x-x_2)^p f^{(2p)}(x)\,dx. \tag{2}$$

În lucrarea de față se face o generalizare a acestor formule la integrale duble, urmînd metoda dată de noi [3] pentru generalizarea formulei de cuadratură a trapezului la integrale duble.

În § 1 se dă formula preliminară (6), din care se deduce în § 2 o problemă la limită asupra sistemului de ecuații cu derivate parțiale (7) cu condiții la limită convenabil alese, astfel ca din formula preliminară (6) să se deducă formula de cubatură (17). Tratarea problemei la limită în cazul general prezentînd dificultăți, deocamdată în § 3 și 4 se rezolvă problema la limită pentru cazurile $n = 3$ și 4. Se ajunge astfel la formulele de cubatură care fac obiectul acestei lucrări.

Avem

$$\iint_D f\,dx\,dy = \frac{S}{12}[7f(x_2, y_2) + f(x_1, y_2) + f(x_2, y_1) + 3f(x_1, y_1)] -$$

$$-\frac{S}{6}\left[(x_2 - x_1)\frac{\partial f}{\partial x}(x_2, y_2) + (y_2 - y_1)\frac{\partial f}{\partial y}(x_2, y_2)\right] + R, \tag{I}$$

unde D este dreptunghiul format de dreptele $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, $S = (x_2 - x_1)(y_2 - y_1)$, iar restul R este dat de formula

$$R = -\iint_D\left(\varphi_0\frac{\partial^3 f}{\partial x^3} + \varphi_1\frac{\partial^3 f}{\partial x^2\partial y} + \varphi_2\frac{\partial^3 f}{\partial x\partial y^2} + \varphi_3\frac{\partial^3 f}{\partial y^3}\right)dx\,dy \tag{I'}$$

și în lucrare se determină funcțiile $\varphi_i(x, y)$, $i = 0, 1, 2, 3$.

De asemenea avem

$$\iint f\,dx\,dy = -\frac{S}{24}\left[S\frac{\partial^2 f}{\partial x\partial y}(x_2, y_2) + 2(x_2 - x_1)\frac{\partial f}{\partial x}(x_2, y_2) + \right.$$

$$\left. + 2(y_2 - y_1)\frac{\partial f}{\partial y}(x_2, y_2) - 12f(x_2, y_2)\right]$$

$$-\frac{S}{24}\left[S\frac{\partial^2 f}{\partial x\partial y}(x_1, y_1) - 2(x_2 - x_1)\frac{\partial f}{\partial x}(x_1, y_1) - \right.$$

$$\left. - 2(y_2 - y_1)\frac{\partial f}{\partial y}(x_1, y_1) - 12f(x_1, y_1)\right] + R, \tag{II}$$

unde

$$R = -\iint_D\left(\varphi_0\frac{\partial^4 f}{\partial x^4} + \varphi_1\frac{\partial^4 f}{\partial x^3\partial y} + \varphi_2\frac{\partial^4 f}{\partial x^2\partial y^2} + \varphi_3\frac{\partial^4 f}{\partial x\partial y^3} + \varphi_4\frac{\partial^4 f}{\partial y^4}\right)dx\,dy \tag{II'}$$

și în lucrare se determină funcțiile $\varphi_i(x, y)$, $i = 0, 1, 2, 3, 4$.

§ 1. Formule preliminarii

1. Fie $f(x, y)$ o funcție definită și continuă în dreptunghiul D definit de inegalitățile

$$x_1 \leqslant x \leqslant x_2, \qquad y_1 \leqslant y \leqslant y_2 \tag{3}$$

și care are în acest dreptunghi derivate parțiale succesive în raport cu x și cu y pînă la ordinul $n = p + q$, continue. La derivatele parțiale

$$\frac{\partial^n f}{\partial x^{n-k}\partial y^k},$$

unde $k = 0, 1, \ldots, n$ atașăm funcțiile $\varphi_k(x, y)$ care de asemenea sînt continue în dreptunghiul D, împreună cu derivatele lor parțiale în raport cu x și y, pînă la ordinul n. Prin integrări prin părți convenabile, vrem să dăm formula de transformare pentru integralele duble

$$\iint_D \varphi_k\frac{\partial^n f}{\partial x^{n-k}\partial y^k}dx\,dy.$$

1°. Pentru $k = 0$, avem

$$\iint_D \varphi_0\frac{\partial^n f}{\partial x^n}dx\,dy = \int_{y_1}^{y_2}dy\int_{x_1}^{x_2}\varphi_0\frac{\partial^n f}{\partial x^n}dx.$$

Însă, după formula generalizată de integrare prin părți, avem

$$\int_{x_1}^{x_2}\varphi_0\frac{\partial^n f}{\partial x^n}dx = \left[\sum_{j=0}^{n-1}(-1)^j\frac{\partial^j\varphi_0}{\partial x^j}\frac{\partial^{n-j-1}f}{\partial x^{n-j-1}}\right]_{x_1}^{x_2} + (-1)^n\int_{x_1}^{x_2}\varphi_0\frac{\partial^n f}{\partial x^n}dx,$$

astfel că

$$\iint_D\varphi_0\frac{\partial^n f}{\partial x^n}dx\,dy = \sum_{j=0}^{n-1}(-1)^j\int_{y_1}^{y_2}\frac{\partial^j\varphi_0}{\partial x^j}(x_2, y)\frac{\partial^{n-j-1}f}{\partial x^{n-j-1}}(x_2, y)dy -$$

$$-\sum_{j=0}^{n-1}(-1)^j\int_{y_1}^{y_2}\frac{\partial^j\varphi_0}{\partial x^j}(x_1, y)\frac{\partial^{n-j-1}f}{\partial x^{n-j-1}}(x_1, y)\,dy + (-1)^n\iint_D\frac{\partial^n\varphi_0}{\partial x^n}f\,dx\,dy. \tag{4}$$

2°. De asemenea pentru $k = n$, avem formula

$$\iint_D\varphi_n\frac{\partial^n f}{\partial y^n}dx\,dy = \sum_{l=0}^{n-1}(-1)^l\int_{x_1}^{x_2}\frac{\partial^l\varphi_n}{\partial y^l}(x, y_2)\frac{\partial^{n-l-1}f}{\partial y^{n-l-1}}(x, y_2)\,dx -$$

$$-\sum_{l=0}^{n-1}(-1)^l\int_{x_1}^{x_2}\frac{\partial^l\varphi_n}{\partial y^l}(x, y_1)\frac{\partial^{n-l-1}f}{\partial y^{n-l-1}}(x, y_1)\,dx + (-1)^n\iint\frac{\partial^n\varphi_n}{\partial y^n}f\,dx\,dy. \tag{5}$$

3°. Să considerăm acum integrala

$$\iint_D\varphi_k\frac{\partial^n f}{\partial x^{n-k}\partial y^k}dx\,dy,$$

în care k este unul din numerele $1, 2, \ldots, n-1$. Aplicînd formula generalizată de integrare prin părți avem

$$\iint_D \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\,dx\,dy = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} \frac{\partial}{\partial y}\left[\sum_{l=0}^{k-1}(-1)^l \frac{\partial^l \varphi_k}{\partial y^l}\,\frac{\partial^{n-l-1} f}{\partial x^{n-k}\partial y^{k-l-1}}\right] dy +$$
$$+ (-1)^k \iint_D \frac{\partial^k \varphi_k}{\partial y^k}\,\frac{\partial^{n-k} f}{\partial x^{n-k}}\,dx\,dy.$$

Aplicînd din nou formula generalizată de integrare prin părți, aceasta se mai scrie

$$\iint_D \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\,dx\,dy = \sum_{l=0}^{k-1}(-1)^l \int_{x_1}^{x_2} \frac{\partial^l \varphi_k}{\partial y^l}(x, y_2)\,\frac{\partial^{n-l-1} f}{\partial x^{n-k}\partial y^{k-l-1}}(x, y_2)\,dx -$$
$$- \sum_{l=0}^{k-1}(-1)^l \int_{x_1}^{x_2} \frac{\partial^l \varphi_k}{\partial y^l}(x, y_1)\,\frac{\partial^{n-l-1} f}{\partial x^{n-k}\partial y^{k-l-1}}(x, y_1)\,dx +$$
$$+ (-1)^k \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left[\sum_{j=0}^{n-k-1}(-1)^j \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}\,\frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}\right] dx +$$
$$+ (-1)^n \iint_D \frac{\partial^n \varphi_k}{\partial x^{n-k}\partial y^k} f\,dx\,dy.$$

La integralele simple din membrul al doilea, aplicăm din nou formula generalizată de integrare prin părți și obținem

$$\iint_D \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\,dx\,dy =$$
$$= \sum_{l=0}^{k-1}(-1)^l \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left[\sum_{j=0}^{n-k-1}(-1)^j \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x, y_2)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x, y_2)\right] dx +\right.$$
$$\left. + (-1)^{n-k}\int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_2)\,\frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_2)\,dx \right\} -$$
$$- \sum_{l=0}^{k-1}(-1)^l \left\{ \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left[\sum_{j=0}^{n-k-1}(-1)^j \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x, y_1)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x, y_1)\right] dx +\right.$$
$$\left. + (-1)^{n-k}\int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_1)\,\frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_1)\,dx \right\} +$$

$$+ (-1)^k \int_{y_1}^{y_2} \sum_{j=0}^{n-k-1}(-1)^j \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_2, y)\,\frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_2, y)\,dy -$$
$$- (-1)^k \int_{y_1}^{y_2} \sum_{j=0}^{n-k-1}(-1)^j \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_1, y)\,\frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_1, y)\,dy +$$
$$+ (-1)^n \iint_D \frac{\partial^n \varphi_k}{\partial x^{n-k}\partial y^k} f\,dx\,dy,$$

de unde rezultă următoarea formulă definitivă

$$\iint_D \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\,dx\,dy =$$
$$= \sum_{j=0}^{n-k-1}\sum_{l=0}^{k-1}(-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_2, y_2)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_2, y_2) -$$
$$- \sum_{j=0}^{n-k-1}\sum_{l=0}^{k-1}(-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_1, y_2)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_1, y_2) -$$
$$- \sum_{j=0}^{n-k-1}\sum_{l=0}^{k-1}(-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_2, y_1)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_2, y_1) +$$
$$+ \sum_{j=0}^{n-k-1}\sum_{l=0}^{k-1}(-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_1, y_1)\,\frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_1, y_1) +$$
$$+ \sum_{l=1}^{k-1}(-1)^{n-k+l}\int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_2)\,\frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_2)\,dx -$$
$$- \sum_{l=0}^{k-1}(-1)^{n-k+l}\int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_1)\,\frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_1)\,dx +$$
$$+ \sum_{j=0}^{n-k-1}(-1)^{k+j}\int_{y_1}^{y_2} \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_2, y)\,\frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_2, y)\,dy -$$
$$- \sum_{j=0}^{n-k-0}(-1)^{k+j}\int_{y_1}^{y_2} \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_1, y)\,\frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_1, y)\,dy +$$
$$+ (-1)^n \iint_D \frac{\partial^n \varphi_k}{\partial x^{n-k}\partial y^k} f\,dx\,dy. \qquad (6)$$

Formula (6) este fundamentală în această lucrare. Ea va conduce la o problemă la limită din care va rezulta generalizarea formulei lui Obreschkoff.

§ 2. Problemă la limită

2. În formulele (4), (5) şi (6) în care presupunem $k = 1, 2, \ldots, n-1$ să alegem funcţiile $\varphi_0(x, y), \varphi_1(x, y), \ldots, \varphi_n(x, y)$ astfel încît ele să fie integrale ale ecuaţiilor cu derivate parţiale

$$\frac{\partial^n \varphi_0}{\partial x^n} = \alpha_0, \quad \frac{\partial^n \varphi_1}{\partial x^{n-1}\partial y} = \alpha_1, \quad \ldots, \quad \frac{\partial^n \varphi_n}{\partial y^n} = \alpha_n, \tag{7}$$

unde $\alpha_0, \alpha_1, \ldots, \alpha_n$ sînt constante care vor fi precizate mai departe. Adunînd toate formulele (4), (5), (6) membru cu membru, vom obţine formula

$$\begin{aligned}
\iint\limits_D \left(\sum_{k=0}^{n} \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\right) dx\, dy = \\
= \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} \sum_{l=0}^{k-1} (-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_2, y_2) \frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_2, y_2) - \\
- \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} \sum_{l=0}^{k-1} (-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_1, y_2) \frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_1, y_2) - \\
- \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} \sum_{l=0}^{k-1} (-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_2, y_1) \frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_2, y_1) + \\
+ \sum_{k=1}^{n} \sum_{j=0}^{n-k-1} \sum_{l=0}^{k-1} (-1)^{j+l} \frac{\partial^{j+l}\varphi_k}{\partial x^j \partial y^l}(x_1, y_1) \frac{\partial^{n-j-l-2} f}{\partial x^{n-j-k-1}\partial y^{k-l-1}}(x_1, y_1) + \\
+ \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (-1)^{k+j} \int_{y_1}^{y_2} \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_2, y) \frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_2, y)\, dy - \\
- \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (-1)^{k+j} \int_{y_1}^{y_2} \frac{\partial^{k+j}\varphi_k}{\partial x^j \partial y^k}(x_1, y) \frac{\partial^{n-k-j-1} f}{\partial x^{n-k-j-1}}(x_1, y)\, dy + \\
+ \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^{n-k+l} \int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_2) \frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_2)\, dx - \\
- \sum_{k=1}^{n} \sum_{l=0}^{k-1} (-1)^{n-k+l} \int_{x_1}^{x_2} \frac{\partial^{n-k+l}\varphi_k}{\partial x^{n-k}\partial y^l}(x, y_1) \frac{\partial^{k-l-1} f}{\partial y^{k-l-1}}(x, y_1)\, dx + \\
+ (-1)^n \left(\sum_{k=0}^{n} \alpha_k\right) \iint\limits_D f\, dx\, dy.
\end{aligned} \tag{8}$$

Să notăm

$$A_j^l(x, y) = \frac{\partial^l \varphi_{j+1}}{\partial x^l} + \frac{\partial^l \varphi_{j+2}}{\partial x^{l-1}\partial y} + \cdots + \frac{\partial^l \varphi_{j+l+1}}{\partial y^l}, \tag{9}$$

unde $l = 0, 1, 2, \ldots, n-2$ şi $j = 0, 1, 2, \ldots, n-l-2$.

Cu aceste notaţii formula (8) se scrie sub forma următoare

$$\begin{aligned}
\iint\limits_D \left(\sum_{k=0}^{n} \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\right) dx\, dy = A_0^0(x_2, y_2) \frac{\partial^{n-2} f}{\partial x^{n-2}}(x_2, y_2) + \\
+ A_1^0(x_2, y_2) \frac{\partial^{n-2} f}{\partial x^{n-3}\partial y}(x_2, y_2) + \cdots + A_{n-2}^0(x_2, y_2) \frac{\partial^{n-2} f}{\partial y^{n-2}}(x_2, y_2) - \\
- \Big[A_0^0(x_1, y_2) \frac{\partial^{n-2} f}{\partial x^{n-2}}(x_1, y_2) + A_1^0(x_1, y_2) \frac{\partial^{n-2} f}{\partial x^{n-3}\partial y}(x_1, y_2) + \cdots + \\
+ \cdots + A_{n-2}^0(x_1, y_2) \frac{\partial^{n-2} f}{\partial y^{n-2}}(x_1, y_2)\Big] - \\
- \Big[A_0^0(x_2, y_1) \frac{\partial^{n-2} f}{\partial x^{n-2}}(x_2, y_1) + A_1^0(x_2, y_1) \frac{\partial^{n-2} f}{\partial x^{n-3}\partial y}(x_2, y_1) + \cdots + \\
+ \cdots + A_{n-2}^0(x_2, y_1) \frac{\partial^{n-2} f}{\partial y^{n-2}}(x_2, y_1)\Big] + \\
+ \Big[A_0^0(x_1, y_1) \frac{\partial^{n-2} f}{\partial x^{n-2}}(x_1, y_1) + A_1^0(x_1, y_1) \frac{\partial^{n-2} f}{\partial x^{n-3}\partial y}(x_1, y_1) + \cdots + \\
+ \cdots + A_{n-2}^0(x_1, y_1) \frac{\partial^{n-2} f}{\partial y^{n-2}}(x_1, y_1)\Big] - \\
- \Big[A_0^1(x_2, y_2) \frac{\partial^{n-3} f}{\partial x^{n-3}}(x_2, y_2) + A_1^1(x_2, y_2) \frac{\partial^{n-3} f}{\partial x^{n-4}\partial y}(x_2, y_2) + \cdots + \\
+ A_{n-3}^1(x_2, y_2) \frac{\partial^{n-3} f}{\partial y^{n-3}}(x_2, y_2)\Big] + \\
+ \Big[A_0^1(x_1, y_2) \frac{\partial^{n-3} f}{\partial x^{n-3}}(x_1, y_2) + A_1^1(x_1, y_2) \frac{\partial^{n-3} f}{\partial x^{n-4}\partial y}(x_1, y_2) + \cdots + \\
+ A_{n-3}^1(x_1, y_2) \frac{\partial^{n-3} f}{\partial y^{n-3}}(x_1, y_2)\Big] + \\
+ \Big[A_0^1(x_2, y_1) \frac{\partial^{n-3} f}{\partial x^{n-3}}(x_2, y_1) + A_1^1(x_2, y_1) \frac{\partial^{n-3} f}{\partial x^{n-4}\partial y}(x_1, y_1) + \cdots +
\end{aligned}$$

$$+ A^1_{n-3}(x_2, y_1)\,\frac{\partial^{n-3}f}{\partial y^{n-3}}(x_2, y_1)\Big] -$$

$$-\Big[A^1_0(x_1, y_1)\,\frac{\partial^{n-2}f}{\partial x^{n-3}}(x_1, y_1) + A^1_1(x_1, y_1)\,\frac{\partial^{n-3}f}{\partial x^{n-4}\partial y}(x_1, y_1) + \cdots +$$

$$+ A^1_{n-3}(x_1, y_1)\,\frac{\partial^{n-3}f}{\partial y^{n-3}}(x_1, y_1)\Big] +$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{n-3}\Big[A_0^{n-3}(x_2, y_2)\,\frac{\partial f}{\partial x}(x_2, y_2) + A_1^{n-3}(x_2, y_2)\,\frac{\partial f}{\partial y}(x_2, y_2)\Big] -$$

$$-(-1)^{n-3}\Big[A_0^{n-3}(x, y_1)\,\frac{\partial f}{\partial x}(x_2, y_1) + A_1^{n-3}(x_1, y_2)\,\frac{\partial f}{\partial y}(x_1, y_2)\Big] -$$

$$-(-1)^{n-3}\Big[A_0^{n-3}(x_2, y_1)\,\frac{\partial f}{\partial x}(x_2, y_1) + A_1^{n-3}(x_2, y_1)\,\frac{\partial f}{\partial y}(x_2, y_1)\Big] +$$

$$+ (-1)^{n-3}\Big[A_0^{n-3}(x_1, y_1)\,\frac{\partial f}{\partial x}(x_1, y_1) + A_1^{n-3}(x_1, y_1)\,\frac{\partial f}{\partial y}(x_1, y_1)\Big] +$$

$$+ (-1)^{n-2}A_0^{n-2}(x_2, y_2)f(x_2, y_2) -$$

$$-(-1)^{n-2}A_0^{n-2}(x_1, y_2)f(x_1, y_2) -$$

$$-(-1)^{n-2}A_0^{n-2}(x_2, y_1)f(x_2, y_1) +$$

$$+ (-1)^{n-2}A_0^{n-2}(x_1, y_1)f(x_1, y_1) +$$

$$+ \int_{y_1}^{y_2}\varphi_0(x_2, y)\,\frac{\partial^{n-1}f}{\partial x^{n-1}}(x_2, y)\,dy - \int_{y_1}^{y_2}\Big[\frac{\partial\varphi_0}{\partial x}(x_2, y) + \frac{\partial\varphi_1}{\partial y}(x_2, y)\Big]\frac{\partial^{n-2}f}{\partial x^{n-2}}\; x_2, y)\,dy +$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{n-1}\int_{y_1}^{y_2}\Big[\frac{\partial^{n-1}\varphi_0}{\partial x^{n-1}}(x_2, y) + \frac{\partial^{n-1}\varphi_1}{\partial y^{n-2}\partial y}(x_2, y) + \dots + \frac{\partial^{n-1}\varphi_{n-1}}{\partial y^{n-1}}(x_2, y)\Big]f(x_2, y)\,dy -$$

$$-\int_{y_1}^{y_2}\varphi_0(x_1, y)\,\frac{\partial^{n-1}f}{\partial x^{n-1}}(x_1, y)\,dy + \int_{y_1}^{y_2}\Big[\frac{\partial\varphi_0}{\partial x}(x_1, y) + \frac{\partial\varphi_1}{\partial y}(x_1, y)\Big]\frac{\partial^{n-2}f}{\partial x^{n-2}}(x_1, y)\,dy +$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{n}\int_{y_1}^{y_2}\Big[\frac{\partial^{n-1}\varphi_0}{\partial x^{n-1}}(x_1, y) + \frac{\partial^{n-1}\varphi_1}{\partial x^{n-2}\partial y}(x_1, y) + \dots + \frac{\partial^{n-1}\varphi_{n-1}}{\partial y^{n-1}}(x_1, y)\Big]f(x_1, y)\,dy +$$

$$+ \int_{x_1}^{x_2}\varphi_n(x, y_2)\,\frac{\partial^{n-1}f}{\partial y^{n-1}}(x, y_2)\,dx - \int_{x_1}^{x_2}\Big[\frac{\partial\varphi_{n-1}}{\partial x}(x, y_2) + \frac{\partial\varphi_n}{\partial y}(x, y_2)\Big]\frac{\partial^{n-2}f}{\partial y^{n-2}}(x, y_2)\,dx +$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{n-1}\int_{x_1}^{x_2}\Big[\frac{\partial^{n-1}\varphi_2}{\partial x^{n-1}}(x, y_2) + \frac{\partial^{n-1}\varphi_2}{\partial x^{n-2}\partial y}(x, y_2) + \dots + \frac{\partial^{n-1}\varphi_n}{\partial y^{n-1}}(x, y_2)\Big]f(x, y_2)\,dx -$$

$$-\int_{x_1}^{x_2}\varphi_n(x, y_1)\,\frac{\partial^{n-1}f}{\partial y^{n-1}}(x, y_1)\,dx + \int_{x_1}^{x_2}\Big[\frac{\partial\varphi_{n-1}}{\partial x}(x, y_1) + \frac{\partial\varphi_n}{\partial y}(x, y_1)\Big]\frac{\partial^{n-2}f}{\partial y^{n-2}}(x, y_1)\,dx +$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots +$$

$$+ (-1)^{n}\int_{x_1}^{x_2}\Big[\frac{\partial^{n-1}\varphi_1}{\partial x^{n-1}}(x, y_1) + \frac{\partial^{n-1}\varphi_2}{\partial x^{n-2}\partial y}(x, y_1) + \dots + \frac{\partial^{n-1}\varphi_n}{\partial y^{n-1}}(x, y_1)\Big]f(x, y_1)\,dx +$$

$$+ (-1)^n\Big(\sum_{k=0}^{n}\alpha_k\Big)\iint_D f\,dx\,dy. \tag{10}$$

3. Să introducem condiţii la limită care să simplifice formula (10). Pentru aceasta să determinăm constantele $\alpha_0, \alpha_1, \dots, \alpha_n$ şi funcţiile $\varphi_0(x,y), \varphi_1(x,y), \dots, \varphi_n(x,y)$ astfel ca în formula (10), în integralele din membrul al doilea, coeficienţii lui

$$\frac{\partial^j f}{\partial x^j}(x_2, y), \frac{\partial^j f}{\partial x^j}(x_1, y)\,; \quad \frac{\partial^l f}{\partial y^l}(x, y_2), \frac{\partial^l f}{\partial y^l}(x, y_1),$$

pentru $j = 0, 1, \dots, n-1$ şi $l = 0, 1, \dots, n-1$ să fie nuli.

Prin urmare introducem condiţiile la limită

$$\left.\begin{aligned} &\varphi_0(x_2, y) = 0\\ &\frac{\partial\varphi_0}{\partial x}(x_2, y) + \frac{\partial\varphi_1}{\partial y}(x_2, y) = 0\\ &\dots\dots\dots\\ &\frac{\partial^{n-1}\varphi_0}{\partial x^{n-1}}(x_2, y) + \frac{\partial^{n-1}\varphi_1}{\partial x^{n-2}\partial y}(x_2, y) + \dots + \frac{\partial^{n-1}\varphi_{n-1}}{\partial y^{n-1}}(x_2, y) = 0 \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} &\varphi_0(x_1, y) = 0\\ &\frac{\partial\varphi_0}{\partial x}(x_1, y) + \frac{\partial\varphi_1}{\partial y}(x_1, y) = 0\\ &\dots\dots\dots\\ &\frac{\partial^{n-1}\varphi_0}{\partial x^{n-1}}(x_1, y) + \frac{\partial^{n-1}\varphi_1}{\partial x^{n-2}\partial y}(x_1, y) + \dots + \frac{\partial^{n-1}\varphi_{n-1}}{\partial y^{n-1}}(x_1, y) = 0 \end{aligned}\right\} \tag{12}$$

$$\left.\begin{array}{l}
\varphi_n(x, y_2) = 0 \\
\dfrac{\partial \varphi_{n-1}}{\partial x}(x, y_2) + \dfrac{\partial \varphi_n}{\partial y}(x, y_2) = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{\partial^{n-1}\varphi_1}{\partial x^{n-1}}(x, y_2) + \dfrac{\partial^{n-1}\varphi_2}{\partial x^{n-2}\partial y}(x, y_2) + \cdots + \dfrac{\partial^{n-1}\varphi_n}{\partial y^{n-1}}(x, y_2) = 0
\end{array}\right\} \quad (13)$$

$$\left.\begin{array}{l}
\varphi_n(x, y_1) = 0 \\
\dfrac{\partial \varphi_{n-1}}{\partial x}(x, y_1) + \dfrac{\partial \varphi_n}{\partial y}(x, y_2) = 0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\dfrac{\partial^{n-1}\varphi_1}{\partial x^{n-1}}(x, y_1) + \dfrac{\partial^{n-1}\varphi_2}{\partial x^{n-2}\partial y}(x, y_1) + \cdots + \dfrac{\partial^{n-1}\varphi_n}{\partial y^{n-1}}(x, y_1) = 0
\end{array}\right\} \quad (14)$$

Dacă funcţia $f(x, y)$ depinde numai de x, aceleaşi transformări de calcul care au condus la formula (10), dau formula de forma

$$\iint\limits_D \varphi_0(x, y) f^{(n)}(x)\, dx\, dy = A_0 f(x_2) + A_1 f'(x_2) + \cdots + A_{p-1} f^{(p-1)}(x_2) +$$
$$+ B_0 f(x_1) + B_1 f'(x_1) + \cdots + B_{q-1} f^{(q-1)}(x_1) + (-1)^n \iint\limits_D \frac{\partial^n \varphi_0}{\partial x^n} f(x)\, dx\, dy.$$

Pentru ca această formulă să se reducă la o formulă a lui N. Obreschkoff, vom alege integrala ecuaţiei

$$\frac{\partial^n \varphi_0}{\partial x^n} = \alpha_0,$$

de forma

$$\varphi_0(x, y) = \frac{\alpha_0}{(p+q)!}(x - x_1)^p (x - x_2)^q \quad (15)$$

unde $p + q = n$.

De asemenea dacă funcţia $f(x, y)$ depinde numai de y, avem formula

$$\iint\limits_D \varphi_n(x, y) g^{(n)}(y)\, dx\, dy = C_0 g(y_2) + C_1 g'(y_2) + \cdots + C_{p-1} g^{(p-1)}(y_2) +$$
$$+ D_0 g(y_1) + D_1 g'(y_1) + \cdots + D_{q-1} g^{(q-1)}(y_1) + (-1)^n \iint\limits_D \frac{\partial^n \varphi_n}{\partial y^n} g(y)\, dx\, dy$$

şi pentru ca aceasta să se reducă la o formulă a lui Obreschkoff, vom alege integrala ecuaţiei

$$\frac{\partial^n \varphi_n}{\partial y^n} = \alpha_n$$

de forma

$$\varphi_n(x, y) = \frac{\alpha_n}{(p+q)!}(y - y_1)^p (y - y_2)^q. \quad (16)$$

Sîntem astfel conduşi la următoarea problemă la limită :

Să se integreze ecuaţiile cu derivate parţiale (7) cu condiţiile la limită (11), (12), (13), (14) *ţinînd seama că* $\varphi_0(x, y)$ *şi* $\varphi_n(x, y)$ *sînt date de formulele* (15) *şi* (16).

Constantele $\alpha_0, \alpha_1, \ldots, \alpha_n$ din ecuaţiile (7) se aleg astfel ca această problemă să fie posibilă.

Cu aceasta, formula (10) se reduce la următoarea formă

$$\iint\limits_D \left(\sum_{k=0}^{n} \varphi_k \frac{\partial^n f}{\partial x^{n-k}\partial y^k}\right) dx\, dy = \sum_{j=0}^{n-2}\Bigg[A_j^0(x_2, y_2)\frac{\partial^{n-2} f}{\partial x^{n-2-j}\partial y^j}(x_2, y_2) -$$
$$- A_j^0(x_1, y_2)\frac{\partial^{n-2} f}{\partial x^{n-2-j}\partial y^j}(x_1, y_2) - A_j^0(x_2, y_1)\frac{\partial^{n-2} f}{\partial x^{n-2-j}\partial y^j}(x_2, y_1) +$$
$$+ A_j^0(x_1, y_1)\frac{\partial^{n-2} f}{\partial x^{n-2-j}\partial y^j}(x_1, y_1)\Bigg] - \sum_{j=0}^{n-3}\Bigg[A_j^1(x_2, y_2)\frac{\partial^{n-3} f}{\partial x^{n-3-j}\partial y^j}(x_2, y_2) -$$
$$- A_j^1(x_1, y_2)\frac{\partial^{n-3} f}{\partial x^{n-3-j}\partial y^j}(x_1, y_2) - A_j^1(x_2, y_1)\frac{\partial^{n-3} f}{\partial x^{n-3-j}\partial y^j}(x_2, y_1) +$$
$$+ A_j^1(x_1, y_1)\frac{\partial^{n-3} f}{\partial x^{n-3-j}\partial y^j}(x_1, y_1)\Bigg] +$$
$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot +$$
$$+ (-1)^{n-2}\Big[A_0^{n-2}(x_2, y_2) f(x_2, y_2) - A_0^{n-2}(x_1, y_2) f(x_1, y_2) -$$
$$- A_0^{n-2}(x_2, y_1) f(x_2, y_1) + A_0^{n-2}(x_1, y_1) f(x_1, y_1)\Big] + (-1)^n \left(\sum_{k=0}^{n} \alpha_k\right) \iint\limits_D f\, dx dy. \quad (17)$$

Vom vedea mai departe în tratarea problemei la limită pusă mai sus, că se întroduc constante pe care le vom determina astfel ca formulele (17) să se mai simplifice.

§ 3. Cazul $n = 3$

4. Avem de tratat următoarea problemă la limită :

Să se determine funcţiile $\varphi_1(x, y)$, $\varphi_2(x, y)$ *integrale ale ecuaţiilor cu derivate parţiale*

$$\frac{\partial^3 \varphi_1}{\partial x^2 \partial y} = \alpha_1, \qquad \frac{\partial^3 \varphi_2}{\partial x \partial y^2} = \alpha_2, \quad (18)$$

cu următoarele condiţii la limită :

$$\left.\frac{\partial \varphi_0}{\partial x} + \frac{\partial \varphi_1}{\partial y}\right|_{x=x_1} = 0, \qquad \left.\frac{\partial \varphi_0}{\partial x} + \frac{\partial \varphi_1}{\partial y}\right|_{x=x_2} = 0, \quad (19)$$

$$\left.\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial x \partial y} + \frac{\partial^2 \varphi_2}{\partial y^2}\right|_{x=x_1} = 0, \qquad \left.\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial x \partial y} + \frac{\partial^2 \varphi_2}{\partial y^2}\right|_{x=x_2} = 0, \quad (20)$$

$$\left.\frac{\partial \varphi_2}{\partial x}+\frac{\partial \varphi_3}{\partial y}\right|_{y=y_1}=0, \qquad \left.\frac{\partial \varphi_2}{\partial x}+\frac{\partial \varphi_3}{\partial y}\right|_{y=y_2}=0, \tag{21}$$

$$\left.\frac{\partial^2 \varphi_1}{\partial x^2}+\frac{\partial^2 \varphi_2}{\partial x \partial y}+\frac{\partial^2 \varphi_3}{\partial y^2}\right|_{y=y_1}=0, \qquad \left.\frac{\partial^2 \varphi_1}{\partial x^2}+\frac{\partial^2 \varphi_2}{\partial x \partial y}+\frac{\partial^2 \varphi_3}{\partial y^2}\right|_{y=y_2}=0, \tag{22}$$

știind că

$$\varphi_0(x, y)=\frac{\alpha_0}{6}(x-x_1)^2(x-x_2), \tag{23}$$

$$\varphi_3(x, y)=\frac{\alpha_3}{6}(y-y_1)^2(y-y_2). \tag{24}$$

1°. Avem

$$\frac{\partial \varphi_0}{\partial x}=\frac{\alpha_0}{6}\left[2(x-x_1)(x-x_2)+(x-x_1)^2\right]$$

și din condițiile la limită (19), deducem că

$$\left.\frac{\partial \varphi_1}{\partial y}\right|_{x=x_1}=0, \qquad \left.\frac{\partial \varphi_1}{\partial y}\right|_{x=x_2}=-\alpha_0 \frac{(x_2-x_1)^2}{6}. \tag{19'}$$

Să integrăm ecuația

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial \varphi}{\partial y}\right)=\alpha_1,$$

ținînd seama de condițiile (19′). Avem

$$\frac{\partial \varphi_1}{\partial y}=\frac{\alpha_1}{2}(x-x_1)(x-x_2)+A_1(y)(x-x_1)+A_2(y)$$

cu funcțiile $A_1(y)$, $A_2(y)$ arbitrare. Scriind că condițiile (19′) sînt satisfăcute, avem

$$A_2(y)=0, \qquad -\alpha_0 \frac{x_2-x_1}{6}=A_1(y)$$

și prin urmare

$$\frac{\partial \varphi_1}{\partial y}=\frac{\alpha_1}{2}(x-x_1)(x-x_2)-\alpha_0 \frac{x_2-x_1}{6}(x-x_1). \tag{25}$$

2°. Avem

$$\frac{\partial^2 \varphi_0}{\partial x^2}=\frac{\alpha_0}{3}\left[2(x-x_1)+(x-x_2)\right],$$

$$\frac{\partial^2 \varphi_1}{\partial x \partial y}=\alpha_1\left(x-\frac{x_1+x_2}{2}\right)-\alpha_0 \frac{x_2-x_1}{6}$$

și condițiile la limită (20), arată că

$$\left.\frac{\partial^2 \varphi_2}{\partial y^2}\right|_{x=x_1}=\frac{\alpha_0+\alpha_1}{2}(x_2-x_1), \qquad \left.\frac{\partial^2 \varphi_2}{\partial y^2}\right|_{x=x_2}=-\frac{\alpha_0+\alpha_1}{2}(x_2-x_1). \tag{20'}$$

Să integrăm ecuația

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \varphi_2}{\partial y^2}\right)=\alpha_2$$

cu condițiile la limită (20′). Avem

$$\frac{\partial^2 \varphi_2}{\partial y^2}=\alpha_2(x-x_1)+B_1(y) \tag{26}$$

și scriind că condițiile la limită (20′) sînt satisfăcute, avem

$$\frac{\alpha_0+\alpha_1}{2}(x_2-x_1)=B_1(y),$$

$$-\frac{\alpha_0+\alpha_1}{2}(x_2-x_1)=B_1(y)+\alpha_2(x_2-x_1),$$

de unde rezultă că între α_0, α_1, α_2 trebuie să existe relația

$$\alpha_0+\alpha_1+\alpha_2=0 \tag{27}$$

și ecuația (26) devine, ținînd seama de aceasta

$$\frac{\partial^2 \varphi_2}{\partial y^2}=\alpha_2(x-x_1)-\frac{\alpha_2}{2}(x_2-x_1),$$

sau

$$\frac{\partial^2 \varphi_2}{\partial y^2}=\alpha_2\left(x-\frac{x_1+x_2}{2}\right). \tag{28}$$

3°. Să ținem seama de formulele (24) și să integrăm din nou ecuațiile (18) cu condițiile la limită (21), (22).
Avem

$$\frac{\partial \varphi_3}{\partial y}=\frac{\alpha_3}{6}\left[2(y-y_1)(y-y_2)+(y-y_1)^2\right]$$

și condițiile (21) arată că

$$\left.\frac{\partial \varphi_2}{\partial x}\right|_{y=y_1}=0, \qquad \left.\frac{\partial \varphi_2}{\partial x}\right|_{y=y_2}=-\frac{\alpha_3}{6}(y_2-y_1)^2. \tag{29}$$

Integrăm ecuația

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial \varphi_2}{\partial x}\right)=\alpha_2,$$

ținînd seama de condițiile (29). Vom avea

$$\frac{\partial \varphi_2}{\partial x}=\frac{\alpha_2}{2}(y-y_1)(y-y_2)+a_1(x)(y-y_1)+a_2(x).$$

Condițiile la limită (29) arată că

$$a_2(x)=0, \qquad -\frac{\alpha_3}{6}(y_2-y_1)=a_1(x),$$

astfel că

$$\frac{\partial \varphi_2}{\partial x}=\frac{\alpha_2}{2}(y-y_1)(y-y_2)-\frac{\alpha_3}{6}(y_2-y_1)(y-y_1).$$

Integrăm aceasta şi deducem

$$\varphi_2(x,y) = \frac{\alpha_2}{2}(x-x_1)(y-y_1)(y-y_2) - \frac{\alpha_3}{6}(y_2-y_1)(x-x_1)(y-y_1) + F(y),$$

unde $F(y)$ este o funcţie arbitrară. Derivînd de două ori în raport cu y şi ţinînd seama de ecuaţia (28), trebuie să avem

$$\alpha_2(x-x_1) + F''(y) = \alpha_2\left(x - \frac{x_1+x_2}{x}\right),$$

de unde rezultă că

$$F''(y) = -\frac{\alpha_2}{2}(x_2-x_1).$$

Integrînd, găsim

$$F(y) = -\frac{\alpha_2}{2}(x_2-x_1)(y-y_1)(y-y_2) + P_1(y-y_1) + P_2,$$

unde P_1 şi P_2 sînt două constante arbitrare.

Rezultă că

$$\varphi_2(x,y) = \frac{\alpha_2}{2}(x-x_1)(y-y_1)(y-y_2) - \frac{\alpha_2}{2}(x_2-x_1)(y-y_1)(y-y_2) -$$
$$- \frac{\alpha_3}{6}(y_2-y_1)(x-x_1)(y-y_1) + P_1(y-y_1) + P_2,$$

sau

$$\varphi_2(x,y) = \frac{\alpha_2}{2}\left(x - \frac{x_1+x_2}{2}\right)(y-y_1)(y-y_2) -$$
$$- \frac{\alpha_3}{6}(y_2-y_1)(x-x_1)(y-y_1) + P_1(y-y_1) + P_2. \tag{30}$$

4°. Avem

$$\frac{\partial^2\varphi_3}{\partial y^2} = \frac{\alpha_3}{3}[2(y-y_1) + (y-y_2)],$$

$$\frac{\partial^2\varphi_2}{\partial x\partial y} = \alpha_2\left(y - \frac{y_1+y_2}{2}\right) - \frac{\alpha_3}{6}(y_2-y_1).$$

Condiţiile la limită (22) arată că

$$\left.\frac{\partial^2\varphi_1}{\partial x^2}\right|_{y=y_1} = \frac{\alpha_2+\alpha_3}{2}(y_2-y_1), \qquad \left.\frac{\partial^2\varphi_1}{\partial x^2}\right|_{y=y_2} = -\frac{\alpha_2+\alpha_3}{2}(y_2-y_1). \tag{31}$$

Să integrăm acum ecuaţia

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\varphi_1}{\partial x^2}\right) = \alpha_1,$$

ţinînd seama de condiţiile (31). Avem

$$\frac{\partial^2\varphi_1}{\partial x^2} = \alpha_1(y-y_1) + b_1(x) \tag{32}$$

şi scriind că condiţiile (31) sînt satisfăcute, avem

$$\frac{\alpha_2+\alpha_3}{2}(y_2-y_1) = b_1(x),$$

$$-\frac{\alpha_2+\alpha_3}{2}(y_2-y_1) = b_1(x) + \alpha_1(y_2-y_1),$$

de unde rezultă că între α_1, α_2, α_3 trebuie să existe relaţia

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{33}$$

şi ecuaţia (32) devine

$$\frac{\partial^2\varphi_1}{\partial x^2} = \alpha_1\left(y - \frac{y_1+y_2}{2}\right).$$

Să integrăm această ecuaţie. Vom avea

$$\varphi_1(x,y) = \frac{\alpha_1}{2}\left(y - \frac{y_1+y_2}{2}\right)(x-x_1)(x-x_2) + G_1(y)(x-x_1) + G_2(y),$$

unde $G_1(y)$ şi $G_2(y)$ sînt funcţii arbitrare.

Scriind că derivata în raport cu y, coincide cu membrul al doilea al formulei (25), avem

$$\frac{\alpha_1}{2}(x-x_1)(x-x_2) + G_1'(y)(x-x_1) + G_2'(y) =$$
$$= \frac{\alpha_1}{2}(x-x_1)(x-x_2) - \frac{\alpha_0}{6}(x_2-x_1)(x-x_1),$$

de unde deducem că

$$G_1'(y) = -\frac{\alpha_0}{6}(x_2-x_1),$$

$$G_2'(y) = 0$$

şi integrînd avem

$$G_1(y) = -\frac{\alpha_0}{6}(x_2-x_1)(y-y_1) + Q_1,$$

$$G_2(y) = Q_2,$$

unde Q_1 şi Q_2 sînt constante arbitrare. Vom avea deci

$$\varphi_1(x,y) = \frac{\alpha_1}{2}\left(y - \frac{y_1+y_2}{2}\right)(x-x_1)(x-x_2) - \frac{\alpha_0}{6}(x_2-x_1)(y-y_1) +$$
$$+ Q_1(x-x_1) + Q_2. \tag{34}$$

Astfel s-a făcut integrarea ecuaţiilor cu derivate parţiale (18) cu condiţiile la limită (19), (20), (21), (22), obţinîndu-se formulele (30), (34) şi condiţiile (27), (33) care arată că trebuie să luăm $\alpha_0 = \alpha_3$.

5. Formula de cubatură corespunzătoare este

$$\iint_D \left(\varphi_0 \frac{\partial^3 f}{\partial x^3} + \varphi_1 \frac{\partial^3 f}{\partial x^2 \partial y} + \varphi_2 \frac{\partial^3 f}{\partial x \partial y^2} + \varphi_3 \frac{\partial^3 f}{\partial y^3}\right) dx\, dy =$$

$$= A_0^0(x_2, y_2) \frac{\partial f}{\partial x}(x_2, y_2) + A_1^0(x_2, y_2) \frac{\partial f}{\partial y}(x_2, y_2) -$$

$$- A_0^0(x_1, y_2) \frac{\partial f}{\partial x}(x_1, y_2) - A_1^0(x_1, y_2) \frac{\partial f}{\partial y}(x_1, y_2) -$$

$$- A_0^0(x_2, y_1) \frac{\partial f}{\partial x}(x_2, y_1) - A_1^0(x_2, y_1) \frac{\partial f}{\partial y}(x_2, y_1) +$$

$$+ A_0^0(x_1, y_1) \frac{\partial f}{\partial x}(x_1, y_1) + A_1^0(x_1, y_1) \frac{\partial f}{\partial y}(x_1, y_1) -$$

$$- A_0^1(x_2, y_2) f(x_2, y_2) + A_0^1(x_1, y_2) f(x_1, y_2) + A_0^1(x_2, y_1) f(x_2, y_1) -$$

$$- A_0^1(x_1, y_1) f(x_1, y_1) - (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) \iint_D f\, dx\, dy, \tag{35}$$

unde

$$A_0^0(x, y) = \varphi_1(x, y), \quad A_1^0(x, y) = \varphi_2(x, y), \quad A_0^1(x, y) = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y}.$$

Ţinînd seama de formulele (30) şi (34) avem

$$A_0^0(x_2, y_2) = -\frac{\alpha_0}{6}(x_2 - x_1)^2(y_2 - y_1) + Q_1(x_2 - x_1) + Q_2,$$

$$A_0^0(x_1, y_2) = Q_2,$$

$$A_0^0(x_2, y_1) = Q_1(x_2 - x_1) + Q_2,$$

$$A_0^0(x_1, y_1) = Q_2,$$

$$A_1^0(x_2, y_1) = -\frac{\alpha_3}{6}(y_2 - y_1)^2(x_2 - x_1) + P_1(y_2 - y_1) + P_2$$

$$A_1^0(x_1, y_2) = P_1(y_2 - y_1) + P_2,$$

$$A_1^0(x_2, y_1) = P_2,$$

$$A_1^0(x_1, y_1) = P_2$$

şi ţinînd seamă că $\alpha_1 + \alpha_2 = -\alpha_0 = -\alpha_3$, avem

$$A_0^1(x_2, y_2) = -\frac{7\alpha_0}{12}(x_2 - x_1)(y_2 - y_1) + P_1 + Q_1,$$

$$A_0^1(x_1, y_2) = \frac{\alpha_0}{12}(x_2 - x_1)(y_2 - y_1) + P_1 + Q_1,$$

$$A_0^1(x_2, y_1) = \frac{\alpha_0}{12}(x_2 - x_1)(y_2 - y_1) + P_1 + Q_1,$$

$$A_0^1(x_1, y_1) = -\frac{3\alpha_0}{12}(x_2 - x_1)(y_2 - y_1) + P_1 + Q_1.$$

Vom alege

$$\alpha_0 = 1, \ \alpha_1 = -\frac{1}{2}, \ \alpha_2 = -\frac{1}{2}, \ \alpha_3 = 1 \tag{36}$$

şi cu aceasta, formula de cubatură (35) se scrie

$$\iint_D f\, dx\, dy = \left[\frac{7}{12}(x_2 - x_1)(y_2 - y_1) - P_1 - Q_1\right] f(x_2, y_2) +$$

$$+ \left[\frac{1}{12}(x_2 - x_1)(y_2 - y_1) + P_1 + Q_1\right] f(x_1, y_2) +$$

$$+ \left[\frac{1}{12}(x_1 - x_1)(y_2 - y_1) + P_1 + Q_1\right] f(x_2, y_1) +$$

$$+ \left[\frac{3}{12}(x_2 - x_1)(y_2 - y_1) - P_1 - Q_1\right] f(x_1, y_1) +$$

$$+ \left[-\frac{1}{6}(x_2 - x_1)^2(y_2 - y_1) + Q_1(x_2 - x_1) + Q_2\right] \frac{\partial f}{\partial x}(x_2, y_2) - Q_2 \frac{\partial f}{\partial x}(x_1, y_2) -$$

$$- [Q_1(x_2 - x_1) + Q_2] \frac{\partial f}{\partial x}(x_2, y_1) + Q_2 \frac{\partial f}{\partial x}(x_1, y_1) + \left[-\frac{1}{6}(x_1 - x_1)(y_2 - y_1)^2 +\right.$$

$$\left. + P_1(y_2 - y_1) + P_2\right] \frac{\partial f}{\partial y}(x_2, y_2) - [P_1(y_2 - y_1) + P_2] \frac{\partial f}{\partial y}(x_1, y_2) - P_2 \frac{\partial f}{\partial y}(x_2, y_1) +$$

$$+ P_2 \frac{\partial f}{\partial y}(x_1, y_1) + \iint_D \left\{-\frac{1}{6}(x - x_2)^2(x - x_2)\frac{\partial^3 f}{\partial x^3} + \left[\frac{1}{4}(x - x_1)(x - x_2)\right.\right.$$

$$\left(y - \frac{y_1 + y_2}{2}\right) + \frac{1}{6}(x_2 - x_1)(x - x_1)(y - y_1) - Q_1(x - x_1) - Q_2\Big] \frac{\partial^3 f}{\partial x^2 \partial y} +$$

$$+ \left[\frac{1}{4}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right) + \frac{1}{6}(y_2 - y_1)(x - x_1)(y - y_1) -\right.$$

$$\left.\left. - P_1(y - y_1) - P_2\right] \frac{\partial^3 f}{\partial x \partial y^2} - \frac{1}{6}(y - y_1)^2(y - y_2)\frac{\partial^3 f}{\partial y^3}\right\} dx\, dy. \tag{37}$$

În această formulă intră constantele arbitrare P_1, P_2, Q_1, Q_2, însă se constată uşor că coeficienţii lor sînt nuli.

Se poate profita de aceasta pentru ca formula (37) să ia diferite forme.

Se pot impune condiţiile

$$
\begin{aligned}
A_0^0(x_1, y_2) &= 0, & A_0^0(x_2, y_1) &= 0,\\
A_1^0(x_1, y_2) &= 0, & A_1^0(x_2, y_1) &= 0.
\end{aligned}
\tag{38}
$$

care conduc la $P_1 = P_2 = Q_1 = Q_2 = 0$, şi se ajunge la formula de cubatură

$$
\begin{aligned}
\iint_D f\,dx\,dy &= \frac{(x_2-x_1)(y_2-y_1)}{12}\,[7f(x_2, y_2) + f(x_1, y_2) + f(x_2\, y_1) + 3f(x_1, y_1)] -\\
&\quad - \frac{(x_2-x_1)(y_2-y_1)}{6}\left[(x_2-x_1)\frac{\partial f}{\partial x}(x_2, y_2) + (y_2-y_1)\frac{\partial f}{\partial y}(x_2, y_2)\right] -\\
&\quad - \iint_D\left(\varphi_0\frac{\partial^3 f}{\partial x^3} + \varphi_1\frac{\partial^3 f}{\partial x^2\partial y} + \varphi_2\frac{\partial^3 f}{\partial x\partial y^2} + \varphi_3\frac{\partial^3 f}{\partial y^3}\right)dx\,dy,
\end{aligned}
\tag{39}
$$

în care

$$
\begin{aligned}
\varphi_0(x, y) &= \frac{1}{6}(x-x_1)^2(x-x_2),\\
\varphi_1(x, y) &= -\frac{1}{4}(x-x_1)(x-x_2)\left(y-\frac{y_1+y_2}{2}\right) - \frac{1}{6}(x_2-x_1)(x-x_1)(y-y_1),\\
\varphi_2(x, y) &= -\frac{1}{4}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right) - \frac{1}{6}(y_2-y_1)(x-x_1)(y-y_1),\\
\varphi_3(x, y) &= \frac{1}{6}(y-y_1)^2(y-y_2).
\end{aligned}
\tag{39'}
$$

Se poate căuta formula de forma (37), unde în membrul al doilea să nu figureze termeni în nodul (x_2, y_2). Vom lua atunci

$$P_1(y_2-y_1) + P_2 = \frac{1}{6}(x_2-x_1)(y_2-y_1)^2,$$

$$Q_1(x_2-x_1) + Q_2 = \frac{1}{6}(x_2-x_1)^2(y_2-y_1),$$

$$P_1 + Q_1 = \frac{7}{12}(x_2-x_1)(y_2-y_1)$$

şi vom mai adăuga

$$P_1 - Q_1 = 0.$$

Aceasta conduce la

$$P_1 = Q_1 = \frac{7}{24}(x_2-x_1)(y_2-y_1)$$

şi

$$P_2 = -\frac{1}{8}(x_2-x_1)(y_2-y_1)^2, \qquad Q_2 = -\frac{1}{8}(x_2-x_1)^2(y_2-y_1).$$

Ajungem astfel la formula de cubatură

$$
\begin{aligned}
\iint_D f\,dx\,dy &= \frac{(x_2-x_1)(y_2-y_1)}{3}\,[2f(x_1, y_2) + 2f(x_2, y_1) - f(x_1, y_1)] +\\
&\quad + \frac{(x_2-x)^2(y_2-y_1)}{24}\left[3\frac{\partial f}{\partial x}(x_1, y_2) - 4\frac{\partial f}{\partial x}(x_2, y_1) - 3\frac{\partial f}{\partial x}(x_1, y_1)\right] +\\
&\quad + \frac{(x_2-x_1)(y_2-y_1)^2}{24}\left[-4\frac{\partial f}{\partial y}(x_1, y_2) + 3\frac{\partial f}{\partial y}(x_2, y_1) - 3\frac{\partial f}{\partial y}(x_1, y_1)\right] -\\
&\quad - \iint_D\left(\varphi_0\frac{\partial^3 f}{\partial x^3} + \varphi_1\frac{\partial^3 f}{\partial x^2\partial y} + \varphi_2\frac{\partial^3 f}{\partial x\partial y^2} + \varphi_3\frac{\partial^3 f}{\partial y^3}\right)dx\,dy,
\end{aligned}
\tag{40}
$$

în care

$$
\begin{aligned}
\varphi_0(x, y) &= \frac{1}{6}(x-x_1)^2(x-x_2),\\
\varphi_1(x, y) &= -\frac{1}{4}(x-x_1)(x-x_2)\left(y-\frac{y_1+y_2}{2}\right) - \frac{1}{6}(x_2-x_1)(x-x_1)(y-y_1) +\\
&\quad + \frac{7}{24}(x_2-x_1)(y_2-y_1)(x-x_1) - \frac{1}{8}(x_2-x_1)^2(y_2-y_1),\\
\varphi_2(x, y) &= -\frac{1}{4}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right) - \frac{1}{6}(y_2-y_1)(y-y_1)(x-x_1) +\\
&\quad + \frac{7}{24}(x_2-x_1)(y_2-y_1)(y-y_1) - \frac{1}{8}(x_2-x_1)(y_2-y_1)^2,\\
\varphi_3(x, y) &= \frac{1}{6}(y-y_1)^2(y-y_2).
\end{aligned}
\tag{40'}
$$

Se observă că dacă funcţia $f(x, y)$ depinde numai de x, atît formula (39) cît şi formula (40) se reduc la formula lui N. Obreschkoff.

6. Este important să observăm că în formulele (39'), *funcţiile* $\varphi_0(x, y)$, $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ *sînt toate negative în dreptunghiul D.*

Pentru a dovedi aceasta, să facem schimbarea de variabile

$$x = \frac{x_1+x_2}{2} + X, \qquad y = \frac{y_1+y_2}{2} + Y$$

şi să notăm

$$h = \frac{x_2-x_1}{2}, \qquad k = \frac{y_2-y_1}{2}.$$

Vom avea

$$\varphi_1(x, y) = -\frac{Y}{4}(X+h)(X-h) - \frac{h}{3}(X+h)(Y+k) =$$

$$= -\frac{X+h}{12}[3Y(X-h)+4h(Y+k)\ ]=$$

$$= -\frac{X+h}{12}[3(XY+hk)+h(Y+k)].$$

Însă în dreptunghiul D, avem $X+h>0$, $Y+k>0$, $|XY|<hk$ şi prin urmare $\varphi_1(x,y)<0$.

La fel se arată că

$$\varphi_2(x,y) = -\frac{Y+k}{12}[3(XY+hk)+k(X+h)]$$

şi deci $\varphi_2(x,y)<0$ în dreptunghiul D.

7. Proprietatea precedentă a funcţiilor $\varphi_0(x,y), \ldots, \varphi_3(x,y)$ ne permite să evaluăm restul din formula de cubatură (39).

Fie M_{30}, M_{21}, M_{12}, M_{03} marginile superioare ale valorilor absolute ale lui $\frac{\partial^3 f}{\partial x^3}$, $\frac{\partial^3 f}{\partial x^2\partial y}$, $\frac{\partial^3 f}{\partial x\partial y^2}$, $\frac{\partial^3 f}{\partial y^3}$ în dreptunghiul D, adică

$$\left|\frac{\partial^3 f}{\partial x^3}\right| \leqslant M_{30}, \quad \left|\frac{\partial^3 f}{\partial x^2\partial y}\right| \leqslant M_{21}, \quad \left|\frac{\partial^3 f}{\partial x\partial y^2}\right| \leqslant M_{12}, \quad \left|\frac{\partial^3 f}{\partial y^3}\right| \leqslant M_{03}. \tag{41}$$

Se deduce că vom avea

$$|R| \leqslant -M_{30}\iint_D \varphi_0\,dx\,dy - M_{21}\iint_D \varphi_1\,dx\,dy - M_{12}\iint_D \varphi_2\,dx\,dy - M_{03}\iint_D \varphi_3\,dx\,dy.$$

Însă

$$\iint_D \varphi_0\,dx\,dy = -\frac{h^4k}{9}, \quad \iint_D \varphi_1\,dx\,dy = -\frac{h^3k^2}{3}, \quad \iint_D \varphi_2\,dx\,dy = -\frac{h^2k^3}{3}, \quad \iint_D \varphi_3\,dx\,dy = -\frac{hk^4}{9}.$$

Deducem că

$$|R| \leqslant \frac{hk}{9}(M_{30}h^3 + 3M_{21}h^2k + 3M_{12}hk^2 + M_{03}k^3). \tag{42}$$

§ 4. Cazul $n = 4$

8. Avem de tratat următoarea problemă la limită:

Să se determine funcţiile $\varphi_1(x,y)$, $\varphi_2(x,y)$, $\varphi_3(x,y)$ integrale ale ecuaţiilor cu derivate parţiale

$$\frac{\partial^4\varphi_1}{\partial x^3\partial y} = \alpha_1, \quad \frac{\partial^4\varphi_2}{\partial x^2\partial y^2} = \alpha_2, \quad \frac{\partial^4\varphi_3}{\partial x\partial y^3} = \alpha_3, \tag{43}$$

cu următoarele condiţii la limită

$$\frac{\partial\varphi_0}{\partial x} + \frac{\partial\varphi_1}{\partial y}\bigg|_{x=x_1} = 0, \quad \frac{\partial\varphi_0}{\partial x} + \frac{\partial\varphi_1}{\partial y}\bigg|_{x=x_2} = 0, \tag{44}$$

$$\frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial x\partial y} + \frac{\partial^2\varphi_2}{\partial y^2}\bigg|_{x=x_1} = 0, \quad \frac{\partial^2\varphi_0}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial x\partial y} + \frac{\partial^2\varphi_2}{\partial y_2}\bigg|_{x=x_2} = 0, \tag{45}$$

$$\frac{\partial^3\varphi_0}{\partial x^3} + \frac{\partial^3\varphi_1}{\partial x^2\partial y} + \frac{\partial^3\varphi_2}{\partial x\partial y^2} + \frac{\partial^3\varphi_3}{\partial y^3}\bigg|_{x=x_1} = 0, \quad \frac{\partial^3\varphi_0}{\partial x^3} + \frac{\partial^3\varphi_1}{\partial x^2\partial y} + \frac{\partial^3\varphi_2}{\partial x\partial y^2} + \frac{\partial^3\varphi_3}{\partial y^3}\bigg|_{x=x_2} = 0, \tag{46}$$

$$\frac{\partial\varphi_3}{\partial x} + \frac{\partial\varphi_4}{\partial y}\bigg|_{y=y_1} = 0, \quad \frac{\partial\varphi_3}{\partial x} + \frac{\partial\varphi_4}{\partial y}\bigg|_{y=y_2} = 0, \tag{47}$$

$$\frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_3}{\partial x\partial y} + \frac{\partial^2\varphi_4}{\partial y^2}\bigg|_{y=y_1} = 0, \quad \frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_3}{\partial x\partial y} + \frac{\partial^2\varphi_4}{\partial y^2}\bigg|_{y=y_2} = 0, \tag{48}$$

$$\frac{\partial^3\varphi_1}{\partial x^3} + \frac{\partial^3\varphi_2}{\partial x^2\partial y} + \frac{\partial^3\varphi_3}{\partial x\partial y^2} + \frac{\partial^3\varphi_4}{\partial y^3}\bigg|_{x=y_1} = 0, \quad \frac{\partial^3\varphi_1}{\partial x^3} + \frac{\partial^3\varphi_2}{\partial x^2\partial y} + \frac{\partial^3\varphi_3}{\partial x\partial y^2} + \frac{\partial^3\varphi_4}{\partial y^3}\bigg|_{y=y_2} = 0, \tag{49}$$

ştiind că

$$\varphi_0(x,y) = \frac{\alpha_0}{24}(x-x_1)^3(x-x_2), \tag{50}$$

$$\varphi_4(x,y) = \frac{\alpha_4}{24}(y-y_1)^3(y-y_2). \tag{51}$$

1°. Avem

$$\frac{\partial\varphi_0}{\partial x} = \frac{\alpha_0}{24}[3(x-x_1)^2(x-x_2) + (x-x_1)^3]$$

şi din condiţiile la limită (44), deducem că

$$\frac{\partial\varphi_1}{\partial y}\bigg|_{x=x_1} = 0, \quad \frac{\partial\varphi_1}{\partial y}\bigg|_{x=x_2} = -\frac{\alpha_0}{24}(x_2-x_1)^3. \tag{52}$$

Să integrăm ecuaţia

$$\frac{\partial^3}{\partial x^3}\left(\frac{\partial\varphi_1}{\partial y}\right) = \alpha_1$$

cu condiţiile (52). Vom avea

$$\frac{\partial\varphi_1}{\partial y} = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x - \frac{x_1+x_2}{2}\right) + A_1''(y)(x-x_1)(x-x_2) +$$

$$+ A_2(y)(x-x_1) + A_3(y),$$

unde $A_1''(y)$ este o funcţie arbitrară, iar $A_2(y)$, $A_3(y)$ sînt funcţii care se determină prin condiţiile (52). Vom avea

$$A_3(y) = 0, \qquad A_2(y) = -\frac{\alpha_0}{24}(x_2-x_1)^2,$$

şi prin urmare

$$\frac{\partial\varphi_1}{\partial y} = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right) + A_1''(y)(x-x_1)(x-x_2) - \tag{53}$$

$$- \frac{\alpha_0}{24}(x_2-x_1)^2(x-x_1).$$

2°. Avem

$$\frac{\partial^2\varphi_0}{\partial x^2}=\frac{\alpha_0}{4}[(x-x_1)(x-x_2)+(x-x_1)^2],$$

$$\frac{\partial^2\varphi_1}{\partial x\partial y}=\frac{\alpha_1}{3}\left(x-\frac{x_1+x_2}{2}\right)^2+\frac{\alpha_1}{6}(x-x_1)(x-x_2)+$$

$$+2A_1''(y)\left(x-\frac{x_1+x_2}{2}\right)-\frac{\alpha_0}{24}(x_2-x_1)^2.$$

Condițiile la limită (45) arată că

$$\left.\frac{\partial^2\varphi_2}{\partial y^2}\right|_{x=x_1}=-\left[\frac{\alpha_1}{12}(x_2-x_1)^2-A_1''(y)(x_2-x_1)-\frac{\alpha_0}{24}(x_2-x_1)^2\right],$$

$$\left.\frac{\partial^2\varphi_2}{\partial y^2}\right|_{x=x_2}=-\left[\frac{\alpha_0}{4}(x_2-x_1)^2+\frac{\alpha_1}{12}(x_2-x_1)^2+A_1''(y)(x_2-x_1)-\right.$$

$$\left.-\frac{\alpha_0}{24}(x_2-x_1)^2\right],$$

adică

$$\left.\frac{\partial^2\varphi_2}{\partial y^2}\right|_{x=x_1}=\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2+A_1''(y)(x_2-x_1),$$

$$\left.\frac{\partial^2\varphi_2}{\partial y^2}\right|_{x=x_2}=\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2-A_1''(y)(x_2-x_1)-\frac{\alpha_0}{4}(x_2-x_1)^2. \tag{54}$$

Să integrăm acum ecuația

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\varphi_2}{\partial y^2}\right)=\alpha_2$$

cu condițiile (54). Vom avea

$$\frac{\partial^2\varphi_2}{\partial y^2}=\frac{\alpha_2}{2}(x-x_1)(x-x_2)+B_1(y)(x-x_1)+B_2(y)$$

și funcțiile $B_1(y)$, $B_2(y)$ sînt determinate de ecuațiile

$$\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2+A_1''(y)(x_2-x_1)=B_2(y),$$

$$-\frac{\alpha_0}{4}(x_2-x_1)^2+\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2-A_1''(y)(x_2-x_1)=$$

$$=B_1(y)(x_2-x_1)+B_2(y),$$

de unde rezultă că

$$B_1(y)=-\frac{\alpha_0}{4}(x_2-x_1)-2A_1''(y)$$

și prin urmare

$$\frac{\partial^2\varphi_2}{\partial y^2}=\frac{\alpha_2}{2}(x-x_1)(x-x_2)-\left[\frac{\alpha_0}{4}(x_2-x_1)+2A_1''(y)\right](x-x_1)+$$

$$+\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2+A_1''(y)(x_2-x_1),$$

adică

$$\frac{\partial^2\varphi_2}{\partial y^2}=\frac{\alpha_2}{2}(x-x_1)(x-x_2)-2A_1''(y)\left(x-\frac{x_1+x_2}{2}\right)+$$

$$+\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2-\frac{\alpha_0}{4}(x_2-x_1)(x-x_1). \tag{55}$$

3°. Avem

$$\frac{\partial^3\varphi_0}{\partial x^3}=\frac{\alpha_0}{4}[3(x-x_1)+(x-x_2)],$$

$$\frac{\partial^3\varphi_1}{\partial x^2\partial y}=\alpha_1\left(x-\frac{x_1+x_2}{2}\right)+2A_1''(y),$$

$$\frac{\partial^3\varphi_2}{\partial x\partial y^2}=\alpha_2\left(x-\frac{x_1+x_2}{2}\right)-2A_1''(y)-\frac{\alpha_3}{4}(x_2-x_1).$$

Condițiile (46) arată că

$$\left.\frac{\partial^3\varphi_3}{\partial y^3}\right|_{x=x_1}=\frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1),$$

$$\left.\frac{\partial^3\varphi_3}{\partial y^3}\right|_{x=x_2}=-\frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1). \tag{56}$$

Să integrăm ecuația

$$\frac{\partial}{\partial x}\left(\frac{\partial^3\varphi_3}{\partial y^3}\right)=\alpha_3$$

cu condițiile (56). Vom avea

$$\frac{\partial^3\varphi_3}{\partial y^3}=\alpha_3(x-x_1)+C_1(y).$$

Condițiile (56) arată că

$$\frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1)=C_1(y).$$

$$-\frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1)=\alpha_3(x_2-x_1)+C_1(y).$$

Funcția $C_1(y)$ este astfel determinată. În plus se stabilește între α_0, α_1, α_2, α_3 relația

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3=0 \tag{57}$$

și vom avea

$$C_1(y)=-\frac{\alpha_3}{2}(x_2-x_1),$$

adică

$$\frac{\partial^3\varphi_3}{\partial y^3}=\alpha_3\left(x-\frac{x_1+x_2}{2}\right). \tag{58}$$

4°. Să ţinem seama acum de formula (51) şi să integrăm din nou ecuaţiile (43) cu condiţiile (47), (48), (49).

Avem

$$\frac{\partial \varphi_4}{\partial y} = \frac{\alpha_4}{24}\left[3(y-y_1)^2(y-y_2) + (y-y_1)^3\right].$$

Condiţiile (47) arată că

$$\left.\frac{\partial \varphi_3}{\partial x}\right|_{y=y_1} = 0, \qquad \left.\frac{\partial \varphi_3}{\partial x}\right|_{y=y_2} = -\frac{\alpha_4}{24}(y_2-y_1)^3. \tag{59}$$

Ecuaţia

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial \varphi_3}{\partial x}\right) = \alpha_3$$

se integrează cu condiţiile (59) şi avem

$$\frac{\partial \varphi_3}{\partial x} = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(y - \frac{y_1+y_2}{2}\right) +$$
$$+ a_1''(x)(y-y_1)(y-y_2) + a_2(x)(y-y_1) + a_3(x),$$

unde $a_1''(x)$ este o funcţie arbitrară, iar $a_2(x)$, $a_3(x)$ sînt funcţii care se determină cu condiţiile (59). Vom avea

$$a_3(x) = 0, \qquad a_2(x) = -\frac{\alpha_4}{24}(y_2-y_1)^2,$$

şi deci

$$\frac{\partial \varphi_3}{\partial x} = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(y - \frac{y_1+y_2}{2}\right) +$$
$$+ a_1''(x)(y-y_2)(y-y_1) - \frac{\alpha_4}{24}(y_2-y_1)^2(y-y_1). \tag{60}$$

Integrăm în raport cu x, pentru a obţine pe $\varphi_3(x, y)$. Vom avea

$$\varphi_3(x, y) = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(y - \frac{x_1+x_2}{2}\right)(x-x_1) + a_1''(x)(y-y_1)(y-y_2) -$$
$$- \frac{\alpha_4}{24}(y_2-y_1)^2(y-y_1)(x-x_1) + F(y), \tag{61}$$

unde $F(y)$ este o funcţie arbitrară, care se determină scriind că $\varphi_3(x, y)$ verifică ecuaţia (58). Vom avea

$$\alpha_3(x-x_1) + F'''(y) = \alpha_3\left(x - \frac{x_1+x_2}{2}\right)$$

şi deci

$$F'''(y) = -\alpha_3\frac{x_1+x_2}{2}.$$

Se deduce că:

$$F(y) = -\frac{\alpha_3}{12}(x_2-x_1)(y-y_1)(y-y_2)\left(y - \frac{y_1+y_2}{2}\right) +$$
$$+ P_1(y-y_1)(y-y_2) + P_2(y-y_1) + P_3,$$

unde P_1, P_2, P_3 sînt constante arbitrare.

Purtînd pe $F(y)$ în formula (61), vom avea

$$\varphi_3(x, y) = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(x - \frac{x_1+x_2}{2}\right)\left(y - \frac{y_1+y_2}{2}\right) +$$
$$+ a_1'(x)(y-y_1)(y-y_2) - \frac{\alpha_4}{24}(y_2-y_1)^2(x-x_1)(y-y_1) + \tag{62}$$
$$+ P_1(y-y_1)(y-y_2) + P_2(y-y_1) + P_3.$$

5°. Avem

$$\frac{\partial^3 \varphi_4}{\partial y^2} = \frac{\alpha_4}{4}\left[(y-y_1)(y-y_2) + (y-y_1)^2\right],$$

$$\frac{\partial^2 \varphi_3}{\partial x \partial y} = \frac{\alpha_3}{3}\left(y - \frac{y_1+y_2}{2}\right)^2 + \frac{\alpha_3}{6}(y-y_1)(y-y_2) + 2a_1''(x)\left(y - \frac{y_1+y_2}{2}\right) -$$
$$- \frac{\alpha_4}{24}(y_2-y_1)^2$$

şi condiţiile (48) ne dau

$$\left.\frac{\partial^2 \varphi_2}{\partial x^2}\right|_{y=y_1} = -\left[\frac{\alpha_3}{12}(y_2-y_1)^2 - a_1''(x)(y_2-y_1) - \frac{\alpha_4}{24}(y_2-y_1)^2\right],$$

$$\left.\frac{\partial^2 \varphi_2}{\partial x^2}\right|_{y=y_2} = -\left[\frac{\alpha_3}{12}(y_2-y_1)^2 + a_1''(x)(y_2-y_1) - \frac{\alpha_4}{24}(y_2-y_1)^2 + \frac{\alpha_4}{4}(y_2-y_1)^2\right],$$

adică

$$\left.\frac{\partial^2 \varphi_2}{\partial x^2}\right|_{y=y_1} = \frac{\alpha_4 - 2\alpha_3}{24}(y_2-y_1)^2 + a_1''(x)(y_2-y_1),$$
$$\left.\frac{\partial^2 \varphi_2}{\partial x^2}\right|_{y=y_2} = \frac{\alpha_4 - 2\alpha_3}{24}(y_2-y_1)^2 - a_1''(x)(y_2-y_1) - \frac{\alpha_4}{4}(y_2-y_1)^2. \tag{63}$$

Integrînd ecuaţia

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 \varphi_2}{\partial x^2}\right) = \alpha_2$$

cu condiţiile (63), avem

$$\frac{\partial^2 \varphi_2}{\partial x^3} = \alpha_2(y-y_1)(y-y_2) + b_1(x)(y-y_1) + b_2(x),$$

unde funcţiile $b_1(x)$, $b_2(x)$ sînt determinate de ecuaţiile

$$\frac{\alpha_4 - 2\alpha_3}{24}(y_2-y_1)^2 + a_1''(x)(y_2-y_1) = b_2(x),$$

$$-\frac{\alpha_4}{4}(y_2-y_1)^2 + \frac{\alpha_4 - 2\alpha_3}{24}(y_2-y_1)^2 - a_1''(x)(y_2-y_1) = b_1(x)(y_2-y_1) + b_2(x),$$

de unde rezultă că

$$-\frac{\alpha_4}{4}(y_2-y_1)-2a_1''(x)=b_1(x).$$

Vom avea deci

$$\frac{\partial^2\varphi_2}{\partial x^2}=\frac{\alpha_2}{2}(y-y_1)(y-y_2)-\left[\frac{\alpha_4}{4}(y_2-y_1)+2a_1''(x)\right](y-y_1)+$$
$$+\frac{\alpha_4-2\alpha_3}{24}(y_2-y_1)^2+a_1''(x)(y_2-y_1),$$

sau

$$\frac{\partial^2\varphi_2}{\partial x^2}=\frac{\alpha_2}{2}(y-y_1)(y-y_2)-2a_1''(x)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+\frac{\alpha_4-2\alpha_3}{24}(y_2-y_1)^2-\frac{\alpha_4}{4}(y_2-y_1)(y-y_1). \tag{64}$$

Pentru a obține pe $\varphi_2(x, y)$ integrăm de două ori în raport cu x și obținem

$$\varphi_2(x, y)=\frac{\alpha_2}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-2a_1(x)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+\frac{\alpha_4-2\alpha_3}{48}(y_2-y_1)^2(x-x_1)(x-x_2)- \tag{65}$$
$$-\frac{\alpha_4}{8}(y_2-y_1(y-y_1)(x-x_1)(x-x_2)+G_1(y)(x-x_1)+G_2(y),$$

unde $G_1(y)$ și $G_2(y)$ sînt funcții arbitrare care se determină scriind că derivata $\frac{\partial^2\varphi_2}{\partial y^2}$ coincide cu membrul al doilea al formulei (55). Vom avea

$$\frac{\alpha_2}{2}(x-x_1)(x-x_2)+G_1''(y)(x-x_1)+G_2''(y)=\frac{\alpha_2}{2}(x-x_2)(x-x_2)-$$
$$-2A_1''(y)\left[x-x_1-\frac{x_2-x_1}{2}\right]+\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2-\frac{\alpha_0}{4}(x_2-x_1)(x-x_1),$$

de unde rezultă că

$$G_1''(y)=-2A_1''(y),$$
$$G_2''(y)=A_1''(y)(x_2-x_1)+\frac{\alpha_0-2\alpha_1}{24}(x_2-x_1)^2-\frac{\alpha_0}{4}(x_2-x_1)(x-x_1).$$

Integrînd avem

$$G_1(y)=-2A_1(y)+Q_1(y-y_1)+Q_2,$$
$$G_2(y)=A_1(y)(x_2-x_1)+\frac{\alpha_0-2\alpha_1}{48}(x_2-x_1)^2(y-y_1)(y-y_2)-$$
$$-\frac{\alpha_0}{8}(x_2-x_1)(x-x_1)(y-y_1)(y-y_2)+Q_3(y-y_1)+Q_4,$$

unde Q_1, Q_2, Q_3, Q_4 sînt constante arbitrare.

Înlocuind pe $G_1(y)$ și pe $G_2(y)$ în formula (65), obținem

$$\varphi_2(x, y)=\frac{\alpha_2}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-2a_1(x)\left(y-\frac{y_1+y_2}{2}\right)-$$
$$-2A_1(y)\left(x-\frac{x_1+x_2}{2}\right)+\frac{\alpha_4-2\alpha_3}{48}(y_2-y_1)^2(x-x_1)(x-x_2)+$$
$$+\frac{\alpha_0-2\alpha_1}{48}(x_2-x_1)^2(y-y_1)(y-y_2)-\frac{\alpha_4}{8}(y_2-y_1)(y-y_1)(x-x_1)(x-x_2)-$$
$$\frac{\alpha_0}{8}(x_2-x_1)(x-x_1)(y-y_1)(y-y_2)+$$
$$+Q_1(x-x_1)(y-y_1)+Q_2(x-x_1)+Q_3(y-y_1)+Q_4. \tag{66}$$

6⁰. Avem

$$\frac{\partial^3\varphi_4}{\partial y^3}=\frac{\alpha_4}{4}[3(y-y_1)+(y-y_2)],$$
$$\frac{\partial^3\varphi_3}{\partial x\partial y^2}=\alpha_3\left(y-\frac{y_1+y_2}{2}\right)+2a_1''(x),$$
$$\frac{\partial^3\varphi_2}{\partial x^2\partial y}=\alpha_2\left(y-\frac{y_1+y_2}{2}\right)-2a_1''(x)-\frac{\alpha_4}{4}(y_2-y_1).$$

Din condițiile (49) rezultă că

$$\left.\frac{\partial^3\varphi_1}{\partial x^3}\right|_{y=y_1}=\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1),$$
$$\left.\frac{\partial^3\varphi_1}{\partial x^3}\right|_{y=y_2}=-\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1). \tag{67}$$

Integrînd ecuația

$$\frac{\partial}{\partial y}\left(\frac{\partial^3\varphi_1}{\partial x^3}\right)=\alpha_1$$

cu condițiile (67), obținem

$$\frac{\partial^3\varphi_1}{\partial x^3}=\alpha_1(y-y_1)+c_1(x), \tag{68}$$

unde condițiile (67) determină funcția $c_1(x)$ și stabilesc o relație între α_1, α_2, α_3, α_4. Vom avea

$$\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1)=c_1(x),$$
$$-\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1)=\alpha_1(y-y_1)+c_1(x),$$

de unde rezultă că

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=0. \tag{69}$$

Ţinînd seama de aceasta, vom avea

$$\frac{\partial^3 \varphi_1}{\partial x^3} = \alpha_1\left(y - \frac{y_1 + y_2}{2}\right). \tag{70}$$

Pentru a obţine pe $\varphi_1(x, y)$ integrăm de trei ori în raport cu x şi vom avea

$$\begin{aligned}\varphi_1(x, y) &= \frac{\alpha_1}{6}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) + \\ &+ H_1(y)(x - x_1)(x - x_2) + H_2(y)(x - x_1) + H_3(y),\end{aligned} \tag{71}$$

unde $H_1(y)$, $H_2(y)$, $H_3(y)$ sînt funcţii arbitrare care se determină scriind că $\frac{\partial \varphi_1}{\partial y}$ coincide cu membrul al doilea al formulei (53).

Vom avea

$$\begin{aligned}&\frac{\alpha_1}{6}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right) + H_1'(y)(x - x_1)(x - x_2) + \\ &+ H_2'(y)(x - x_1) + H_3'(y) = \frac{\alpha_1}{6}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right) + \\ &+ A_1''(y)(x - x_1)(x - x_2) - \frac{\alpha_0}{24}(x_2 - x_1)^2(x - x_1),\end{aligned}$$

de unde rezultă ecuaţiile

$$\begin{aligned}H_1'(y) &= A_1''(y), \\ H_2'(y) &= -\frac{\alpha_0}{24}(x_2 - x_1)^2, \\ H_3'(y) &= 0,\end{aligned}$$

şi deci

$$\begin{aligned}H_1(y) &= A_1'(y) + R_1, \\ H_2(y) &= -\frac{\alpha_0}{24}(x_2 - x_1)^2(y - y_1) + R_2, \\ H_3(y) &= R_3,\end{aligned}$$

unde R_1, R_2, R_3 sînt constante arbitrare.

Purtînd pe $H_1(y)$, $H_2(y)$, $H_3(y)$ în formula (71), vom avea

$$\begin{aligned}\varphi_1(x, y) &= \frac{\alpha_1}{6}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) + \\ &+ A_1'(y)(x - x_1)(x - x_2) - \frac{\alpha_0}{24}(x_2 - x_1)^2(x - x_1)(y - y_1) + \\ &+ R_1(x - x_1)(x - x_2) + R_2(x - x_1) + R_3.\end{aligned} \tag{72}$$

9. Funcţiile $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ au fost determinate. Vom fixa acum constantele α_0, α_1, α_2, α_3, α_4 astfel ca condiţiile (57), (69) să fie îndeplinite şi ca formula (66) care determină pe $\varphi_2(x, y)$ să fie mai simplă. Vom alege

$$\alpha_0 = 1, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = -2, \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = 1 \tag{73}$$

şi cu această alegere vom avea

$$\begin{aligned}\varphi_1(x, y) &= \frac{1}{12}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) - \\ &- \frac{1}{24}(x_2 - x_1)^2(x - x_1)(y - y_1) + A_1'(y)(x - x_1)(x - x_2) + \\ &+ R_1(x - x_1)(x - x_2) + R_2(x - x_1) + R_3 \\ \varphi_2(x, y) &= -\frac{1}{2}(x - x_1)(x - x_2)(y - y_1)(y - y_2) - \\ &- \frac{1}{8}(x_2 - x_1)(x - x_1)(y - y_1)(y - y_2) - \\ &- \frac{1}{8}(y_2 - y_1)(y - y_1)(x - x_1)(x - x_2) - \\ &- 2A_1(y)\left(x - \frac{x_1 + x_2}{2}\right) - 2a_1(x)\left(y - \frac{y_1 + y_2}{2}\right) + \\ &+ Q_1(x - x_1)(y - y_1) + Q_2(x - x_1) + Q_3(y - y_1) + Q_4 \\ \varphi_3(x, y) &= \frac{1}{12}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) - \\ &- \frac{1}{24}(y_2 - y_1)^2(x - x_1)(y - y_1) + a_1'(x)(y - y_1)(y - y_2) + \\ &+ P_1(y - y_1), (y - y_2) + P_2(y - y_1) + P_3.\end{aligned} \tag{74}$$

Rămîne să introducem noi condiţii care să determine constantele P_1, P_2, P_3, Q_1, Q_2, Q_3, Q_4, R_1, R_2, R_3 şi să vedem rolul funcţiilor arbitrare $A_1(y)$ şi $a_1(x)$ în aceste formule.

10. Formula de cubatură (17) se scrie în cazul $n = 4$ sub forma

$$\begin{aligned}&\iint_D \left(\varphi_0 \frac{\partial^4 f}{\partial x^4} + \varphi_1 \frac{\partial^4 f}{\partial x^3 \partial y} + \varphi_2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + \varphi_3 \frac{\partial^4 f}{\partial x \partial y^3} + \varphi_4 \frac{\partial^4 f}{\partial y^4}\right) dx\, dy = \\ &= A_0^0(x_2, y_2)\frac{\partial^2 f}{\partial x^2}(x_2, y_2) + A_1^0(x_2, y_2)\frac{\partial^2 f}{\partial x \partial y}(x_2, y_2) + A_2^0(x_2, y_2)\frac{\partial^2 f}{\partial y^2}(x_2, y_2) - \\ &- A_0^0(x_1, y_2)\frac{\partial^2 f}{\partial x^2}(x_1, y_2) - A_1^0(x_1, y_2)\frac{\partial^2 f}{\partial x \partial y}(x_1, y_2) - A_2^0(x_1, y_2)\frac{\partial f^2}{\partial y^2}(x_1, y_2) - \\ &- A_0^0(x_2, y_1)\frac{\partial^2 f}{\partial x^2}(x_2, y_1) - A_1^0(x_2, y_1)\frac{\partial^2 f}{\partial x \partial y}(x_2, y_1) - A_2^0(x_2, y_1)\frac{\partial^2 f}{\partial y^2}(x_2, y_1) + \\ &+ A_0^0(x_1, y_1)\frac{\partial^2 f}{\partial x^2}(x_1, y_1) + A_1^0(x_1, y_1)\frac{\partial^2 f}{\partial x \partial y}(x_1, y_1) + A_2^0(x_1, y_1)\frac{\partial^2 f}{\partial y^2}(x_1, y_1) -\end{aligned}$$

$$
\begin{aligned}
&- A_0^1(x_2, y_2)\frac{\partial f}{\partial x}(x_2, y_2) - A_1^1(x_2, y_2)\frac{\partial f}{\partial y}(x_2, y_2) + \\
&+ A_0^1(x_1, y_2)\frac{\partial f}{\partial x}(x_1, y_2) + A_1^1(x_1, y_2)\frac{\partial f}{\partial y}(x_1, y_2) + \\
&+ A_0^1(x_2, y_1)\frac{\partial f}{\partial x}(x_2, y_1) + A_1^1(x_2, y_1)\frac{\partial f}{\partial y}(x_2, y_1) - \\
&- A_0^1(x_1, y_1)\frac{\partial f}{\partial x}(x_1, y_1) - A_1^1(x_1, y_1)\frac{\partial f}{\partial y}(x_1, y_1) + \\
&+ A_0^2(x_2, y_2) f(x_2, y_2) + A_0^2(x_1, y_2) f(x_1, y_2) + \\
&+ A_0^2(x_2, y_1) f(x_2, y_1) + A_0^2(x_1, y_1) f(x_1, y_1) + \iint_D f\,dx\,dy,
\end{aligned}
\tag{75}
$$

unde

$$
\begin{gathered}
A_0^0(x, y) = \varphi_1(x, y),\ A_1^0(x, y) = \varphi_2(x, y),\ A_2^0(x, y) = \varphi_3(x, y), \\
A_0^1(x, y) = \frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y},\ A_1^1(x, y) = \frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_3}{\partial y}, \\
A_0^2(x, y) = \frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial x \partial y} + \frac{\partial^2 \varphi_3}{\partial y^2}.
\end{gathered}
\tag{76}
$$

Introducem noi condiții la limită

$$
\begin{gathered}
A_0^0(x_1, y_2) = 0,\ A_1^0(x_1, y_2) = 0,\ A_2^0(x_1, y_2) = 0, \\
A_0^0(x_2, y_1) = 0,\ A_1^0(x_2, y_1) = 0,\ A_2^0(x_2, y_1) = 0.
\end{gathered}
\tag{77}
$$

$$
\begin{gathered}
A_0^1(x_1, y_2) = 0,\ A_1^1(x_1, y_2) = 0, \\
A_0^1(x_2, y_1) = 0,\ A_1^1(x_2, y_1) = 0.
\end{gathered}
\tag{77'}
$$

$$
\begin{gathered}
A_0^2(x_1, y_2) = 0, \\
A_0^2(x_2, y_1) = 0.
\end{gathered}
\tag{77''}
$$

Vom vedea mai departe că aceste condiții determină constantele $P_1, P_2, P_3, Q_1, Q_2, Q_3, Q_4, R_1, R_2, R_3$.

Ecuațiile (75) ne dau

$$R_2 = 0,\ R_3 = 0,\ P_2 = 0,\ P_3 = 0 \tag{78}$$

și

$$
\left.
\begin{aligned}
A_1(y_2)(x_2 - x_1) - a_1(x_1)(y_2 - y_1) + Q_3(y_2 - y_1) + Q_4 = 0 \\
- A_1(y_1)(x_2 - x_1) + a_1(x_2)(y_2 - y_1) + Q_2(x_2 - x_1) + Q_4 = 0
\end{aligned}
\right\}
\tag{79}
$$

Avem

$$\frac{\partial \varphi_1}{\partial x} = \frac{1}{6}\left(x - \frac{x_1 + x_2}{2}\right)^2\left(y - \frac{y_1 + y_2}{2}\right) + \frac{1}{12}(x - x_1)(x - x_2)\left(y - \frac{y_1 + y_2}{2}\right) -$$

$$- \frac{1}{24}(x_2 - x_1)^2(y - y_1) + 2A_1(y)\left(-x\frac{x_1 + x_2}{2}\right) + 2R_1\left(x - \frac{x_1 + x_2}{2}\right),$$

$$
\begin{aligned}
\frac{\partial \varphi_2}{\partial y} = &- (x - x_1)(x - x_2)\left(y - \frac{y_1 + y_2}{2}\right) - \frac{1}{4}(x_2 - x_1)(x - x_1)\left(y - \frac{y_1 + y_2}{2}\right) - \\
&- \frac{1}{8}(y_2 - y_1)(x - x_1)(x - x_2) - \\
&- 2A_1'(y)\left(x - \frac{x_1 + x_2}{2}\right) - 2a_1(x) + Q_1(x - x_1) + Q_3.
\end{aligned}
$$

Rezultă că primele două ecuații (76) sînt

$$
\left.
\begin{aligned}
&- \frac{1}{48}(x_2 - x_1)^2(y_2 - y_1) - 2a_1(x_1) + Q_3 - R_1(x_2 - x_1) = 0 \\
&\frac{5}{48}(x_2 - x_1)^2(y_2 - y_1) - 2a_1(x_2) + Q_3 + R_1(x_2 - x_1) + Q_1(x_2 - x_1) = 0
\end{aligned}
\right\}
\tag{80}
$$

De asemenea avem

$$
\begin{aligned}
\frac{\partial \varphi_2}{\partial x} = &- (y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right) - \frac{1}{8}(x_2 - x_1)(y - y_1)(y - y_2) - \\
&- \frac{1}{4}(y_2 - y_1)(y - y_1)\left(x - \frac{x_1 + x_2}{2}\right) - \\
&- 2A_1(y) - 2a_1'(x)\left(y - \frac{y_1 + y_2}{2}\right) + Q_1(y - y_1) + Q_2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_3}{\partial y} = &\frac{1}{6}\left(y - \frac{y_1 + y_2}{2}\right)^2\left(x - \frac{x_1 + x_2}{2}\right) + \frac{1}{12}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right) - \\
&- \frac{1}{24}(y_2 - y_1)^2(x - x_1) + 2a_1'(x)\left(y - \frac{y_1 + y_2}{2}\right) + 2P_1\left(y - \frac{y_1 + y_2}{2}\right),
\end{aligned}
$$

de unde rezultă că celelalte ecuații (76) sînt

$$
\left.
\begin{aligned}
&\frac{5}{48}(x_2 - x_1)(y_2 - y_1)^2 - 2A_1(y_2) + P_1(y - y_1) + Q_2 + Q_1(y_2 - y_1) = 0 \\
&- \frac{1}{48}(x_2 - x_1)(y_2 - y_1)^2 - 2A_1(y_1) - P_1(y_2 - y_1) + Q_2 = 0.
\end{aligned}
\right\}
\tag{81}
$$

În fine avem

$$\frac{\partial^2 \varphi_1}{\partial x^2} = \frac{1}{2}\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) + 2A_1'(y) + 2R_1$$

$$
\begin{aligned}
\frac{\partial^2 \varphi_2}{\partial x \partial y} = &- 2\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) - \frac{1}{4}(x_2 - x_1)\left(y - \frac{y_1 + y_2}{2}\right) - \\
&- \frac{1}{4}(y_2 - y_1)\left(x - \frac{x_1 + x_2}{2}\right) + 2A_1'(y) - 2a_1'(x) + Q_1
\end{aligned}
$$

$$\frac{\partial^2 \varphi_3}{\partial y^2} = \frac{1}{2}\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) + 2a_1'(x) + 2P_1$$

de unde rezultă că ecuațiile (77) *se reduc la una singură*

$$\frac{(x_2 - x_1)(y_2 - y_1)}{4} + 2P_1 + Q_1 + 2R_1 = 0. \tag{82}$$

Rămîne să rezolvăm ecuațiile (79), (80), (81), (82), în raport cu P_1, Q_1, Q_2, Q_3, Q_4, R_1. Vom vedea că *acest sistem este compatibil.*

11. Din ecuațiile (79) se deduce

$$\left.\begin{aligned} Q_3 &= - A_1(y_2)\frac{x_2 - x_1}{y_2 - y_1} + a_1(x) - \frac{Q_4}{y_2 - y_1}, \\ Q_2 &= A_1(y) - a_1(x_2)\frac{y_2 - y_1}{x_2 - x_1} - \frac{Q_4}{x_2 - x_1}. \end{aligned}\right\} \tag{83}$$

Din ecuațiile (80), se deduce ținînd seamă de prima ecuație (83)

$$\left.\begin{aligned} R_2 &= - \frac{1}{48}(x_2 - x_1)(y_2 - y_1) - \frac{Q_2(x_1)}{x_2 - x_1} - \frac{A_1(y_2)}{y_2 - y_1} - \frac{Q_4}{(x_2 - x_1)(y_2 - y_1)}, \\ Q_1 &= - \frac{1}{12}(x_2 - x_1)(x_2 - y_1) + \frac{2a_1(x_2)}{x_2 - x_1} + \frac{2A_1(y_2)}{y_2 - y_1} + \frac{2Q_4}{(x_2 - x_1)(y_2 - y_1)}. \end{aligned}\right\} \tag{84}$$

A doua ecuație (81) ne dă

$$P_1 = - \frac{1}{48}(x_2 - x_1)(y_2 - y_1) - \frac{a_1(x_2)}{x_2 - x_1} - \frac{A_1(y_1)}{y_2 - y_1} - \frac{Q_4}{(x_2 - x_1)(y_2 - y_1)}. \tag{85}$$

Prima ecuație (81) se reduce la o identitate în baza formulelor (83), (84), (85).

În fine ecuația (82) dă

$$Q_4 = \frac{(x_2 - x_1)^2 (y_2 - y_1)^2}{24} - [a_1(x_1)(y_2 - y_1) + A_1(y_1)(x_2 - x_1)]. \tag{86}$$

Purtînd pe Q_4 în formulele (83), (84), (85), avem

$$\left.\begin{aligned} Q_1 &= 2\,\frac{a_1(x_2) - a_1(x_1)}{x_2 - x_1} + 2\,\frac{A_1(y_2) - A_1(y_1)}{y_2 - y_1}, \\ Q_2 &= - \frac{(x_2 - x_1)(y_2 - y_1)^2}{24} + 2A_1(y_1) + [a_1(x_1) - a_1(x_2)]\frac{y_2 - y_1}{x_2 - x_1}, \\ Q_3 &= - \frac{(x_2 - x_1)^2 (y_2 - y_1)}{24} + 2a_1(x_1) + [A_1(y_1) - A_1(y_2)]\frac{x_2 - x_1}{y_2 - y_1}, \\ P_1 &= - \frac{(x_2 - x_1)\ (y_2 - y_1)}{16} - \frac{a_1(x_2) - a_1(x_1)}{x_2 - x_1}, \\ R_1 &= - \frac{(x_2 - x_1)\ (y_2 - y_1)}{16} - \frac{A_1(y_2) - A_1(y_1)}{y_2 - y_1}. \end{aligned}\right\} \tag{87}$$

12. Ținînd seamă de formulele (86) și (87), forma definitivă a funcțiilor $\varphi_1(x_1 y)$, $\varphi_2(x,y)$, $\varphi^{\varepsilon}(x,y)$ este

$$\left.\begin{aligned} \varphi_1(x, y) = {} & \frac{1}{12}(x - x_1)(x - x_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) - \\ & - \frac{1}{24}(x_2 - x_1)^2 (x - x_1)(y - y_1) - \\ & - \frac{1}{16}(x_2 - x_1)(y_2 - y_1)(x - x_1)(x - x_2) + \\ & + \left[A_1'(y) - \frac{A_1(y_2) - A_1(y_1)}{y_2 - y_1}\right](x - x_1)(x - x_2), \\ \varphi_1(x_1\ y) = {} & - \frac{1}{2}(x - x_1)(x - x_2)(y - y_1)(y - y_2) - \\ & - \frac{1}{8}(x_2 - x_1)(x - x_1)(y - y_1)(y - y_2) - \\ & - \frac{1}{8}(y_2 - y_1)(y - y_1)(x - x_1)(x - x_2) - \\ & - \frac{1}{24}(y_2 - y_1)^2 (x_2 - x_1)(x - x_1) - \\ & - \frac{1}{24}(x_2 - x_1)^2 (y_2 - y_1)(y - y_1) + \\ & + \frac{1}{24}(x_2 - x_1)^2 (y_2 - y_1)^2 - 2A_1(y)\left(x - \frac{x_1 + x_2}{2}\right) - \\ & - 2a_1(x)\left(y - \frac{y_1 + y_2}{2}\right) + 2A_1(y_1)\left(x - \frac{x_1 + x_2}{2}\right) + \\ & + 2a_1(x_1)\left(y - \frac{y_1 + y_2}{2}\right) + 2\,\frac{A_1(y_2) - A_1(y_1)}{y_2 - y_1}(y - y_1)\left(x - \frac{x_1 + x_2}{2}\right) + \\ & + 2\,\frac{a_1(x_2) - a_1(x_1)}{x_2 - x_1}(x - x_1)\left(y - \frac{y_1 + y_2}{2}\right), \\ \varphi_3(x, y) = {} & \frac{1}{12}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) - \\ & - \frac{1}{24}(y_2 - y_1)^2 (x - x_1)(y - y_1) - \\ & - \frac{1}{16}(x_2 - x_1)(y_2 - y_1)(y - y_1)(y - y_2) + \\ & + \left[a_1'(x) - \frac{a_1(x_2) - a_1(x_1)}{x_2 - x_1}\right](y - y_1)(y - y_2). \end{aligned}\right\} \tag{88}$$

13. Ținînd seama de formulele (88), formula de cubatură (75) devine

$$\iint_D f\,dx\,dy = \frac{(x_2-x_1)(y_2-y_1)}{24}\left[(x_2-x_1)^2\frac{\partial^2 f}{\partial x^2}(x_2,\ y_2)+\right.$$

$$\left.+(x_2-x_1)(y_2-y_1)\frac{\partial^2 f}{\partial x\partial y}(x_2, y_2)+(y_3-y_1)^2\frac{\partial f}{\partial y^2}(x_2,\ y_2)\right]-$$

$$-\frac{(x_2-x_1)^2(y_2-y_1)^2}{24}\frac{\partial^2 f}{\partial x\partial y}(x_1, y_1)- \tag{89}$$

$$-\frac{(x_2-x_1)(y_2-y_1)}{4}\left[(x_2-x_1)\frac{\partial f}{\partial x}(x_2, y_2)+(y_2-y_1)\frac{\partial f}{\partial y}(x_2, y_2)\right]+$$

$$+\frac{(x_2-x_1)(y_2-y_1)}{4}[3f(x_2, y_2)+f(x_1, y_1)]+R,$$

unde restul R este dat de formula

$$R=\iint_D\left(\varphi_0\frac{\partial^4 f}{\partial x^4}+\varphi_1\frac{\partial^4 f}{\partial x^3\partial y}+\varphi_2\frac{\partial^4 f}{\partial x^2\partial y^2}+\varphi_3\frac{\partial^4 f}{\partial x\partial y^3}+\varphi_4\frac{\partial^4 f}{\partial y^4}\right)dx\,dy. \tag{90}$$

Se verifică ușor că dacă $f(x, y)$ este o funcție numai de x, formula (89) se reduce la formula lui N. Obreschkoff.

14. Pentru a vedea rolul funcției $A_1(y)$ care figurează în expresiile lui $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y)$ să calculăm integrala dublă

$$I=\iint_D\left[A_1'(y)-\frac{A_1(y_2)-A_1(y_1)}{y_2-y_1}\right](x-x_1)(x-x_2)\frac{\partial^4 f}{\partial x^3\partial y}-$$

$$-2\iint_D\left[A_1(y)-A_1(y_1)-\frac{A_1(y_2)-A_1(y_1)}{y_2-y_1}(y-y_1)\right]\left(x-\frac{x_1+x_2}{2}\right)\frac{\partial^4 f}{\partial x^2\partial y^2}\,dxdy.$$

Aceasta se mai scrie

$$I=\int_{y_1}^{y_2}\left[A_1'(y)-\frac{A_1(y_2)-A_1(y_1)}{y_2-y_1}\right]dy\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left[(x-x_1)(x-x_2)\frac{\partial^3 f}{\partial x^2\partial y}-\right.$$

$$\left.-(2x-x_1-x_2)\frac{\partial^2 f}{\partial x\partial y}+2\frac{\partial f}{\partial y}\right]dx-2\int_{y_1}^{y_2}\left[A_1(y)-A_1(y_1)-\right.$$

$$\left.-\frac{A_1(y_2)-A_1(y_1)}{y_1-y_1}(y-y_1)\right]dy\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left[\left(x-\frac{x_1+x_2}{2}\right)\frac{\partial^3 f}{\partial x\partial y^2}-\frac{\partial^2 f}{\partial y^2}\right]dx,$$

sau

$$I=-\int_{y_1}^{y_2}\left[A_1'(y)-\frac{A_1(y_2)-A_1(y_1)}{y_2-y_1}\right]\left\{(x_2-x_1)\left[\frac{\partial^2 f}{\partial x\partial y}(x_2, y)+\frac{\partial^2 f}{\partial x\partial y}(x_1, y)\right]-\right.$$

$$\left.-2\left[\frac{\partial f}{\partial y}(x_2, y)-\frac{\partial f}{\partial y}(x_1, y)\right]\right\}dy-\int_{y_1}^{y_2}\left[A_1(y)-A_1(y_1)-\right.$$

$$\left.-\frac{A_1(y_2)-A_1(y_1)}{y_1-y_1}\right]\left\{(x_2-x_1)\left[\frac{\partial^3 f}{\partial x\partial y^2}(x_2, y)+\frac{\partial^3 f}{\partial x\partial y^2}(x_1, y)\right]-\right.$$

$$\left.-2\left[\frac{\partial^2 f}{\partial y^2}(x_2, y)-\frac{\partial^2 f}{\partial y^2}(x_1, y)\right]\right\}dy.$$

Notînd

$$u(y)=A_1(y)-A_1(y_1)-\frac{A_1(y_2)-A_1(y_1)}{y_2-y_1}(y-y_1),$$

$$v(y)=(x_2-x_1)\left[\frac{\partial^2 f}{\partial x\partial y}(x_2, y)+\frac{\partial^2 f}{\partial x\partial y}(x_1, y)\right]-2\left[\frac{\partial f}{\partial y}(x_2, y)-\frac{\partial f}{\partial y}(x_1,\ y)\right],$$

integrala I se mai scrie sub forma

$$I=-\int_{y_1}^{y_2}d(uv)=-u(y_2)\,v(y_2)+u(y_1)\,v(y_1)$$

și se vede că avem $I=0$, oricare ar fi funcția $A_1(y)$, deoarece $u(y_1)=0$, $u(y_2)=0$.

De asemenea funcția $a_1(x)$ nu are nici un rol în expresia restului R din formula (89), căci dacă se calculează partea din R în care figurează $a_1(x)$, se constată că această parte este nulă.

De aceea putem lua pe $A_1(y)=0$ și $a_1(x)=0$, ceea ce simplifică formulele (88).

15. Să reluăm problema la limită de la nr. **8**, adică integrarea ecuațiilor cu derivate parțiale (43) cu condițiile la limită (44)—(49), presupunînd însă că

$$\varphi_0(x, y)=\frac{\alpha_0}{24}(x-x_1)\ (x-x_2)^2, \tag{91}$$

$$\varphi_4(x, y)=\frac{\alpha_5}{24}(y-y_1)\cdot(y-y_2)^2. \tag{92}$$

1⁰. Avem

$$\frac{\partial\varphi_0}{\partial x}=\frac{\alpha_0}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)$$

și condițiile la limită (44) arată că

$$\left.\frac{\partial\varphi_1}{\partial y}\right|_{x=x_1}=0,\qquad \left.\frac{\partial\varphi_1}{\partial y}\right|_{x=x_2}=0. \tag{93}$$

Integrînd ecuația

$$\frac{\partial^3}{\partial x^3}\left(\frac{\partial\varphi_1}{\partial y}\right)=\alpha_1,$$

cu condiţiile (93), se găseşte

$$\frac{\partial\varphi_1}{\partial y} = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(\left(x-\frac{x_1+x_2}{2}\right) + A_1''(y)\,\frac{(x-x_1)(x-x_2)}{2}, \qquad (94)$$

unde $A_1''(y)$ este o funcţie arbitrară.

2°. Avem

$$\frac{\partial^2\varphi_0}{\partial x^2} = \frac{\alpha_0}{3}\left(x-\frac{x_1+x_2}{2}\right)^2 + \frac{\alpha_0}{6}(x-x_1)(x-x_2),$$

$$\frac{\partial^2\varphi_1}{\partial x\partial y} = \frac{\alpha_1}{3}\left(x-\frac{x_1+x_2}{2}\right)^2 + \frac{\alpha_1}{6}(x-x_1)(x-x_2) + A_1''(y)\left(y-\frac{y_1+y_2}{2}\right).$$

Condiţiile la limită (45) arată că

$$\left.\frac{\partial^2\varphi_2}{\partial y_2}\right|_{x=x_1} = -\frac{\alpha_0+\alpha_1}{12}(x_2-x_1)^2 + \frac{A_1''(y)}{2}(x_2-x_1)$$
$$\left.\frac{\partial^2\varphi_2}{\partial y^2}\right|_{x=x_2} = -\frac{\alpha_0+\alpha_1}{12}(x_2-x_1)^2 - \frac{A_1''(y)}{2}(x_2-x_1). \qquad (95)$$

Integrăm acum ecuaţia

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\varphi}{\partial y^2}\right) = \alpha_2$$

cu condiţiile (95). Se găseşte

$$\frac{\partial^2\varphi_2}{\partial y^2} = \frac{\alpha_2}{2}(x-x_1)(x-x_2) - \frac{\alpha_0+\alpha_1}{12}(x_2-x_1)^2 - A_1''(y)\Big)x-\frac{x_1+x_2}{3}\Big). \qquad (96)$$

3°. Avem

$$\frac{\partial^3\varphi_0}{\partial x^3} = \alpha_0\left(x-\frac{x_1+x_2}{2}\right),$$

$$\frac{\partial^3\varphi_1}{\partial x^2\partial y} = \alpha_1\left(x-\frac{x_1+x}{2}\right) + A_1''(y),$$

$$\frac{\partial^3\varphi_2}{\partial x\partial y^2} = \alpha_2\left(x-\frac{x_1+x_2}{2}\right) - A_1''(y)$$

şi condiţiile (46) arată că

$$\left.\frac{\partial^3\varphi_3}{\partial y^3}\right|_{x=x_1} = \frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1), \qquad \left.\frac{\partial\varphi_3}{\partial y^3}\right|_{x=x_2} = -\frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1). \qquad (97)$$

Să integrăm ecuaţia

$$\frac{\partial}{\partial x}\left(\frac{\partial^3\varphi_3}{\partial y^3}\right) = \alpha_3.$$

Avem

$$\frac{\partial^3\varphi_3}{\partial y^3} = \alpha_3(x-x_1) + C_1(y)$$

şi condiţiile (97) determină funcţia $C_1(y)$ şi stabileşte o legătură între α_0, α_1, α_2, α_3. Vom avea

$$C_1(y) = \frac{\alpha_0+\alpha_1+\alpha_2}{2}(x_2-x_1)$$

şi

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3 = 0. \qquad (98)$$

Rezultă că putem scrie

$$\frac{\partial^3\varphi_3}{\partial y^3} = \alpha_3\left(x-\frac{x_1+x_2}{2}\right). \qquad (99)$$

4°. Să revenim la ecuaţiile cu derivate parţiale (43) şi să le integrăm ţinînd seamă de condiţiile (47), (48), (49) şi formula (92). Avem

$$\frac{\partial\varphi_4}{\partial y} = \frac{\alpha_4}{6}(y-y_1)(y-y_2)\left(y-\frac{y_1+y_2}{2}\right),$$

de unde rezultă datorită condiţiilor (47) că

$$\left.\frac{\partial\varphi_3}{\partial x}\right|_{y=y_1} = 0, \qquad \left.\frac{\partial\varphi_3}{\partial x}\right|_{y=y_2} = 0. \qquad (100)$$

Integrăm ecuaţia

$$\frac{\partial^3}{\partial y^3}\left(\frac{\partial\varphi_3}{\partial x}\right) = \alpha_3,$$

cu condiţiile (100). Se găseşte

$$\frac{\partial\varphi_3}{\partial x} = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(y-\frac{y_1+y_2}{2}\right) + \frac{a_1''(x)}{2}(y-y_1)(y-y_2), \qquad (101)$$

unde $a''_1(x)$ este o funcţie arbitrară.

Integrăm în raport cu x şi obţinem

$$\varphi_3(x,y) = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y+y_2}{2}\right) +$$
$$+ \frac{a_1'(x)}{2}(y-y_2)(y-y_2) + F(y)$$

unde $F(y)$ este o funcţie care se determină scriind că $\frac{\partial^3\varphi_3}{\partial y^3}$ este dat de formula (99).

Vom avea

$$\alpha_3\left(x-\frac{x_1+x_2}{2}\right) + F'''(y) = \alpha_3\left(x-\frac{x_1+x_2}{2}\right),$$

adică

$$F'''(y) = 0$$

şi prin urmare

$$F(y) = P_1(y-y_1)(y-y_2) + P_2(y-y_1) + P_3,$$

unde P_1, P_2, P_3 sînt constante arbitrare.

Avem deci

$$\varphi_3(x, y) = \frac{\alpha_3}{6}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$

$$+\frac{a_1'(x)}{2}(y-y_1)(y-y_2)+P_1(y-y_1)(y-y_2)+P_2(y-y_1)+P_3. \quad (102)$$

5°. Avem

$$\frac{\partial^2\varphi_4}{\partial y^2}=\frac{\alpha_4}{3}\left(y-\frac{y_1+y_2}{2}\right)^2+\frac{\alpha_4}{6}(y-y_1)(y-y_2),$$

$$\frac{\partial^2\varphi_3}{\partial x\partial y}=\frac{\alpha_3}{3}\left(y-\frac{y_1+y_2}{2}\right)^2+\frac{\alpha_3}{6}(y-y_1)(y-y_2)+a_1''(x)\left(y-\frac{y_1+y_2}{2}\right)$$

şi condiţiile (48) arată că

$$\left.\frac{\partial^2\varphi_2}{\partial x^2}\right|_{y=y_1} = -\frac{\alpha_3+\alpha_4}{12}(y_2-y_1)^2+a_1''(x)\frac{y_2-y_1}{2},$$

$$\left.\frac{\partial^2\varphi_2}{\partial x^2}\right|_{y=y_2} = -\frac{\alpha_3+\alpha_4}{12}(y_2-y_1)^2-a_1''(x)\frac{y_2-y_1}{2}. \quad (103)$$

Integrînd ecuaţia

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\varphi_2}{\partial x^2}\right)=\alpha_2$$

cu condiţiile (103), avem

$$\frac{\partial^2\varphi_2}{\partial x^2}=\frac{\alpha_2}{2}(y-y_1)(y-y_2)-\frac{\alpha_3+\alpha_4}{12}(y_2-y_1)^2-a_1''(x)\left(y-\frac{y_1+y_2}{2}\right). \quad (104)$$

Integrînd de două ori în raport cu x, obţinem

$$\varphi_2(x, y)=\frac{\alpha_2}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-\frac{\alpha_3+\alpha_4}{24}(y_2-y_1)^2(x-x_1)(x-x_2)-$$

$$-a_1(x)\left(y-\frac{y_1+y_2}{2}\right)+G_1(y)\left(x-\frac{x_1+x_2}{2}\right)+G_2(y), \quad (105)$$

unde $G_1(y)$ şi $G_2(y)$ sînt funcţii care se determină scriind că $\frac{\partial^2\varphi_2}{\partial y^2}$ este dat de formula (96). Vom avea

$$\frac{\alpha_2}{2}(x-x_1)(x-x_2)+G_1''(y)\left(x-\frac{x_1+x_2}{2}\right)+G_2''(y)=\frac{\alpha_2}{2}(x-x_1)(x-x_2)-$$

$$-A_1''(y)\left(x-\frac{x_1+x_2}{2}\right)-\frac{\alpha_0+\alpha_1}{12}(x_2-x_1)^2,$$

de unde rezultă ecuaţiile

$$G_1''(y)=-A_1''(y),$$

$$G_2''(y)=-\frac{\alpha_0+\alpha_1}{12}(x_2-x_1)^2.$$

Rezultă că

$$G_1(y)=-A_1(y)+Q_1\left(y-\frac{y_1+y_2}{2}\right)+Q_2,$$

$$G_2(y)=-\frac{\alpha_0+\alpha_1}{24}(x_2-x_1)^2(y-y_1)(y-y_2)+Q_3\left(y-\frac{y_1+y_2}{2}\right)+Q_4$$

şi purtînd acestea în formula (105), vom avea

$$\varphi_2(x,y)=\frac{\alpha_2}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-$$

$$-\frac{\alpha_3+\alpha_4}{24}(y_2-y_1)^2(x-x_1)(x-x_2)-\frac{\alpha_0+\alpha_1}{24}(x_2-x_1)^2(y-y_1)(y-y_2)-$$

$$-a_1(x)\left(y-\frac{y_1+y_2}{2}\right)-A_1(y)\left(x-\frac{x_1+x_2}{2}\right)+Q_1\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$

$$+Q_2\left(x-\frac{x_1+x_2}{2}\right)+Q_3\left(y-\frac{y_1+y_2}{2}\right)+Q_4, \quad (106)$$

unde Q_1, Q_2, Q_3, Q_4 sînt constante arbitrare.

6°. Avem

$$\frac{\partial^3\varphi_4}{\partial y^3}=\alpha_4\left(y-\frac{y_1+y_2}{2}\right),$$

$$\frac{\partial^3\varphi_3}{\partial x\partial y^2}=\alpha_3\left(y-\frac{y_1+y_2}{2}\right)+a_1''(x),$$

$$\frac{\partial^3\varphi_2}{\partial x^2\partial y}=\alpha_2\left(y-\frac{y_1+y_2}{2}\right)-a_1''(x).$$

Condiţiile (49) ne arată că

$$\left.\frac{\partial^3\varphi_1}{\partial x^3}\right|_{y=y_1}=\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1), \qquad \left.\frac{\partial^3\varphi_1}{\partial x^3}\right|_{y=y_2}=-\frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1). \quad (107)$$

Integrala ecuaţiei

$$\frac{\partial}{\partial y}\left(\frac{\partial^3\varphi_1}{\partial x^3}\right)=\alpha_1$$

este

$$\frac{\partial^3\varphi_1}{\partial x^3}=\alpha_1(y-y_1)+c_1(x)$$

şi condiţiile (107) determină funcţia $c_1(x)$ şi o relaţie între $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Avem

$$c_1(x) = \frac{\alpha_2+\alpha_3+\alpha_4}{2}(y_2-y_1)$$

şi

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0, \tag{108}$$

de unde se deduce că

$$\frac{\partial^3\varphi_1}{\partial x^3} = \alpha_1\left(y - \frac{y_1+y_2}{2}\right).$$

Integrînd de trei ori în raport cu x, deducem

$$\varphi_1(x,y) = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+H_1(y)(x-x_1)(x-x_2)+H_2(y)(x-x_1)+H_3(y), \tag{109}$$

unde funcţiile $H_1(y)$, $H_2(y)$, $H_3(y)$ se determină scriind că $\frac{\partial\varphi_1}{\partial y}$ este dat de formula (94). Avem

$$\frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)+H_1'(y)(x-x_1)(x-x_2)+$$
$$H_2'(y)(x-x_1)+H_2'(y) = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)+$$
$$+A_1'(y)\frac{(x-x_1)(x-x_2)}{2}.$$

Rezultă că

$$H_1'(y) = \frac{1}{2}A_1''(y),$$
$$H_2'(y) = 0,$$
$$H_3'(y) = 0,$$

de unde

$$H_1(y) = \frac{1}{2}A_1'(y)+R_1,\quad H_2(y) = R_2,\quad H_3(y) = R_3,$$

unde R_1, R_2, R_3 sînt constante arbitrare.

Purtînd pe $H_1(y)$, $H_2(y)$, $H_3(y)$ în formula (109), vom avea

$$\varphi_1(x,y) = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+\frac{A_1'(y)}{2}(x-x_1)(x-x_2)+R_1(x-x_1)(x-x_2)+R_2(x-x_2)+R_3. \tag{110}$$

Să observăm că dacă introducem funcţiile

$$\frac{A_1(y)}{2}+R_1\left(y-\frac{y_1+y_2}{2}\right) = A_2(y),$$
$$\frac{a_1(x)}{2}+P_1\left(x-\frac{x_1+x_2}{2}\right) = a_2(x)$$

şi constanta

$$\lambda_1 = 2P_1+Q_1+2R_1,$$

formulele (110), (106), (102) se mai scriu sub forma

$$\varphi_1(x,y) = \frac{\alpha_1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+A_2'(y)(x-x_1)(x-x_2)+R_2(x-x_1)+R_3,$$
$$\varphi_2(x,y) = \frac{\alpha_2}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-$$
$$-\frac{\alpha_3+\alpha_4}{24}(y_2-y_1)^2(x-x_1)(x-x_2)-\frac{\alpha_0+\alpha_1}{24}(x_2-x_1)^2(y-y_1(y-y_2)-$$
$$-2a_2(x)\left(y-\frac{y_1+y_2}{2}\right)-2A_2(y)\left(x-\frac{x_1+x_2}{2}\right)+\lambda_1\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+ \tag{111}$$
$$+Q_2\left(x-\frac{x_1+x_2}{2}\right)+Q_3\left(y-\frac{y_1+y_2}{2}\right)+Q_4$$
$$\gamma_3+(x,y) = \frac{\alpha_3}{6}(y-y_1)(y-y)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+a_2'(x)(y-y_1)(y-y_2)+P_2(y-y_1)+P_3$$

şi se vede că aceste formule depind de funcţiile arbitrare $a_2(x)$, $A_2(y)$ şi de constantele arbitrare R_2, R_3, P_2, P_3, λ_1, Q_2, Q_3, Q_4.

16. Să fixăm constantele $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ astfel ca formulele (111) să fie mai simple. Ţinînd seamă de formulele (98) şi (108) vom alege

$$\alpha_0 = 1,\quad \alpha_1 = -1,\quad \alpha_2 = 1,\quad \alpha_3 = -1,\quad \alpha_4 = 1$$

şi atunci formulele (111) devin

$$\varphi_1(x,y) = -\frac{1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+$$
$$+A_2'(y)(x-x_1)(x-x_2)+R_2(x-x_1)+R_3.$$
$$\varphi_2(x,y) = \frac{1}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-2a_2(x)\left(y-\frac{y_1+y_2}{2}\right)-$$

$$- 2A_2(y)\left(x - \frac{x_1 + x_2}{2}\right) + \lambda_1\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) +$$

$$+ Q_2\left(x - \frac{x_1 + x_2}{2}\right) + Q_3\left(y - \frac{y_1 + y_2}{2}\right) + Q_4. \tag{112}$$

$$\varphi_3(x,y) = -\frac{1}{6}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}\right) +$$

$$+ a_2'(x)(y - y_1)(y - y_2) + P_2(y - y_1) + P_3.$$

Rămîne să introducem noi condiții care să determine constantele R_2, R_3, λ_1, Q_2, Q_3, Q_4, P_2, P_3 și să vedem rolul funcțiilor $a_2(x)$, $A_2(y)$ în formula de cubatură.

17. Vom introduce următoarele condiții la limită :

$$\begin{matrix} \varphi_1(x_1,y_2) = 0 & ; & \varphi_2(x_1,y_2) = 0 & ; & \varphi_3(x_1,y_2) = 0 \\ \varphi_1(x_2,y_1) = 0 & & \varphi_2(x_2,y_1) = 0 & & \varphi_3(x_2,y_1) = 0. \end{matrix} \tag{113}$$

$$\begin{matrix} \left.\dfrac{\partial\varphi_1}{\partial x} + \dfrac{\partial\varphi_2}{\partial y}\right|_{\substack{x=x_1\\y=y_2}} = 0\,; & \left.\dfrac{\partial\varphi_2}{\partial x} + \dfrac{\partial\varphi_3}{\partial y}\right|_{\substack{x=x_1\\y=y_2}} = 0 \\ \left.\dfrac{\partial\varphi_1}{\partial x} + \dfrac{\partial\varphi_2}{\partial y}\right|_{\substack{x=x_2\\y=y_1}} = 0\,; & \left.\dfrac{\partial\varphi_2}{\partial x} + \dfrac{\partial\varphi_3}{\partial y}\right|_{\substack{x=x_2\\y=y_1}} = 0. \end{matrix} \tag{114}$$

Condițiile $\varphi_1(x_1,y_2) = 0$, $\varphi_1(x_2,y_1) = 0$ arată că $R_2 = 0$, $R_3 = 0$.
Condițiile $\varphi_3(x_1,y_2) = 0$, $\varphi_3(x_2,y_1) = 0$, arată că $P_2 = 0$, $P_3 = 0$.

Avem

$$\varphi_2(x_1,y_2) = - a_2(x_1)(y_2 - y_1) + A_2(y_2)(x_2 - x_1) - \frac{\lambda_1}{4}(x_2 - x_1)(y_2 - y_1) -$$

$$- \frac{Q_2}{2}(x_2 - x_1) + \frac{Q_3}{2}(y_2 - y_1) + Q_4 = 0, \tag{115}$$

$$\varphi_2(x_2 y_1) = a_2(x_2)(y_2 - y_1) - A_2(y_1)(x_2 - x_1) - \frac{\lambda_1}{4}(x_2 - x_1)(y_2 - y_1) +$$

$$+ \frac{Q_2}{2}(x_2 - x_1) - \frac{Q_3}{2}(y_2 - y_1) + Q_4 = 0.$$

Avem

$$\frac{\partial\varphi_1}{\partial x} = -\frac{1}{3}\left(x - \frac{x_1 + x_2}{2}\right)^2\left(y - \frac{y_1 + y_2}{2}\right) - \frac{1}{6}(x - x_1)(x - x_2)\left(y - \frac{y_1 + y_2}{2}\right) +$$

$$+ 2A_2'(y)\left(x - \frac{x_1 + x_2}{2}\right),$$

$$\frac{\partial\varphi_2}{\partial y} = \frac{1}{2}(x - x_1)(x - x_2)\left(y - \frac{y_1 + y_2}{2}\right) - 2a_2(x) - 2A_2'(y)\left(x - \frac{x_1 + x_2}{2}\right) +$$

$$+ \lambda_1\left(x - \frac{x_1 + x_2}{2}\right) + Q_3.$$

Primele condiții (114) dau deci ecuațiile

$$\begin{aligned} -\frac{1}{24}(x_2 - x_1)^2(y_2 - y_1) - 2a_2(x_1) - \frac{\lambda_1}{2}(x_2 - x_1) + Q_3 = 0, \\ \frac{1}{24}(x_2 - x_1)^2(y_2 - y_1) - 2a_2(x_2) + \frac{\lambda_1}{2}(x_2 - x_1) + Q_3 = 0. \end{aligned} \tag{116}$$

Avem

$$\frac{\partial\varphi_2}{\partial x} = \frac{1}{2}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right) - 2a_2'(x)\left(y - \frac{y_1 + y_2}{2}\right) - 2A_2(y) +$$

$$+ \lambda_1\left(y - \frac{y_1 + y_2}{2}\right) + Q_2,$$

$$\frac{\partial\varphi_3}{\partial y} = -\frac{1}{3}\left(x - \frac{x_1 + x_2}{2}\right)\left(y - \frac{y_1 + y_2}{2}^2\right) - \frac{1}{6}(y - y_1)(y - y_2)\left(x - \frac{x_1 + x_2}{2}\right) +$$

$$+ 2a_2'(x)\left(y - \frac{y_1 + y_2}{2}\right).$$

Celelalte condiții (114) dau ecuațiile

$$\begin{aligned} \frac{1}{24}(x - x_1)(y_2 - y_1)^2 - 2A_2(y_2) + \frac{\lambda_1}{2}(y_2 - y_1) + Q_2 = 0, \\ -\frac{1}{24}(x_2 - x_1)(y_2 - y_1)^2 - 2A_2(y_1) - \frac{\lambda_1}{2}(y_2 - y_1) + Q_2 = 0. \end{aligned} \tag{117}$$

Adunînd membru cu membru ecuațiile (116) și apoi ecuațiile (117), obținem

$$\begin{aligned} Q_3 = a_2(x_2) + a_2(x_1), \\ Q_2 = A_2(y_2) + A_2(y_1). \end{aligned} \tag{118}$$

Scăzînd membru cu membru ecuațiile (116) și apoi ecuațiile (117), obținem

$$\frac{1}{12}(x_2 - x_1)^2(y_2 - y_1) - 2\,[a_2(x_2) - a_2(x_1)] + \lambda_2(x_2 - x_1) = 0,$$

$$-\frac{1}{12}(x_2 - x_1)^2(y_2 - y_1) - 2\,[A_2(y_2) - A_2(y_1)] + \lambda_1(y_2 - y_1) = 0,$$

de unde rezultă o condiție de compatibilitate

$$\frac{A_2(y_2) - A_2(y_1)}{y_2 - y_1} = \frac{a_2(x_2) - a_2(x_1)}{x_2 - x_1} \tag{119}$$

și

$$\lambda_1 = -\frac{(x_2 - x_1)(y_2 - y_1)}{12} + 2\,\frac{a_2(x_2) - a_2(x_1)}{x_2 - x_1}. \tag{120}$$

Ținînd seamă de ecuațiile (118) și (120), ecuațiile (115) se reduc la una singură care dă

$$Q_4 = -\frac{(x_2 - x_1)^2(y_2 - y_1)^2}{48} - \frac{A_2(y_2) - A_2(y_1)}{2}(x_2 - x_1). \tag{121}$$

18. Să calculăm acum coeficienţii din formula de cubatură (75).

Avem

$$\left.\begin{aligned}
&\varphi_1(x_2,y_2)=0, \qquad \varphi_3(x_2,y_2)=0,\\
&\varphi_1(x_1,y_1)=0, \qquad \varphi_3(x_1,y_1)=0,\\
&\varphi_2(x_2,y_2)=-\frac{(x_2-x_1)^2(y_2-y_1)^2}{24}-[A_2(y_2)-A_2(y_1)](x_2-x_1),\\
&\varphi_2(x_1,y_1)=-\frac{(x_2-x_1)^2(y_2-y_1)^2}{24}-[A_2(y_2)-A_2(y_1)](x_2-x_1).
\end{aligned}\right\} \tag{122}$$

De asemenea avem

$$\left.\begin{aligned}
&\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\bigg|_{\substack{x=x_2\\y=y_2}}=-\frac{1}{12}(x_2-x|)^2(y_2-y_1),\\
&\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}\bigg|_{\substack{x=x_1\\y=y_1}}=\frac{1}{12}(x_2-x_1)^2(y_2-y_1),
\end{aligned}\right\} \tag{123}$$

$$\left.\begin{aligned}
&\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}\bigg|_{\substack{x=x_2\\y=y_2}}=-\frac{1}{12}(x_2-x_1)(y_2-y_1)^2,\\
&\frac{\partial\varphi_2}{\partial x}+\frac{\partial\varphi_3}{\partial y}\bigg|_{\substack{x=x_1\\y=y_1}}=\frac{1}{12}(x_2-x_1)(y_2-y_1)^2.
\end{aligned}\right\} \tag{124}$$

Pentru calculul lui $A_0^2(x,y)$ avem

$$\frac{\partial^2\varphi_1}{\partial x^2}=-\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+2A_2'(y),$$

$$\frac{\partial^2\varphi_\cdot}{\partial x\partial y}=\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)-2a_2'(x)-2A_2'(y)+\lambda_1,$$

$$\frac{\partial^2\varphi_3}{\partial y^2}=-\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+2a_2'(x),$$

de unde rezultă că

$$A_0^2(x_2,y_2)=-\frac{1}{4}(x_2-x_1)(x_2-y_1)+\lambda_1,$$

$$A_0^2(x_1,y_2)=\frac{1}{4}(x_2-x_1)(y_2-y_1)+\lambda_1,$$

$$A_0^2(x_2,y_1)=\frac{1}{4}(x_2-x_1)(y_2-y_1)+\lambda_1,$$

$$A_0^2(x_1,y_1)=-\frac{1}{4}(x_2-x_1)(y_2-y_1)+\lambda_1$$

şi deoarece trebuie să avem după formula (75)

$$A_0^2(x_2,y_2)+A_0^2(x_1,y_2)+A_2^0(x_2,y_1)+A_0^2(x_1,y_1)+(x_2-x_1)(y_2-y_1)=0,$$

rezultă că în formulele precedente trebuie să avem

$$\lambda_1=-\frac{(x_2-x_1)(y_2-y_1)}{4},$$

ceea ce înseamnă că ţinînd seamă de formula (120), trebuie să avem

$$\frac{a_2(x_2)-a_2(x_1)}{x_2-x_1}=-\frac{(x_2-x_1)(y_2-y_1)}{12}. \tag{125}$$

Formulele precedente arată că vom avea

$$\left.\begin{aligned}
&A_0^2(x_2,y_2)=A_0^2(x_1,y_1)=-\frac{1}{2}(x_2-x_1)(y_2-y_1),\\
&A_0^2(x_1,y_2)=A_0^2(x_2,y_1)=0.
\end{aligned}\right\} \tag{126}$$

Formula de cubatură (75) este în acest caz

$$\begin{aligned}
\iint_D f\,dx\,dy=&-\frac{(x_2-x_1)^2(y_2-y_1)^2}{24}\left[\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)+\frac{\partial^2 f}{\partial x\partial y}(x_1,y_1)\right]-\\
&-\frac{(x_2-x_1)(y_2-y_1)}{12}\left[(x_2-x_1)\frac{\partial f}{\partial x}(x_2,y_2)+(y_2-y_1)\frac{\partial f}{\partial y}(x_2,y_2)\right]+\\
&+\frac{(x_2-x_1)(y_2-y_1)}{12}\left[(x_2-x_1)\frac{\partial f}{\partial x}(x_1,y_1)+(y_2-y_1)\frac{\partial f}{\partial y}(x_1,y_1)\right]+\\
&+\frac{(x_2-x_1)(y_2-y_1)}{2}[f(x_2,y_2)+f(x_1,y_1)]+R,
\end{aligned} \tag{127}$$

unde

$$R=-\iint_D\left(\varphi_0\frac{\partial^4 f}{\partial x^4}+\varphi_1\frac{\partial^4 f}{\partial x^3\partial y}+\varphi_2\frac{\partial^4 f}{\partial x^2\partial y^2}+\varphi_3\frac{\partial^4 f}{\partial x\partial y^3}+\varphi_4\frac{\partial^4 f}{\partial y^4}\right]dxdy, \tag{128}$$

funcţiile $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ fiind date de următoarele formule

$$\left.\begin{aligned}
\varphi_0(x,y)=&\frac{1}{24}(x-x_1)^2(x-x_2)^2,\\
\varphi_1(x,y)=&-\frac{1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&+A_2'(y)(x-x_1)(x-x_2),\\
\varphi_2(x,y)=&\frac{1}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-\\
&-\frac{(x_2-x_1)(y_2-y_1)}{4}\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&+\frac{(x_2-x_1)^2(y_2-y_1)^2}{48}-2\left[A_2(y)-\frac{A_2(y_2)+A_2(y_1)}{2}\right]\left(x-\frac{x_1+x_2}{2}\right)-
\end{aligned}\right\} \tag{129}$$

$$\left.\begin{aligned}
&\quad -2\left[a_2(x)-\frac{a_2(x_2)+a_2(x_1)}{2}\right]\left(y-\frac{y_1+y_2}{2}\right),\\
&\varphi_3(x,y)=-\frac{1}{6}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&\qquad + a_2'(x)(y-y_1)(y-y_2)\\
&\varphi_4(x,y)=\frac{1}{24}(y-y_1)^2(y-y_2)^2.
\end{aligned}\right\}$$

Notînd

$$\begin{aligned}
A_3(y)&=A_2(y)-\frac{A_2(y_2)+A_2(y_1)}{2},\\
a_3(x)&=a_2(x)-\frac{a_2(x_2)+a_2(x_1)}{2},
\end{aligned} \tag{130}$$

condițiile (119) și (125) se reduc la

$$\frac{A_3(y_2)-A_3(y_1)}{y_2-y_1}=\frac{a_3(x_2)-a_3(x_1)}{x_2-x_1}=-\frac{(x_2-x_1)(y_2-y_1)}{12}, \tag{131}$$

iar formulele (129) se scriu

$$\left.\begin{aligned}
&\varphi_0(x,y)=\frac{1}{24}(x-x_1)^2(x-x_2)^2,\\
&\varphi_1(x,y)=-\frac{1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&\qquad + A_3'(y)(x-x_1)(x-x_2),\\
&\varphi_2(x,y)=\frac{1}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2)-\\
&\qquad -\frac{(x_2-x_1)(y_2-y_1)}{4}\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&\qquad +\frac{(x_2-x_1)^2(y_2-y_1)^2}{48}-2A_3(y)\left(x-\frac{x_1+x_2}{2}\right)-\\
&\qquad -2a_3(x)\left(y-\frac{y_1+y_2}{2}\right),\\
&\varphi_3(x,y)=-\frac{1}{6}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right)+\\
&\qquad + a_3'(x)(y-y_1)(y-y_2),\\
&\varphi_4(x,y)=\frac{1}{24}(y-y_1)^2(y-y_2)^2.
\end{aligned}\right\} \tag{129'}$$

Rămîne să vedem rolul funcțiilor $A_3(y)$ și $a_3(x)$ în expresia restului formulei de cubatură (127).

19. Funcția $A_3(y)$ intră în integrala dublă

$$I=-\iint\limits_D A_3'(y)(x-x_1)(x-x_2)\frac{\partial^4 f}{\partial x^3\partial y}\,dx\,dy+2\iint\limits_D A_3(y)\left(x-\frac{x_1+x_2}{2}\right)\frac{\partial^4 f}{\partial x_2\partial y_2}dxdy,$$

iar $a_3(x)$ intră în integrala dublă

$$J=-\iint\limits_D A_3'(x)(y-y_1)(y-y_2)\frac{\partial^4 f}{\partial x\partial y^3}\,dx\,dy+2\iint\limits_D a_3(x)\left(y-\frac{y_1+y_2}{2}\right)\frac{\partial^4 f}{\partial x^2\partial y^2}\,dxdy.$$

Avem

$$\begin{aligned}
I=&-\int_{y_1}^{y_2}A_3'(y)\,dy\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left[(x-x_1)(x-x_2)\frac{\partial^3 f}{\partial x^2\partial y}-2\left(x-\frac{x_1+x_2}{2}\right)\frac{\partial^2 f}{\partial x\partial y}+2\frac{\partial f}{\partial y}\right]dx+\\
&+2\int_{y_1}^{y_2}A_3(y)dy\int_{x_1}^{x_2}\frac{\partial}{\partial x}\left[\left(x-\frac{x_1+x_2}{2}\right)\frac{\partial^3 f}{\partial x\partial y^2}-\frac{\partial^2 f}{\partial y^2}\right]dx
\end{aligned}$$

și deci

$$\begin{aligned}
I=&\int_{y_1}^{y_2}A_3'(y)\left|(x_2-x_1)\left(\frac{\partial^2 f}{\partial x\partial y}(x_2,y)+\frac{\partial^2 f}{\partial x\partial y}(x_1,y)\right)-\right.\\
&\left.-2\left(\frac{\partial f}{\partial y}(x_2,y)-\frac{\partial f}{\partial y}(x_1,y)\right)\right]dy+\int_{y_1}^{y_2}A_3(y)\left[(x_2-x_1)\frac{\partial^3 f}{\partial x\partial y^2}(x_2,y)+\right.\\
&\left.+\frac{\partial^3 f}{\partial x\partial y^2}(x_1,y)\right)-2\left(\frac{\partial^2 f}{\partial y^2}(x_2,y)-\frac{\partial^2 f}{\partial y^2}(x_1,y)\right)\right]dy,
\end{aligned}$$

sau

$$\begin{aligned}
I=&\int_{y_1}^{y_2}\frac{\partial}{\partial y}\left\{A_3(y)\left[(x_2-x_1)\left(\frac{\partial^2 f}{\partial x\partial y}(x_2,y)+\frac{\partial^2 f}{\partial x\partial y}(x_1,y)\right)-\right.\right.\\
&\left.\left.-2\left(\frac{\partial f}{\partial y}(x_2,y)-\frac{\partial f}{\partial y}(x_1,y)\right)\right]\right\}dy,
\end{aligned}$$

adică

$$\begin{aligned}
I=&A_3(y_2)\left[(x_2-x_1)\left(\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2)+\frac{\partial^2 f}{\partial x\partial y}(x_1,y_2)\right)-\right.\\
&\left.-2\left(\frac{\partial f}{\partial y}(x_2,y_2)-\frac{\partial f}{\partial y}(x_1,y_2)\right)\right]-A_3(y_1)\left[(x_2-x_1)\left(\frac{\partial^2 f}{\partial x\partial y}(x_2,y_1)+\right.\right.\\
&\left.\left.+\frac{\partial^2 f}{\partial x\partial y}(x_1,y_1)\right)-2\left(\frac{\partial f}{\partial y}(x_2,y_1)-\frac{\partial f}{\partial y}(x_1,y_1)\right)\right]
\end{aligned}$$

Însă după formulele (130) avem

$$-A_3(y_1) = A_3(y_2) = -\frac{(x_2-x_1)(y_2-y_1)^2}{24},$$

astfel că

$$I = -\frac{(x_2-x_1)^2(y_2-y_1)^2}{24}\left[\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2) + \frac{\partial^2 f}{\partial x\partial y}(x_1,y_2) + \frac{\partial^2 f}{\partial x\partial y}(x_2,y_1) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial x\partial y}(x_1,y_1)\right] + \frac{(x_2-x_1)^2(y_2-y_1)^2}{12}\left[\frac{\frac{\partial f}{\partial y}(x_2,y_2) - \frac{\partial f}{\partial y}(x_1,y_2)}{x_2-x_1} + \right.$$

$$\left. + \frac{\frac{\partial f}{\partial y}(x_2,y_1) - \frac{\partial f}{\partial y}(x_1,y_1)}{x_2-x_1}\right]. \tag{132}$$

În mod analog se găseşte că

$$J = -\frac{(x_2-x_1)^2(y_1-y_1)^2}{24}\left[\frac{\partial^2 f}{\partial x\partial y}(x_2,y_2) + \frac{\partial^2 f}{\partial x\partial y}(x_1,y_2) + \frac{\partial^2 f}{\partial x\partial y}(x_2,y_1) + \right.$$

$$\left. + \frac{\partial^2 f}{\partial x\partial y}(x_1,y_1)\right] + \frac{(x_2-x_1)^2(y_2-y_1)^2}{12}\left[\frac{\frac{\partial f}{\partial x}(x_2,y_2) - \frac{\partial f}{\partial x}(x_2,y_1)}{y_2-y_1} + \right.$$

$$\left. + \frac{\frac{\partial f}{\partial x}(x_1,y_2) - \frac{\partial f}{\partial x}(x_1,y_1)}{y_2-y_1}\right]. \tag{133}$$

Formulele (132), (133) arată că oricare ar fi funcţiile $A_3(y)$, $a_3(x)$, care verifică condiţiile (131), suma $I + J$ rămîne aceeaşi şi este egală cu suma membrilor ai doilea din formulele (132) şi (133). Putem alege

$$\left.\begin{aligned} A_3(y) &= -\frac{(x_2-x_1)(y_2-y_1)}{12}\left(y - \frac{y_1+y_2}{2}\right), \\ a_3(x) &= -\frac{(x_2-x_1)(y_2-y_1)}{12}\left(x - \frac{x_1+x_2}{2}\right) \end{aligned}\right\} \tag{134}$$

şi cu aceasta, formulele (129') iau forma definitivă

$$\left.\begin{aligned} \varphi_0(x,y) &= \frac{1}{24}(x-x_1)^2(x-x_2)^2 \\ \varphi_1(x,y) &= -\frac{1}{6}(x-x_1)(x-x_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right) - \\ &\quad -\frac{(x_2-x_1)(y_2-y_1)}{12}(x-x_1)(x-x_2) \\ \varphi_2(x,y) &= \frac{1}{4}(x-x_1)(x-x_2)(y-y_1)(y-y_2) + \end{aligned}\right\} \tag{129''}$$

$$\left.\begin{aligned} &+\frac{1}{12}(x_2-x_1)(y_2-y_1)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right) + \\ &+\frac{(x_2-y_1)(y_2-y_1)^2}{48} \\ \varphi_3(x,y) &= -\frac{1}{6}(y-y_1)(y-y_2)\left(x-\frac{x_1+x_2}{2}\right)\left(y-\frac{y_1+y_2}{2}\right) - \\ &-\frac{(x_2-x_1)(y_2-y_1)}{12}(y-y_1)(y-y_2) \\ \varphi_4(x,y) &= \frac{1}{24}(y-y_1)^2(y-y_2)^2. \end{aligned}\right\}$$

20. *Funcţiile* $\varphi_0(x,y)$, $\varphi_1(x,y)$, $\varphi_2(x,y)$, $\varphi_3(x,y)$, $\varphi_4(x,y)$ *date de formulele* (129") *sînt pozitive în dreptunghiul* D.

Într-adevăr, punînd

$$x = \frac{x_1+x_2}{2} + X, \quad y = \frac{y_1+y_2}{2} + Y, \quad \frac{x_2-x_1}{2} = h, \quad \frac{y_2-y_1}{2} = k,$$

avem

$$\varphi_1(x,y) = -\frac{X^2-h^2}{6}(XY+2hk),$$

$$\varphi_2(x,y) = \frac{1}{4}(X^2-h^2)(Y^2-k^2) + \frac{hk}{3}(XY+hk),$$

$$\varphi_3(x,y) = -\frac{Y^2-k^2}{6}(XY+2hk)$$

şi deoarece în dreptunghiul D avem

$$X^2-h^2<0, \quad Y^2-k^2<0, \quad |XY|<hk,$$

rezultă că în dreptunghiul D avem

$$\varphi_1(x,y)>0, \quad \varphi_2(x,y)>0, \quad \varphi_3(x,y)>0.$$

21. *Restul în formula de cubatură* (127). Se observă că

$$\iint_D \varphi_0(x,y)\,dx\,dy = \frac{(x_2-x_1)^5(y_2-y_1)}{720}, \quad \iint_D \varphi_1(x,y)\,dx\,dy = \frac{(x_2-x_1)^4(y_2-y_1)^2}{72},$$

$$\iint_D \varphi_2(x,y)\,dx\,dy = \frac{(x_2-x_1)^3(y_2-y_1)^3}{36}, \tag{135}$$

$$\iint_D \varphi_3(x,y)\,dx\,dy = \frac{(x_2-x_1)^2(y_2-y_1)^4}{72}, \quad \iint_D \varphi_4(x,y)\,dx\,dy = \frac{(x_2-x_1)(y_2-y_1)^5}{720}.$$

Ținînd seamă că funcțiile $\varphi_0(x, y), \ldots, \varphi_4(x, y)$ sînt pozitive în dreptunghiul D, din formulele (128) și (135) se deduce că restul formulei de cubatură (127) se mai poate scrie sub forma

$$\begin{aligned} R = -\frac{(x_2 - x_1)(y_2 - y_1)}{720}\Big[& (x_2 - x_1)^4 \frac{\partial^4 f}{\partial x^4}(\xi_1, y_1) + \\ & + 10(x_2 - x_1)^3(y_2 - y_1)\frac{\partial^4 f}{\partial x^3 \partial y}(\xi_2, \eta_2) + \\ & + 20(x_2 - x_1)^2(y_2 - y_1)^2 \frac{\partial^4 f}{\partial x^2 \partial y^2}(\xi_3, \eta_3) + \\ & + 10(x_2 - x_1)(y_2 - y_1)^3 \frac{\partial^4 f}{\partial x \partial y^3}(\xi_4, \eta_4) + \\ & + (y_2 - y_1)^4 \frac{\partial^4 f}{\partial y^4}(\xi_5, \eta_5)\Big], \end{aligned} \tag{136}$$

unde (ξ_i, η_i), $i = 1, 2, 3, 4, 5$ sînt coordonatele unor anumite puncte din domeniul D.

Notînd în general cu M_{ik} margini superioare ale valorilor absolute ale derivatelor parțiale $\frac{\partial^4 f}{\partial x^i \partial y^k}$, $i + k = 4$, deducem din formula (136) următoarea evaluare a valorii absolute a restului :

$$\begin{aligned} |R| \leqslant \frac{(x_2 - x_1)(y_2 - y_1)}{720}\big[& M_{40}(x_1 - x_1)^5 + 10M_{31}(x_2 - x_1)^3(y_2 - y_1) + \\ & + 20M_{22}(x_2 - x_1)^2(y_2 - y_1)^2 + 10M_{13}(x_2 - x_1)(y_2 - y_1)^3 + M_{04}(y_2 - y_1)^4\big]. \end{aligned} \tag{137}$$

ОБОБЩЕНИЕ КВАДРАТУРНОЙ ФОРМУЛЫ Н. ОБРЕШКОВА В СЛУЧАЕ ДВОЙНЫХ ИНТЕГРАЛОВ

КРАТКОЕ СОДЕРЖАНИЕ

В настоящем труде обобщается квадратурная формула (1) Н. Обрешкова [1] в случае двойных интегралов относительно прямоугольной области. Чтобы достичь этого обобщения, применили тот же метод, что и в труде [3] для обобщения квадратурной формулы трапеции, в случае двойных интегралов относительльно прямоугольной области.

В § 1 даётся предварительная формула (6), приведшая нас к трактовке в § 2 предельной задачи относительно системы уравнений с частными производными (7) при выгодно выбранных предельных условиях, таким образом, чтобы из предварительной формулы (6) могли вывести кубатурную формулу (17). Так как решение этой задачи представляло трудности, мы решили задачу в случаях №. 3 и 4.

Таким образом, для $n = 3$ мы получили кубатурную формулу (39) с её остатком, где функции $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ характеризующие эту формулу, даны формулами (39′). Мы доказали что эти функции отрицательны в прямоугольнике D, что привело нас к оценке абсолютного значения остатка при помощи неравенства (42).

В случае $n = 4$ мы дали две кубатурных формулы. Первая — это формула (89) с остатком, данным формулой (90), а функции $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ даются формулами (50), (51), и (88). Вторая кубатурная формула — это формула (127), остаток которой даётся формулой (128), а функции $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ даются формулами (129′). Мы доказали, что все эти функции положительны в прямоугольнике D, что дало нам возможность оценить абсолютное значение остатка при помощи неравенства (137).

В труде важно то, что задача нахождения кубатурной формулы вида (39), (89), или (127) и остатка свелась к решению предельной задачи относительно уравнений с частными производными (7).

LA GÉNÉRALISATION DE LA FORMULE DE QUADRATURE DE N. OBRESCHKOFF À DES INTÉGRALES DOUBLES

RÉSUMÉ

Dans ce travail on généralise la formule de quadrature (1) de N. Obreschkoff [1] à des intégrales doubles relativement à un domaine rectangulaire. Pour aboutir à ces généralisations on a appliqueé la même méthode que dans le travail [3] pour généraliser la formule de quadrature du trapèze, à des intégrales doubles relativement à un domaine rectangulaire.

Au § 1 este donnée la formule préliminaire (6) qui nous a conduit à traiter, au § 2, un problème aux limites relativement au système d'équations aux dérivées partielles (7) aux conditions aux limites convenablement choisies, telles qu'on puisse déduire de la formule préliminaire (6) une formule de cubature de la forme (17). La résolution de ce problème présentant des difficultés, nous avons résolu ces problèmes dans les cas $n=3$ et 4.

Ainsi nous avons obtenu pour $n = 3$ la formule de cubature (39) avec son reste, où les fonctions $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ qui caractérisent cette formule sont donnes par les formules (39′). Nous avons démontré que ces fonctions sont négatives dans le rectangle D, ce qui nous a conduit à l'évaluation de la valeur absolue du reste par l'inégalité (42).

Dans le cas $n = 4$ nous avons donné deux formules de cubatures. La première est la formule (89) dont le reste est donné par la formule (90) les fonctions $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ étant données par les formules (50), (51) et (88). La deuxième formule de cubature est la formule (127) dont le reste est donné par la formule (128) et les fonctions $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ par les formules (129′) Nous avons démontré que toutes ces fonctions sont positives dans

le rectangle D, ce qui nous a permis de donner une évalution de la valeur absolue du reste par l'inégalité (127).

Ce qui importe d'être relevé, c'est que, dans ce travail, nous avons réduit le problème consistant à trouver une formule de cubature de la forme (39), (89) ou (127) et du reste à la résolution d'un problème aux limites relatif aux équations aux dérivées partielles (7).

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