Derivation of one-dimensional hydrodynamic model for stock price evolution

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Abstract

It was proved that balance equations for systems with corpuscular structure can be derived if a kinematic description by piece-wise analytic functions is available [4]. This article presents a rigorous derivation of an one-dimensional hydrodynamic model for the stock price evolution. The kinematic description is given by a set of time functions describing the evolution of the stock price.

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1 Introduction

The movement of the molecules of a fluid as an ensemble is described by macroscopic continuous fields (e.g. the density, velocity or temperature of a fluid) with a deterministic evolution given by hydrodynamic equations. Thus, even if, on a microscopic scale, the motion of each molecule is very complex, at a macroscopic scale the molecules ensemble has a deterministic common motion. In comparison, a financial market contains a set of assets with a very complex price evolution and one can assume that a hydrodynamic-type description is also possible in this case. Such an approach can be found in [1].

The derivation of a hydrodynamic description for a financial market cannot be obtained by means of the usual methods of the nonequilibrium statistical physics. The derivation of the hydrodynamic equations has, as a starting point, the Hamiltonian of all the molecules [2,3]. From the Hamilton equation through the BBGKY chain one derives the Boltzmann equation, and the hydrodynamic equations are obtained by averaging with respect to velocity.

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This approach cannot be applied for a financial market since the evolution of the asset price is not described by a Hamiltonian. That is, for the evolution of the asset price we have no *dynamical law*, the only available information being a *kinematic* one by means of time functions.

However, as shown in [4,5], the hydrodynamic type description can be obtained for any corpuscular system with a kinematic description. The continuous fields are obtained as space-time coarse-grained averages, and they satisfy the hydrodynamic equations. This method can be used for the molecules of a fluid, as well as for the asset price. If the number of the particles is very large and they form a thermodynamic system satisfying local equilibrium, then the usual hydrodynamic description is obtained. But the coarse-grained averages can be calculated for an arbitrary number of particles (even for a single particle) and they give a hydrodynamic description even for mechanical systems having no statistical properties characteristic to a thermodynamic system [6].

In the following we derive the one-dimensional hydrodynamic model of a financial market. Then we exemplify this method for the price of a single stock so that the significance of the various continuous fields could be better understood. At the end, we discuss the possible applications of this new method in finance.

2 Hydrodynamic model for financial markets

Consider a financial market formed by N assets. We study the evolution of this system during the temporal interval $I = [0, T]$. The position of an asset is given by a quantity directly related to its price (for example, the price itself, the logarithm of the price or the normalized value of the price) and it is denoted by the coordinate x. We assume that the system kinematics is known, i.e. the position of each asset $i \leq N$ is a given function of time $x_i : I \to \mathbb{R}$, and its temporal change ("velocity") is $\xi_i = dx_i/dt : I \to \mathbb{R}$.

Usually the price of an asset is not available as an analytic function but as values given at equal temporal intervals Δt. Then the function $x_i(t)$ must approximate the discrete sequence of the price values, for example by spline functions. In the following, we assume that the position $x_i(t)$ has a linear variation between the moments when the price has well defined values. Its derivative $\xi_i(t)$ has a constant value except for the moments $t = n\,\Delta t$ when it has discontinuous variations.

Let $\varphi_i(t)$, $t \in I$, be the real function of time describing the variation of an arbitrary quantity φ attributed to the i-th asset. In the following, φ_i will represent a constant unit function (1) and the price change (ξ_i). Since the only variations of the velocity ξ_i are the jumps from one constant value to another, the temporal derivative of φ_i identically vanishes $\dot{\varphi}_i \equiv 0$ almost everywhere. Consider two real parameters τ and a with $0 < \tau < T/2$ and $a > 0$, and define the function

$$\langle\varphi\rangle(x,t) \;=\; \frac{1}{4\,\tau\,a}\,\sum_{i=1}^{N}\int_{t-\tau}^{t+\tau} G_i(x,t')\;dt', \tag{1}$$

where

$$G_i(x,t) \;= \varphi_i(t)\;H(a - |x_i(t) - x|)\,, \tag{2}$$

and H is the left continuous Heaviside function. A nonvanishing contribution to $\langle\varphi\rangle$ is only due to assets lying in the spatial interval $(x-a, x+a)$ during the temporal interval $(t-\tau, t+\tau)$. Therefore, $\langle\varphi\rangle(x,t)$ characterizes the mean distribution of φ about the point x and the time t. It is a coarse-grained average over the price and time intervals defined by a and τ, i.e. the density of φ. Obviously, $\langle\varphi\rangle$ also depends on the parameters a and τ, but we do not write explicitly this dependence. The average $\langle\varphi\rangle$ is nonvanishing only if the integral interval in Eq. (1) is contained in I, i.e. $t \in (\tau, T-\tau)$.

For a given x, the integrand (2) is a continuous function, except at a finite number of points where it has discontinuities of jump type. Hence G_i is Riemann integrable and the partial derivative with respect to t of $\langle\varphi\rangle$ is

$$\partial_t\,\langle\varphi\rangle \;=\; \frac{1}{4\,\tau\,a}\,\sum_{i=1}^{N}[\,G_i(x,t+\tau) \;-\; G_i(x,t-\tau)\,]. \tag{3}$$

The function $\langle\varphi\rangle$ depends on x through the instants u when the i-th asset enters or leaves the interval $(x-a, x+a)$. These instants are given by the zeros of the equations

$$x_i(u) \;-\; x \;\pm\; a \;=\; 0\,,$$

and using the implicit function theorem we obtain $du\,/\,dx \;=\; 1/\,\xi_i(u)$. If $u \in (t-\tau, t+\tau)$, then u occurs as integration limit in (1) and the derivative of $\langle\varphi\rangle$ with respect to x is

$$\partial_x\,\langle\varphi\rangle \;=\; \frac{1}{4\,\tau\,a}\,\sum_{i=1}^{N}\left[\,\sum_{u\in U_i'}\frac{\varphi_i(u)}{\xi_i(u)} \;-\; \sum_{u\in U_i''}\frac{\varphi_i(u)}{\xi_i(u)}\,\right]\,, \tag{4}$$

where U_i' (U_i'') is the set containing the instants when the i-th asset leaves (enters) the interval $(x-a, x+a)$ during the interval $(t-\tau, t+\tau)$. One can prove that the partial derivatives (3) and (4) are almost everywhere continuous [4].

Relation (3) shows that $\partial_t \langle\varphi\rangle$ is related to the change of G_i from $t-\tau$ to $t+\tau$. Since $\dot{\varphi}_i \equiv 0$ and $\dot{H} \equiv 0$ almost everywhere, the changes of G_i are only jumps. When the i-th asset enters (leaves) the interval $(x-a, x+a)$, the change of G_i is $+\varphi_i(u)$ (respectively $-\varphi_i(u)$). Comparing with (4), the corresponding part of $\partial_t \langle\varphi\rangle$ is equal to $-\partial_x \langle\varphi\,\xi\rangle$. The change of G_i due to φ_i occurs at $t = n\,\Delta t$, when the assets price has a discontinuous variation. The discontinuous part of $\partial_t \langle\varphi\rangle$ is

$$\delta_d\,\varphi \;=\frac{1}{4\,\tau\,a}\sum_{i=1}^{N}\;\sum_{n=0}^{T/\Delta t} H(\tau-|n\Delta t-t|)\,[G_i(x,n\Delta t-0)-G_i(x,n\Delta t+0)], \tag{5}$$

where $G_i(x, n\Delta t{+}0)$ is the limit from above, and $G_i(x, n\Delta t{-}0)$ from below. We deduce that the relation

$$\partial_t \langle\varphi\rangle \;+\; \partial_x \langle\varphi\,\xi\rangle \;=\; \delta_d\,\varphi \tag{6}$$

is always true. In the following we show that this identity is the general form of the balance equations for the corpuscular system considered.

First we apply the identity (6) to the number of assets, i.e. $\varphi_i \equiv 1$. The price-time average (1) becomes the asset number density or the concentration $c = \langle 1\rangle$. The mean velocity field v is defined by $\langle\xi\rangle = c\,v$ if $c \neq 0$ and is zero otherwise. The term $\delta_d\,\varphi$ defined by (5) vanishes, because the discontinuous variations do not imply a variation of the number of assets. Then relation (6) becomes

$$\partial_t\,c \;+\; \partial_x\,(c\,v) \;=\; 0, \tag{7}$$

which is the continuity equation.

For velocity we choose $\varphi_i \;=\; \xi_i$ and get $\langle\varphi\rangle \;=\; c\,v$. The second term in the left-hand side of (6) can be written as the sum of two terms, namely $\langle\,\xi^2\rangle = \,c\,v^2 {+}\, \langle(\xi{-}v)^2\rangle$. The first term is due to the average motion of the assets in price space and the second one can be interpreted as the "microscopic" flux of velocity. In this case (6) represents the balance equation of velocity

$$\partial_t\,(c\,v) \;+\; \partial_x\,(c\,v^2) \;+\; \partial_x\,(c\,\theta) \;=\; \delta_d\,\xi. \tag{8}$$

Here, we have introduced the kinetic temperature $\theta = \langle(\xi-v)^2\rangle\,/\,c$ (when $c \neq 0$) which is the continuous field related to the volatility. As it follows from the definition (6), the term $\delta_d\,\xi$ is the analogue of the exterior force being a measure of the causes which have induced the discontinuous variations of the velocity ξ_i.

3 Hydrodynamic description of a single stock price

To discuss the meaning of the continuous fields defined in the previous section, we present the hydrodynamic model related to a single asset. We use the daily close price of the stocks issued by the Aluminium Company of America for 100 days beginning with the 27-th of December 1979 (Fig. 1). In order to make the result more intuitive we use as coordinate x the price and not its logarithm. The value of the coarse-grained average $\langle\varphi\rangle$ for an arbitrary time t and position x is given by the price trajectory within the rectangle in Fig. 1.

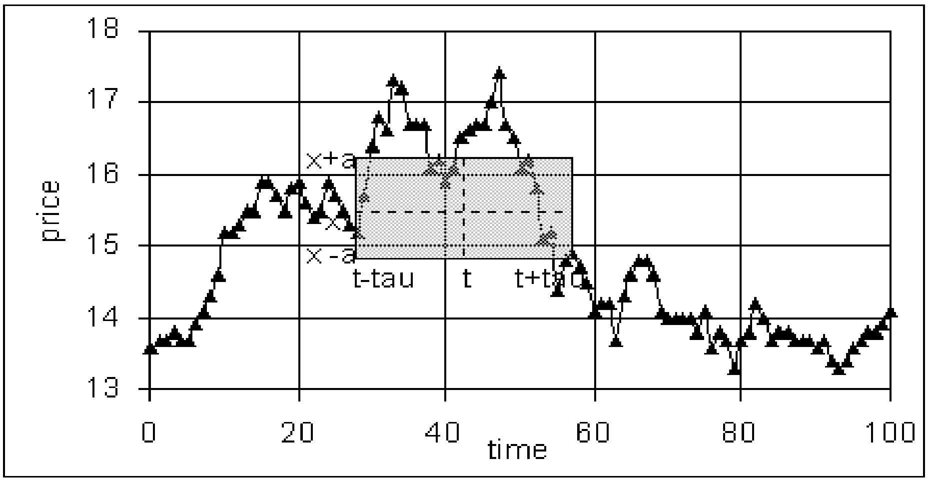

Fig. 1. Stock price recording and averaging domain.

To find the meaning of the concentration $c = \langle 1\rangle$, first we discuss the general case of N assets. If $t \in (\tau, T-\tau)$, then from Eq. (1) it follows

$$\int_{-\infty}^{+\infty} c(x,t)\, dx = N.$$

Hence c describes the distribution of the N assets on the coordinate axis. As a verification we consider the particular case $\tau \ll \Delta t$, $a \ll 1$ and $\tau < a$. Then for $t = n\,\Delta t$, the concentration is nonvanishing only in the neighborhood of the points $x = x_i(n\,\Delta t)$ and it coincides with the discrete repartition of the assets price at time $n\,\Delta t$. When the values of the parameters τ and a increase, the repartition of the assets price becomes more smeared and the concentration c approaches a continuous field. The parameters a and τ characterize the price time scale of the continuous fields. For a single stock, the concentration $c(x,t)$ is proportional to the time interval when the stock price is contained within the rectangle in Fig.1.

Figure 2 presents the support of the concentration for $\tau = 5$d and $a = 0.5$, that is the points (t,x) where $c \neq 0$. The stock mean price at time t is given by $\int_{-\infty}^{+\infty} x\, c(x,t)\, dx$ and corresponds to the circles in Fig. 2. It is easy to verify that when $a \ll 1$, this mean price becomes the usual moving average on the interval $(t-\tau, t+\tau)$.

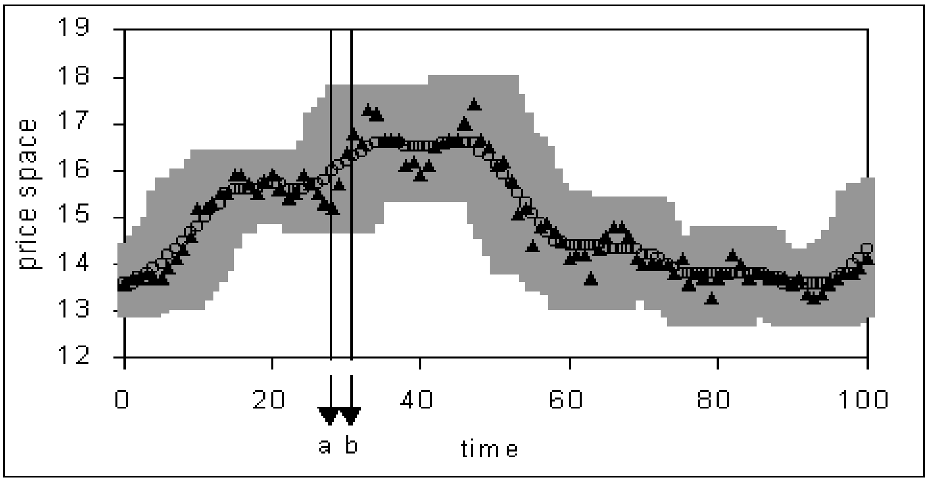

Fig. 2. The support of the concentration field is a price-time band that follows the trend of the mean (○) and the recorded (△) prices.

The velocity field $v = \langle \xi \rangle / c$ is non vanishing only if $c \neq 0$, thus it has the same support as the concentration field. In Fig. 3 we present the regions where the velocity field has the same sign. One can observe that there are situations when the velocity has the same sign for all the prices, whereas in other situations different price trends occur at different values of the price.

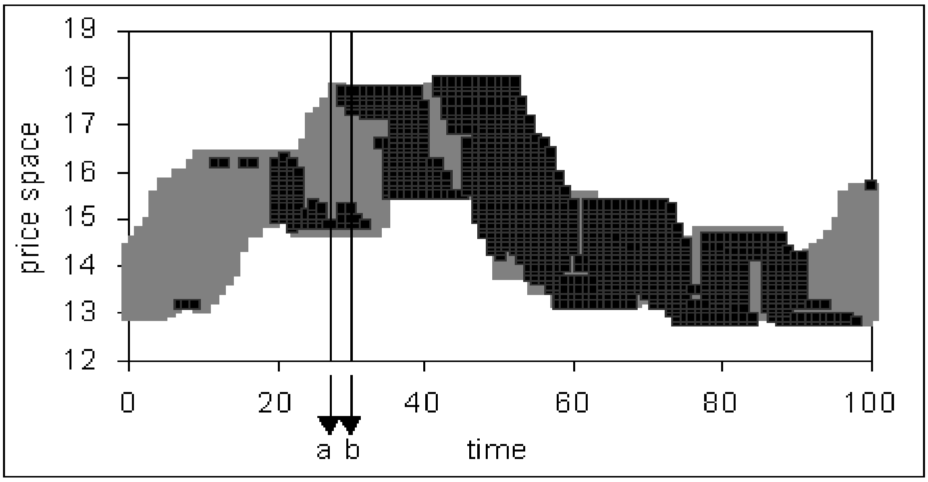

Fig. 3. The support of the velocity field and the areas with rising (grey) and lowering (black) prices.

In Figs. 4 and 5 the four continuous fields in equations (7) and (8) are presented for $t = 27$ and $t = 30$ (corresponding to sections 'a' and 'b' in Figs. 2 and 3).

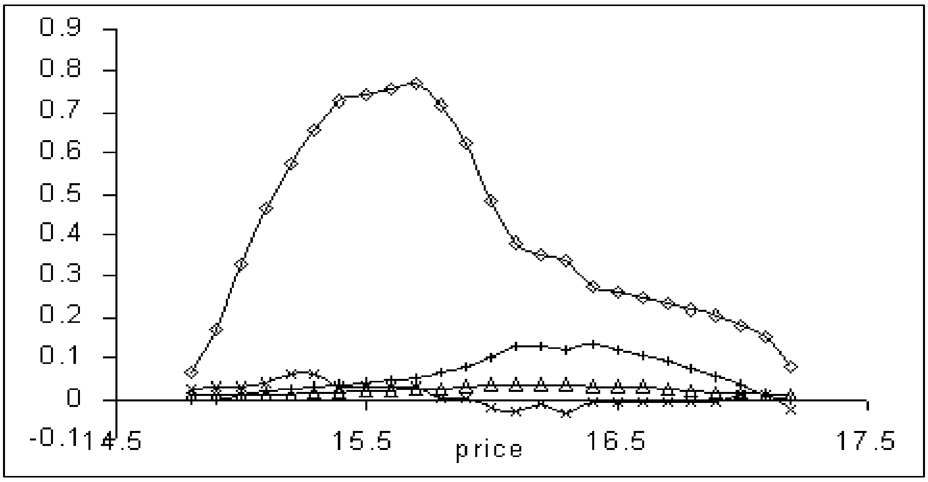

Fig. 4. Hydrodynamic fields at t=27 days: concentration (◇), velocity (+), temperature (△) and force (×)

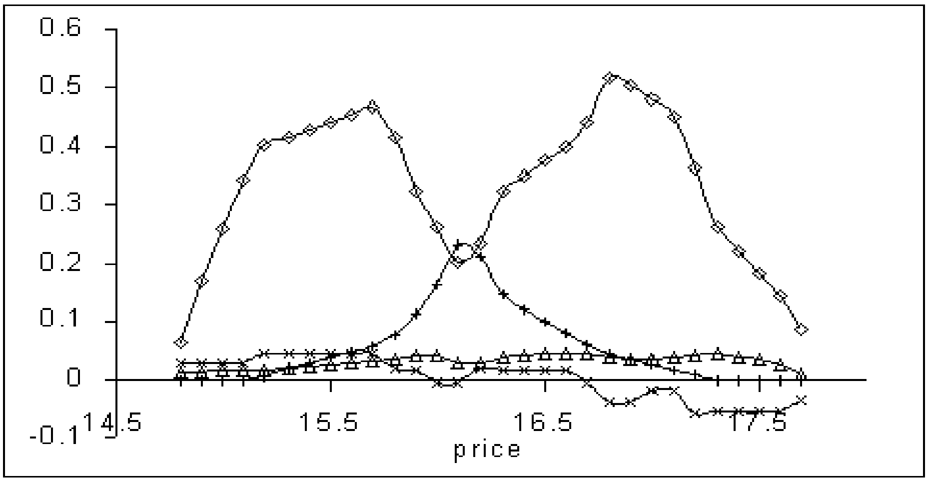

Fig. 5. Hydrodynamic fields at t=30 days.

One can notice how, under the influence of a nonuniform velocity, the concentration develops a complex shape with two maxima. The temperature is not constant having a rather complex variation. As we expect, the exterior force is positive (negative) in the inferior (superior) part of the "fluid", so that it retains the stock price in a bounded domain.

4 Conclusion

As recently shown [7,8], only a small part of the asset price correlation can be attributed to the market evolution, the major part being noise. The method presented in this article could render evident the market evolution in a similar way with the hydrodynamic description of the macroscopic flow of a fluid obtained from the disorderly molecular motion. Moreover, the coarse-grained averages can be calculated from the available information of the asset prices.

In this article we derived only the balance equations for concentration and velocity. The same method can be used to obtain the equation for the kinetic temperature.

Relations (6)-(8) are either identities or equations, according to the available information on the microscopic structure. If the motion of each particle is explicitly known, then (6)-(8) are simple identities containing only known functions. Otherwise, they become the balance equations for the coarse-grained averages $\langle \varphi \rangle$, which now are unknown functions. In general, the number of continuous fields is greater than the number of balance equations and to obtain a solvable problem, additional relations are needed (e.g. the expression of the stress tensor for a specified material). In continuum mechanics, such relations are referred to as "constitutive relations" [9] and represent the second part of the hydrodynamic description. Thus, the hydrodynamic equations always consist of balance equations and constitutive relations. The constitutive relations describe the macroscopic properties of the material and are related to the specific microscopic structure of the corpuscular system. If from the study of a financial market one could formulate such constitutive relations, then the hydrodynamic equations might allow a forecast of the market evolution.

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