Chapter 5

The derivation of the Liouville equation

The probability density ρ in phase space can be defined as the relative number density of the phase points associated with an infinite statistical ensemble. The statistical ensemble have to be infinite because otherwise ρ would be a step function with an a.e. null derivative. Imposing some conditions on the statistical ensemble (which we shall thoroughly discuss in the following) and using the Radon-Nikodym theorem, one can rigorously define ρ as a function with continuous partial derivatives over all the phase space.

The necessity to use an infinite statistical ensemble complicates however the study of the equation satisfied by ρ. The Liouville equation should result from the conservation of the number of the phase points. For example, Landau and Lifshitz ([40], §3) formally consider the set of the phase points as a gas to which the continuity equation can be applied. Using Liouville's theorem for Hamiltonian systems it is easy to show that this continuity equation is equivalent to the Liouville equation. To transform this approach in a rigorous mathematical derivation of the Liouville equation is a difficult task. As the probability density ρ is the limit of a sequence of step functions ρ_N for a statistical ensemble sequence with N representatives, similarly the Liouville equation should be the limit of the discrete equations satisfied by the step functions ρ_N. To our knowledge such a proof has not been presented and the fact that ρ satisfies the Liouville equation is a distinct hypothesis of the statistical mechanics.

In this chapter we present a rigorous derivation of the Liouville equation based on the space-time coarse-grained averages defined in the previous chapters. Instead of a step functions sequence we will obtain a sequence of functions having a.e. continuous first partial derivatives and satisfying usual differential conservation equations. Because we use a kinematic description of the motion of the phase points in phase space, the dynamical laws governing their motion are ignored. We will write the conservation equation for an arbitrary physical quantity in the case of a statistical ensemble with N copies. Then, taking the limit $N \to \infty$, we will obtain the conservation equation for the continuous field in phase space and, as a particular case, the Liouville equation.

In comparison with the case discussed in Chapter 4, the balance equations in the phase space have some special characteristics. The phase space dimension is arbitrary and usually much larger than 3 and the sphere $S(\mathbf{r}, a)$ becomes a hypersphere. In order to avoid the more complicated geometrical properties of a hypersphere we replace it with a hypercube. The edges of the hypercube introduce additional terms in the discrete balance equations. Moreover, in the end it is necessary to take the infinitesimal limit both in time and space, which implies some mathematical problems of limits handling.

The approach in this chapter is more general than that usually used in statistical mechanics. To obtain the Liouville equation, the essential assumption is that the phase space trajectories of a Hamiltonian system are continuous. Then the number of the phase points within a domain changes only when the phase points enter or leave the domain. This process is described as a flux through the domain boundary and a continuity equation can be written. But any dynamical system has continuous trajectories in the phase space and the continuity equation must also exist in the general case. If, as for Hamiltonian systems, the volume in phase space is invariant, then the Liouville equation has the usual form. Furthermore, the continuity equation holds even if the phase points trajectories intersect each other. This situation occurs when the trajectories are projected on a subspace with smaller dimension than the phase space. Thus, we shall consider the projection of an arbitrary dynamical system (not only Hamiltonian) on a subspace of the phase space. As an exemplification of the generality of our results, we shall apply them to the quantum theories with hidden variables.

5.1 Probability density

Let $\mathbb{R}^D$ be the phase space of the dynamical system associated to a physical system. Denote by $q(q_0, t)$ the position at time t of the phase point with the initial position q_0. To avoid mathematical complications, we do not consider that the phase points can be generated or annihilated. A statistical ensemble is a set of phase points $\{M_n \mid 1 \leq n \leq N\}$ with fixed initial positions q_{0n}. The motion of each phase point M_n during the temporal interval $I = [0, T]$ is given by the function $q_n : I \to \mathbb{R}^D$, $q_n(t) = q(q_{0n}, t)$.

Let $P : \mathbb{R}^D \to \mathbb{R}^d$, $d \le D$, be a projection such that the motion of the projection of a phase point M_n is described by an analytic function $x_n = P \circ q_n : I \to \mathbb{R}^d$ and its velocity is $\xi_n = \dot{x}_n$. In the phase space $\mathbb{R}^D$ two trajectories can never intersect each other, but in $\mathbb{R}^d$ the trajectories $x_{n'}$ and $x_{n''}$ intersect at time t and position x if $q_{n'}(t), q_{n''}(t) \in P^{-1}(x)$.

The probability density ρ characterizes the mean distribution of the projections of the representatives of the statistical ensemble about a certain position in $\mathbb{R}^d$. In a Lagrangian description, a physical quantity attached to an arbitrary phase point is a function $\varphi(q_0, t)$. The time variation of φ attached to a representative M_n is given by the analytic time function $\varphi_n(t) = \varphi(q_{0n}, t)$. For a fixed $t \in I$, we define the σ-additive function of the borelian set $A \in \mathcal{B}(\mathbb{R}^d)$

$$\nu_N^{(\varphi)}(t)(A) = \frac{1}{N} \sum_{n=1}^{N} \varphi_n(t)\, \chi_A(x_n(t)), \tag{5.1}$$

where χ_A is the characteristic function of A. Since φ_n is not in general non-negative, $\nu_N^{(\varphi)}(t)$ is not an usual measure, but a real one. We make the following hypothesis:

(A) The sequence $\left\{\nu_N^{(\varphi)}(t)(A)\right\}_N$ *converges for all* $t \in I$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$, *and its limit is denoted by* $\nu^{(\varphi)}(t)(A)$.

The limit $\nu^{(\varphi)}(t)$ is a real measure too. A real measure can be written as an indefinite integral under the conditions of the Radon-Nikodym theorem [34]: *If* $(X, \mathcal{A}, \mu)$ *is a space with* σ*-finite measure and* ν *is a real measure on* $\mathcal{A}$, *absolutely continuous with respect to* μ, *then there exists an integrable function* $f : X \to \mathbb{R}$, *uniquely determined except a null* μ*-measure set, such that* $\nu(A) = \int_A f\, d\mu$ *for all* $A \in \mathcal{A}$. Later on we shall discuss if $\nu^{(\varphi)}(t)$ satisfies the conditions of this theorem. Now we assume that they are satisfied when μ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{A} = \mathcal{B}(\mathbb{R}^d)$. Then there exists a function $\rho^{(\varphi)} : \mathbb{R}^d \to \mathbb{R}$, such that

$$\nu^{(\varphi)}(t)(A) = \int_A \rho^{(\varphi)}(x, t)\, \mu(dx) \tag{5.2}$$

for all $t \in I$ and $A \in \mathcal{B}(\mathbf{R}^d)$.

If $\varphi_n(t) = 1$ for all $t \in I$ and $1 \le n \le N$, then $N\, \nu_N^{(\varphi)}(t)(A)$ is the number of phase points M_n in A at time t and we use the simplified notation $\nu_N(t)(A)$. The measure $\nu_N(t)$ is positive and probabilistic: $\nu_N(t)(\mathbb{R}^d) = 1$. In this case the hypothesis *(A)* is not needed, since the sequence $\{\nu_N(t)(A)\}_N$

is monotone and bounded, so it converges to a probabilistic measure $\nu(t)$. If $\nu(t)$ satisfies the Radon-Nikodym theorem, then relation (5.2) defines the usual probability density $\rho = \rho^{(1)}$. However, not all the measures $\nu(t)$ are absolutely continuous with respect to the Lebesgue measure, for example, when all the phase points M_n lie on sets of null Lebesgue measure. To avoid such situations a new hypothesis is introduced:

(B) The probabilistic measure $\nu(t)$ is absolutely continuous with respect to the Lebesgue measure for all $t \in I$.

Since the functions φ_n are analytic on the compact set $I = [0, T]$, they are bounded and using the hypothesis *(B)* it follows that the measure $\nu^{(\varphi)}(t)$ is absolutely continuous with respect to the Lebesgue measure for any physical quantity φ. Therefore, the function $\rho^{(\varphi)}$ defined by (5.2) exists for any φ.

Consider that in (5.1) the set A is an infinitesimal neighborhood of a point $x \in \mathbb{R}^d$. A nonvanishing contribution to $\nu^{(\varphi)}(t)(A)$ is given only by the phase points M_n which at time t have their projections in the vicinity of x, i.e., the phase points in the vicinity of the set $P^{-1}(x)$. Then, for $N \to \infty$, relation (5.1) can be written $\nu^{(\varphi)}(t)(A) = \bar{\varphi}(x,t)\, \nu(t)(A)$, where $\bar{\varphi}$ is the average of the values of the physical quantity φ attached to the phase points on $P^{-1}(x)$ at time t. Using Eq. (5.2) the continuous field $\bar{\varphi}$ can be defined by the relation $\rho^{(\varphi)} = \rho\bar{\varphi}$ if $\rho \neq 0$ and $\bar{\varphi} = 0$ if $\rho = 0$. When the projection P is the identity mapping, then $\bar{\varphi}(x,t)$ is the value of φ attached to the phase point which at time t has the coordinates given by $P^{-1}(x)$.

The hypotheses *(A)* and *(B)* are an equivalent statement of the usual hypothesis in statistical mechanics that a probability density ρ corresponds to each statistical ensemble. We also make the usual hypothesis on the smoothness of the continuous fields in phase space:

(C) The continuous fields $\rho^{(\varphi)}$ and $\bar{\varphi}$ have continuous second order partial derivatives.

We will use a temporal averaging on the interval $(t^-, t^+) \equiv (t - \tau, t + \tau)$, with $0 < \tau < T/2$ a real parameter defined in (4.5). This temporal averaging and the limit $N \to \infty$ commute

$$\lim_{N\to\infty} \int_{t-\tau}^{t+\tau} \nu_N^{(\varphi)}(s)(A)\, ds = \int_{t-\tau}^{t+\tau} \nu^{(\varphi)}(s)(A) ds \qquad (5.3)$$

if we introduce the additional hypothesis:

(D) The sequence $\left\{\nu_N^{(\varphi)}(t)(A)\right\}_N$ *is uniformly convergent for all* $t \in I$ *and* $A \in \mathcal{B}(\mathbb{R}^d)$.

In the following we consider that A is the open d-dimensional cube with side a denoted by $C_x \equiv \left\{y \in \mathbb{R}^d \mid |y_\alpha - x_\alpha| < a/2, \alpha = 1, 2, ..., d\right\}$ not the sphere $S(\mathbf{r}, a)$ used in (4.5). We define the space-time average generalized to d dimensions

$$\langle\varphi\rangle_N(x,t) = \int_{t-\tau}^{t+\tau} \nu_N^{(\varphi)}(s)(C_x)\, ds \ . \tag{5.4}$$

If $\langle\varphi\rangle_N$ is multiplied by the number of representatives N and divided by the volume of the space-time averaging interval $2\tau a^d$, then the mean density of φ given by (4.5) is obtained. Using (5.1) we have

$$\langle\varphi\rangle_N(x,t) = \sum_{n=1}^{N} \int_{t-\tau}^{t+\tau} G_n(x,s)\, ds, \tag{5.5}$$

where

$$G_n(x,s) = \frac{1}{N}\varphi_n(s) \prod_{\alpha=1}^{d} \chi_\alpha(x_{\alpha n}(s)) \tag{5.6}$$

and χ_α is the characteristic function of the one-dimensional interval $(x_\alpha - a/2, x_\alpha + a/2)$.

We redo the proof of the existence of the balance equations in Section 4.3 for an arbitrary number of dimensions d and another geometrical shape of the averaging domain. For $x \in \mathbb{R}^d$ given, we denote by U_n' (U_n'') the set of the time values for which $G_n(x,t)$ has an increasing (decreasing) discontinuous variation. According to (6), they are among the solutions of the equations

$$x_{\alpha n}(u_\alpha) - x_\alpha \pm \frac{1}{2}\, a = 0, \quad \alpha = 1, 2, ..., d \ . \tag{5.7}$$

Since $x_{\alpha n}$ is an analytic function with respect to u_α, and I is a closed interval, then either Eq. (5.7) has a finite number of solutions, or it is identically satisfied. In the latter case, the characteristic function in (5.6) vanishes and G_n has no discontinuities. Hence for a given x, the integrand $G_n(x,s)$ in (5.5) is a continuous function, except at a finite number of points, where it has discontinuities of jump type. So, it is Riemann integrable and the partial derivative of $\langle\varphi\rangle_N$ with respect to t is

$$\partial_t\langle\varphi\rangle_N = \sum_{n=1}^{N}[G_n(x,t+\tau) - G_n(x,t-\tau)]. \tag{5.8}$$

Now we use a theorem stating that every function with bounded variation may be split into a sum of two functions: one continuous and a jump function. We apply this theorem to G_n considered as a function of t. Except some jump discontinuities due to the characteristic function, G_n is analytic on I and then its continuous part G'_n is also absolutely continuous. We may write $G_n = G'_n + G''_n$, where G''_n is the jump function. Introducing this relation in (5.5), it follows that $\partial_t \langle \varphi \rangle_N$ can also be written as a two-term sum

$$\partial_t \langle \varphi \rangle_N = (\partial_t \langle \varphi \rangle_N)' + (\partial_t \langle \varphi \rangle_N)'' \ . \tag{5.9}$$

According to Lebesgue's theorem, the absolutely continuous part of G_n is equal to the integral of the derivative of G_n and then

$$(\partial_t \langle \varphi \rangle_N)' = \langle \dot{\varphi} \rangle_N \ . \tag{5.10}$$

The discontinuous part of $\partial_t \langle \varphi \rangle_N$ can be written as

$$(\partial_t \langle \varphi \rangle_N)'' = \frac{1}{N} \sum_{n=1}^{N} \sum_{\alpha=1}^{d} \left[\sum_{u_\alpha \in W'_n} \varphi_n(u_\alpha) - \sum_{u_\alpha \in W''_n} \varphi_n(u_\alpha) \right] + \varepsilon_N \ , \tag{5.11}$$

where $W'_n = U'_n \cap (t-\tau, t+\tau)$ and $W''_n = U''_n \cap (t-\tau, t+\tau)$ are the sets of the solutions of Eq. (5.7) in the averaging interval $(t-\tau, t+\tau)$. The term ε_N contains the contributions of the discontinuities of two types. When a phase point intersects an edge of the cube C_x, then several equations (5.7) have the same solution and the contribution of this phase point in (5.11) is taken into account more times. By means of ε_N the additional terms are canceled, such that a phase point contributes only by one value $\varphi_n(u_\alpha)$ to $(\partial_t \langle \varphi \rangle_N)''$. If, for the phase point M_n, $t \pm \tau$ are among the solutions of Eq. (5.7), then the discontinuity is introduced in ε_N. To establish the behavior of ε_N when $N \to \infty$, let us consider $\varphi_n(t) = 1$ for all $t \in I$ and $1 \leq n \leq N$. Then $\varepsilon_N = N'/N$, where N' is the number of phase points intersecting the faces and edges of the cube C_x during the averaging interval $(t-\tau, t+\tau)$, i.e., a null Lebesgue measure set. But the measure $\nu(t)$ is absolutely continuous (hypothesis *(B)*) and then $N'/N \to 0$ for $N \to \infty$. Even if φ_n is arbitrary, it is bounded and ε_N has the same behavior $\varepsilon_N \to 0$ for $N \to \infty$.

Consider the integral

$$J_{\alpha n}(x_\alpha) = \int_{t-\tau}^{t+\tau} \varphi_n(s) \ \xi_{\alpha n}(s) \, \chi_\alpha(x_{\alpha n}(s)) \, ds \ . \tag{5.12}$$

The variation of $J_{\alpha n}$ for a variation δx_α of the argument can be written as the difference of two integrals of type (5.12), but with the characteristic functions of the intervals $(x_\alpha \pm a/2 - \delta x_\alpha/2, x_\alpha \pm a/2 + \delta x_\alpha/2)$. Using the implicit function theorem for Eq. (5.7) and neglecting the powers of δx_α, the two characteristic functions are nonvanishing for a temporal interval of length

$$\delta_\alpha t = \left| \frac{\partial u_\alpha}{\partial x_\alpha} \right| \delta x_\alpha = \frac{\delta x_\alpha}{\mid \xi_{\alpha n} \mid} \ .$$

We denote by $\sigma^{\pm}_{\alpha n} = \xi_{\alpha n}(u^{\pm}_\alpha) \,/\, |\xi_{\alpha n}(u^{\pm}_\alpha)|$ the sign of the α-component of the phase point velocity, where u^{+}_α (u^{-}_α) is the solution of Eq. (7) with plus (minus) sign. Then the mean value theorem gives

$$J'_{\alpha n} = \lim_{\delta x_\alpha \to 0} \frac{\delta J_{\alpha n}}{\delta x_\alpha} = \sigma^{-}_{\alpha n}\, \varphi_n(u^{-}_\alpha) - \sigma^{+}_{\alpha n}\, \varphi_n(u^{+}_\alpha) \ . \tag{5.13}$$

If $\sigma^{-}_{\alpha n} > 0$, then the phase point M_n leaves the cube C_x through the face $x_\alpha + a/2 = 0$ and $G_n(x,t)$ has a discontinuous decreasing, the sign of $\varphi_n(u^{-}_\alpha)$ in (5.11) being minus. Analyzing the other three types of terms in (5.13), it follows that relation (5.11) can be written

$$(\partial_t \langle \varphi \rangle_N)'' = -\frac{1}{N} \sum_{n=1}^{N} \sum_{\alpha=1}^{d} J'_{\alpha n} \prod_{\beta=1 \neq \alpha}^{d} \chi_\beta(q_{\beta n}(s)) + \varepsilon_N \ .$$

From definition (5.1) and relation (5.12) we obtain

$$(\partial_t \langle \varphi \rangle_N)'' = -\sum_{\alpha=1}^{d} \lim_{\delta x_\alpha \to 0} \frac{1}{\delta x_\alpha} \int_{t-\tau}^{t+\tau} \delta \left[\nu_N^{(\varphi \xi_\alpha)}(s)(C_x) \right] ds + \varepsilon_N \ . \tag{5.14}$$

Now consider the limit $N \to \infty$ of the relation (5.9). Analogously to (5.8), the derivative of (5.4) with respect to t is

$$\partial_t \langle \varphi \rangle_N = \left[\nu_N^{(\varphi)}(t+\tau)(C_x) - \nu_N^{(\varphi)}(t-\tau)(C_x) \right] .$$

By means of (5.2), we have for $N \to \infty$

$$\partial_t \langle \varphi \rangle_N \to \int_{C_x} \left[\rho^{(\varphi)}(x, t+\tau) - \rho^{(\varphi)}(x, t-\tau) \right] dx \ . \tag{5.15}$$

The limit of the term (5.10) can be determined by applying (5.3). The order of the limits after N and δx_α can be reversed in (5.14) since, according to the hypothesis *(C)*, the limit function is continuous. Then, using (5.3), we have for $N \to \infty$

$$(\partial_t \langle \varphi \rangle_N)'' \to -\sum_{\alpha=1}^{d} \frac{\partial}{\partial x_\alpha} \int_{t-\tau}^{t+\tau} ds \int_{C_x} \rho^{(\varphi \xi_\alpha)}(x, s)\, dx \, . \tag{5.16}$$

In the limit of (5.9) for $N \to \infty$ we apply the mean value theorem and then for $a \to 0$ and $\tau \to 0$ we obtain

$$\partial_t(\rho\bar{\varphi}) + \sum_{\alpha=1}^{d} \partial_\alpha(\rho\, \overline{\varphi\xi_\alpha}) = \rho\, \bar{\dot{\varphi}} \ . \tag{5.17}$$

This equation can be written as a balance equation in continuum mechanics. For $\varphi_n = \xi_n$, the mean velocity field $v = \bar{\xi}$ is obtained. It is easy to show that (5.17) becomes

$$\partial_t(\rho\bar{\varphi}) + \sum_{\alpha=1}^{d} \partial_\alpha \left(\rho\, v_\alpha\, \overline{\varphi} + \rho\ \overline{\varphi(\xi_\alpha - v_\alpha)}\right) = \rho\, \bar{\dot{\varphi}}\ , \tag{5.18}$$

where according to the discussion in Chapter 1 the terms correspond respectively to: the time variation of the density of φ, the flux of φ due to the mean motion of the phase points, the flux of φ due to the relative motion of the phase points with respect to the mean motion and the production of φ.

We particularize equations (5.17) and (5.18) for two simple cases. If $\varphi_n(t) = 1$ for all n and t, then $\dot{\varphi}_n = 0$ and (5.17) becomes the continuity equation expressing the conservation of the phase points number or, equivalently, of the probability

$$\partial_t\rho + \nabla \cdot (\rho v) = 0 \ . \tag{5.19}$$

If $\varphi_n = \xi_n$, then, using (5.19), the equation (5.18) can be written

$$\partial_t v_\alpha + \sum_{\beta=1}^{d} v_\beta \partial_\beta v_\alpha + \frac{1}{\rho} \sum_{\beta=1}^{d} \partial_\beta(\rho\sigma_{\alpha\beta}) = a_\alpha\ , \tag{5.20}$$

where a_α is the mean acceleration field and the symmetric tensor

$$\sigma_{\alpha\beta} = \overline{(\xi_\alpha - v_\alpha)(\xi_\beta - v_\beta)} \tag{5.21}$$

corresponds to the kinematic part of the stress tensor (3.19). Eq. (5.20) is the analogue of the momentum equation for a continuum (3.21).

The usual Liouville equation is obtained when the projection P is the identity mapping. Then, as we have shown, $v = \xi$ and Eq. (5.19) becomes

$$\partial_t\rho + \nabla \cdot (\rho\xi) = 0 \ . \tag{5.22}$$

For a Hamiltonian system $\nabla \cdot \xi = 0$ and (5.22) takes the usual form of the Liouville equation.

The generality of the hypotheses we have used allows a rigorous treatment of fundamental statistical problems, for example the derivation of the BBGKY hierarchy and the Boltzmann equation. This generality ensures the existence of an equation (5.19) similar to the Liouville equation, even if the volume in phase space is not conserved.

5.2 The Schrödinger equation

The theories with hidden parameters try to explain the quantum properties assuming that there are additional degrees of freedom that are not accessible to the usual measurements. The inexistence of such theories has been proved by von Neumann [71] and experimentally by Aspect and collaborators by means of the Bell inequalities [3]. In the following we present a new theoretical derivation based on the *reductio ad absurdum* method.

We consider a single quantum particle without spin in given exterior conditions, including the force fields, which does not interact with other particles. Let us assume that there exists a hidden parameter theory. Then in the complete phase space the evolution of the particle is described by the set of the possible trajectories compatible with the given exterior conditions. By projecting on the usual three-dimensional Euclidean space (the configurations space) one obtains trajectories that intersect each other but remain distinct and that provide a kinematic description of the quantum system. This way we are under the conditions assumed in the previous section and the derived equations are applicable in this case too.

In the hidden parameters theories the usual interpretation from statistical mechanics is used. A statistic ensemble represents a numerable infinite set of identical copies of the considered physical system. The space probability distribution attached to the statistical ensemble describes the relative number of states that exist in the neighborhood of the considered state. The selection of the representatives of the statistical ensemble is arbitrary and there is an infinity of probability distributions which can be attached to a physical system. However, we do know that for a quantum particle there is a particular quantum probability distribution that is the solution of the Schrödinger equation for the given exterior conditions. This quantum limitation contradicts the total freedom to choose the statistical ensemble from statistical mechanics. By means of this observation Keller [30] rejected the proposition of Bohm's hidden parameters theory in 1952 [7]. Bohm answered this objection introducing the hypothesis that any probability distribution tends, in a very short time, to a solution of the Schrödinger equation. We do not intend to detail this dispute since we consider in the following only probability densities compatible with the Schrödinger equation.

But even for the statistical ensembles compatible with the quantum mechanics principles, features incompatible with the initial hypotheses can occur. This statement results from the fact that the quantum mechanics can be formally obtained by the path integral method introduced by Feynman [17]. The typical quantum behavior results from the hypothesis that the total probability of a process is obtained through the interference of different possible classical trajectories and not through their simple superposition. This "interaction" between the possible trajectories contradicts the basic hypothesis of the statistical ensemble definition that it is formed by identical but *independent* representatives of the considered system. In the following we emphasize this incompatibility using the balance equations from the phase space projection derived in the previous section.

Let us consider that the hypothesis of the hidden variables were true. Then there would exist a phase space $\mathbb{R}^D$ where the evolution of the quantum state would be given by a deterministic dynamical system. As we have discussed, the projection on the configuration space $\mathbb{R}^d$, $d < D$, implies a loss of information and this would explain the quantum behavior. A quantum state would be completely described by the probability density ρ and the velocity of the probability field v. The relation of this description with the usual wave function description $\Psi = A\exp(i\Phi)$ is given by the correspondence principle $\rho = A^2$ and $v = \lambda\nabla\Phi$, where λ is a real constant related to the Plank constant [39]. For a quantum particle in the force field U we have $d = 3$ and replacing ρ and v in the equations (5.19) and (5.20) we obtain the equations

$$\partial_t A^2 + \lambda\nabla \cdot (A^2\nabla\Phi) = 0 \ . \tag{5.23}$$

and

$$\lambda\partial^2_{ti}\Phi + \lambda\sum_{j=1}^{3}\partial_j\Phi\partial_{ij}\Phi + \frac{1}{A^2}\sum_{j=1}^{3}\partial_j(A^2\sigma_{ij}) + \partial_i U = 0\,. \tag{5.24}$$

In these equations the stress tensor form σ_{ij} remains to be specified.

If the equations (5.23) and (5.24) describe the motion of a quantum particle in the force field U, then they have to satisfy the superposition principle of quantum mechanics. If two quantum states Ψ_1 and Ψ_2 are described by A_k and Φ_k, $k = 1, 2$, then the functions

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\Phi_1 - \Phi_2)$$

and

$$\Phi = \mathrm{arctg}\frac{A_1\sin\Phi_1 + A_2\sin\Phi_2}{A_1\cos\Phi_1 + A_2\cos\Phi_2}$$

corresponding to the quantum state $\Psi = \Psi_1 + \Psi_2$, also have to satisfy the same equations. Replacing these quantities in (5.23) through direct computation it follows that each of the two quantum fields has to satisfy the equation

$$\partial_t\Phi_k + \frac{\lambda}{2}(\nabla\Phi_k)^2 - \frac{\lambda}{2A_k}\triangle A_k = U\ , \quad k = 1, 2\ .$$

The gradient of this function coincides with (5.24) if

$$\sigma_{ij} = -\frac{\lambda^2}{4}\rho\partial^2_{ij}\ln\rho \ . \tag{5.25}$$

The equivalence of the Schrödinger equation with the equations (5.23), (5.24), and (5.25) was been obtained for the first time by Takabayasi [57] in the formulation of the “hydrodynamic model of the quantum mechanics”.

The existence of a nonzero stress tensor for the “fluid” formed of the quantum states projection on the configuration spaces is possible only if the statistical ensemble representatives interact, contradicting the initial hypothesis. Therefore the assumption concerning the existence of a hidden parameters description of a quantum particle motion is wrong.