Chapter 9

Financial hydrodynamics

Although the applications of the statistical physics to finance have been developed especially after 1995, however, before 1995 there were some contributions that strongly influenced the statistical theories in finance. In 1900, Bachelier was the first one who modeled the evolution of the logarithm of an asset prices through the Brownian motion. This idea dominated the theoretic approach in finance throughout the entire century. Only in 1963 Mandelbrot noticed that the probability distribution of the prices changes does not decrease exponentially as it is the case with the Gauss distribution, but according to a power law [42] (one says that the probability distribution has "fat tails"). He proposed the Levy stable distribution as a model of the probability distribution of the asset returns. This proposal was improved in 1995 by Mantegna and Stanley by using a truncated Levy distribution [43].

This approach relies on the hypothesis that the financial market is composed of a very large number of complex systems related by nonlinear interactions, similar to many of the systems the statistical physics deals with. In 1996 another approach occurred in which the methods used in turbulent flows study were applied to the financial markets [20]. As in the turbulence there exists an energy cascade from the big scale vortexes toward the small scale ones, in a financial market there is an information cascade from the transactions made by the investors on long term (months or years) toward the speculative transactions on short term (days or even hours and minutes). A dispute started between the supporters of the two approaches. It appealed a larger and larger number of physicists into financial field. Some of them consider that these preoccupations form a new domain, called "econophysics". Lately conferences on this subject have been organized and the first books have been published [44] and [70].

One of the problem that could be approached with the methods of statistical physics is to extract the collective modes of a financial market which can be obtained like in hydrodynamics. The motion of the molecules of a fluid as an ensemble is described by macroscopic continuous fields (for example the density, velocity or temperature of a fluid) with a deterministic evolution given by the hydrodynamic equations. Thus, even if, on a microscopic scale, the motion of each molecule is very complex, at a macroscopic scale the molecules ensemble has a deterministic common motion. In comparison, a financial market contains a set of assets with a very complex price evolution, resembling to that of a molecule. Then we may assume that a hydrodynamic description is possible also for a financial market. Such an attempt can be found in [45] where the hydrodynamic equations for a financial market are written by means of the analogy to the hydrodynamic turbulence. In [14] a fluid is defined in a formal space with many dimensions in which the distance between two types of assets is given by the correlation of the prices variation over a given time interval.

The derivation of a hydrodynamic description for a financial market cannot be obtained by means of the classical methods in nonequilibrium statistical mechanics. The derivation of the hydrodynamic equations has as a starting point the Hamiltonian of all the molecules. From the Hamilton equation in phase space through the BBGKY chain one derives the Boltzmann equation and the hydrodynamic equations are obtained by averaging with respect to the velocity. This approach cannot be applied for a financial market since the evolution of the asset price is not described by a Hamiltonian. That is, for the evolution of the asset price we have no dynamical law, the only available information being a kinematic one by means of the prices time series.

However, as shown in the Chapter 4, the hydrodynamic description can be obtained for any corpuscular system with a kinematic description. The a.e. continuous fields are obtained as space-time coarse-grained averages and they satisfy the hydrodynamic equations. This method can be used for the molecules of a fluid, as well as for the assets prices [66]. Let us consider a financial market formed by N assets. We study the evolution of this corpuscular system during the temporal interval $I = [0, T]$. The position of an asset is given by a quantity directly related to its price (for example, the price itself, the logarithm of the price or the normalized value of the price), and it is denoted by the coordinate x. We assume that the system kinematics is known, i.e., the position of each asset $i \leq N$ is a given function of time $x_i : I \to [\,0\,,\,1\,]$, and its velocity is $\xi_i = dx_i/dt : I \to [\,0\,,\,1\,]$.

Usually the price of an asset is not available as an analytic function but as values given at equal temporal intervals Δt. Then the function $x_i(t)$ have to approximate the discrete sequence of the price values, for example by spline functions. In the following, we assume that the position $x_i(t)$ has a

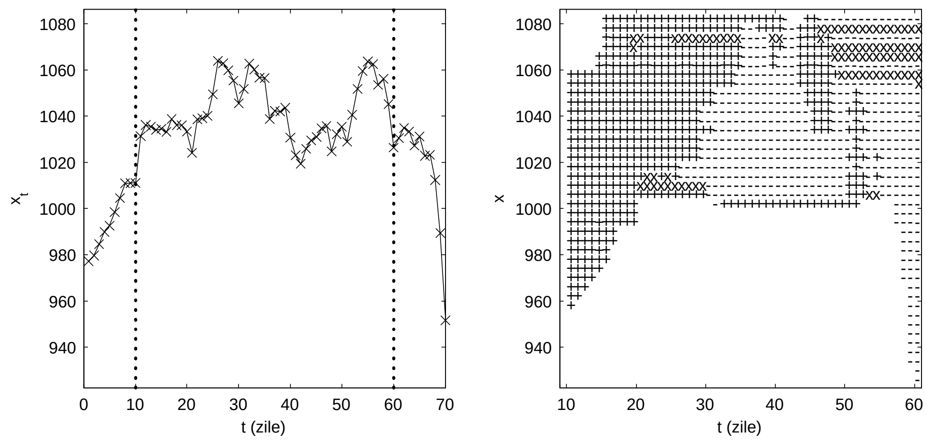

Figure 9.1: The daily values of the Dow-Jones index starting with the 27th of April 1990 and the position of the points (t, x) at which the discrete analogue of the concentration $\langle 1 \rangle$ for $\tau = 10$ and $a = 20$ is nonvanishing. We have denoted by markers the points where the space-time coarse-grained average of the velocity $\langle \xi \rangle$ has positive values (+), negative (-) and null (x) .

linear variation between the moments when the price has well-defined values. Hence $x_i(t)$ is a continuous function and its derivative $\xi_i(t)$ has a constant value except for the moments $t_k = k\,\Delta t$ when it has discontinuous jumps. Thus we have the same situation as in chapter 7, so that we can take over all the results from section 7.1. We rewrite here the equations (7.9) and (7.10) as

$$\partial_t \langle 1 \rangle \; + \; \partial_x \langle \xi \rangle \; = \; 0 \tag{9.1}$$

and

$$\partial_t \langle \xi \rangle \; + \; \partial_x \langle \xi^2 \rangle \; = \; F \; , \tag{9.2}$$

where the right-hand side is the average density of the force acting on the particle at the moments when its velocity varies discontinuously $F = \delta_c \, \xi$.

In order to point out the meaning of the space-time coarse-grained averages we consider the simple case of a single financial series. We have chosen the variation of the daily value ($\Delta t = 1$ day) of the Dow-Jones composite index for 70 days starting from the 27th of April 1990 (Fig. 9.1). Within this period of time the index turned from a continuous increase of more than 10 days to a strong decrease for the last 10 days. Between these two well defined variations the index had a fluctuant evolution with several local minimums

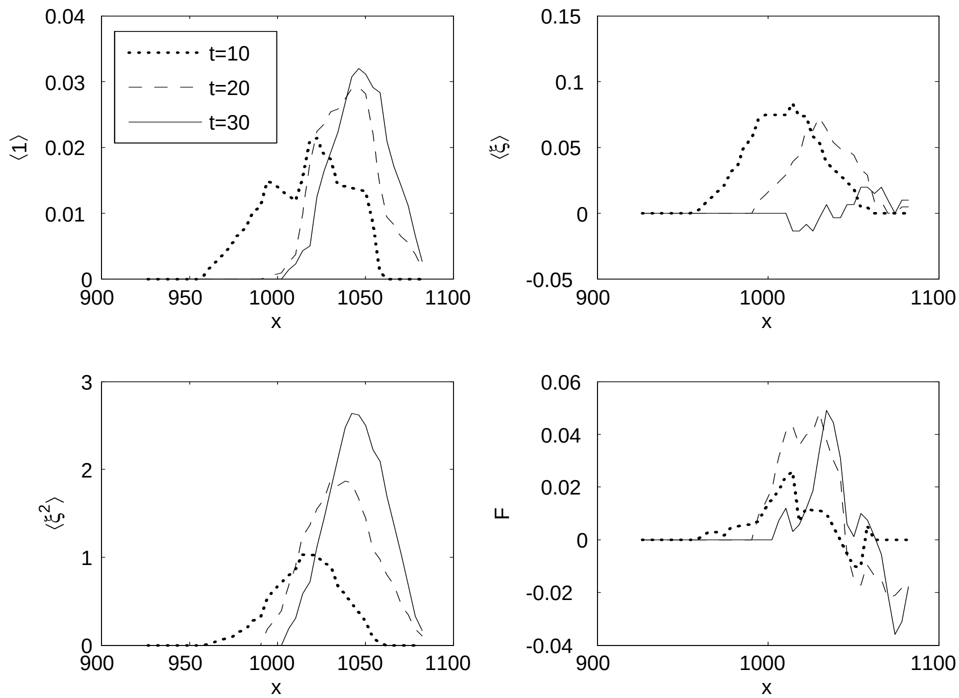

Figure 9.2: The discrete fields $\langle 1 \rangle$, $\langle \xi \rangle$, $\langle \xi^2 \rangle$ and F at the moments $t = 10, 20, 30$.

and maximums. It is interesting to study how this shift between two periods of time with opposite signs variations are reflected by the coarse-grained averages. In order to render more intuitive these results we have chosen the coordinate x equal with the index value, not with its logarithm For an arbitrary moment t and an arbitrary value of the coordinate x, the coarse-grained average value $\langle \varphi \rangle$ is obtained by averaging the contribution of the price trajectory within the interior of the rectangle $(t-\tau, t+\tau) \times (x-a, x+a)$. In this case we have chosen $\tau = 10$ days and $a = 20$.

For a single asset the discrete analogue of the concentration $\langle 1 \rangle$ is proportional with the time interval within which the index price is inside the rectangle $(t-\tau, t+\tau) \times (x-a, x+a)$. In Fig. 9.1 is presented the support of the function $\langle 1 \rangle$, i.e,. the points (t, x) in which $\langle 1 \rangle \neq 0$. Since $\tau = 10$, the time averages over the entire interval $(t-\tau, t+\tau)$ could be performed only between $[\tau, T-\tau]$, i.e, within the interval confined to the area between the vertical dot lines of the plot presenting the index evolution. One can notice that the support of $\langle 1 \rangle$ follows the index evolution form.

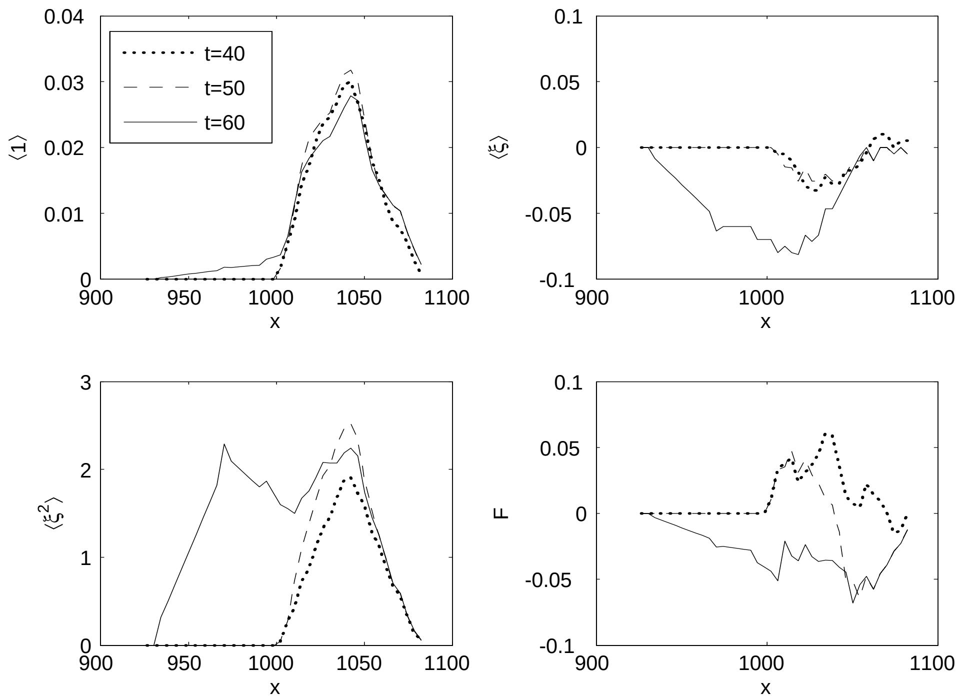

Figure 9.3: The discrete fields $\langle 1\rangle$, $\langle\xi\rangle$, $\langle\xi^2\rangle$ and F at the moments $t = 40, 50, 60$.

The other discrete field occurring in (9.1) is the average density of the velocity $\langle\xi\rangle$ which has the same support as $\langle 1\rangle$. In Fig 9.1 we present the sign of $\langle\xi\rangle(t, x)$. One remarks that at the beginning and the end of the interval when the index has a monotone variation, $\langle\xi\rangle$ has the same sign for t fixed, whereas at the middle of the interval $\langle\xi\rangle$ has opposite signs for the same t. There are situations when $\langle\xi\rangle$ vanishes although $\langle 1\rangle \neq 0$ (marked with x in the figure) when the positive and negative variations of the index mutually compensate. So, the discrete fields structure is more complex in the zones with fluctuant variation of the index and simpler in the intervals where the index has a monotone variation.

In figures 9.2 and 9.3 are represented the fields occurring in the equations (9.1) and (9.2) for the first half of the period analyzed, respectively for the second half. One notices that up to $t = 30$ the discrete concentration $\langle 1\rangle$ becomes concentrated over a smaller interval, its maximum increasing in value. According to the continuity equation (9.1) this shift of $\langle 1\rangle$ is due to the positive average velocity $\langle\xi\rangle$ at $t = 10$ and $t = 20$. At $t = 30$ the average velocity becomes positive at the top of the support of $\langle 1\rangle$ and negative at the bottom, which implies that at $t = 40$ (fig. 9.3) the concentration is distributed over a larger interval. The average flux of the momentum $\langle\xi^2\rangle$ has the same behavior as $\langle 1\rangle$ but the maximum value increase is more significant due to the larger fluctuations of the velocity at the middle of the interval studied. The exterior force F is positive (negative) at the bottom (at the top) of the " fluid" , so that the index value varies within a limited domain. However, for $t = 10$ and $t = 20$ when the shift of the concentration $\langle 1\rangle$ is toward the large values of x, the part with $F > 0$ is larger than the one in which $F < 0$.

In the second half of the analyzed interval (fig. 9.3) the transition toward the state at $t = 60$ is performed, in which the average velocity has only negative values. As a result of this modification the support of averages $\langle 1\rangle$ and $\langle\xi\rangle$ expends toward the smaller values of the index. The average force F also becomes negative for any x, which indicates an abrupt decrease of the index values.

One can numerically verify that the calculated coarse-grained averages identically satisfy the discrete balance equations (9.1) and (9.2). Modifying the values of parameters τ and a continuous descriptions can be obtained at other space-time scales too. For example, for $a \ll 1$, the space-time coarse-grained averages come to the moving average performed over the interval $(t-\tau, t+\tau)$. The main aim of this analysis is to find out the global evolution of more than one asset in a financial market. As recently shown [36] and [49], only a small part of the assets prices correlation can be attributed to the market evolution, the major part being noise. The above method could render evident the market evolution in a similar way with the hydrodynamic description of the macroscopic flow of a fluid obtained from the disorderly molecular motion. Moreover, the coarse-grained averages can be computed from the available information of the assets prices.