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ON SOME ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS

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1. INTRODUCTION

In the papers [3], [4] and [5] are studied nonlinear equations having the from:

$$f(x) + g(x) = 0, \tag{1}$$

where, f, $g : X \to X$, X is a Banach space, f is a differentiable operator and g is continuous but nondifferentiable. For this reason the Newton's method, i.e. the approximation of the solution x^* of the equation (1) by the sequence $(x_n)_{n\geq 0}$ given by

$$x_{n+1} = x_n - \big(f'(x_n)+g'(x_n)\big)^{-1}\big(f(x_n)+g(x_n)\big), \qquad n = 1, 2, \ldots, \; x_0 \in X, \tag{2}$$

cannot be applied.

In the mentioned papers the following Newton-like methods are then considered:

$$x_{n+1} = x_n - f'(x_n)^{-1}\big(f(x_n) + g(x_n)\big), \qquad n = 1, 2, \ldots, \; x_0 \in X, \tag{3}$$

or

$$x_{n+1} = x_n - A(x_n)^{-1}\big(f(x_n) + g(x_n)\big), \qquad n = 1, 2, \ldots, \; x_0 \in X, \tag{3'}$$

where A is a linear operator approximating f'. It is shown that, under certain conditions, these sequences are converging to the solution of (1).

In the present paper, for solving equation (1), we propose the following method:

$$x_{n+1} = x_n - \big(f'(x_n) + [x_{n-1}, x_n; g]\big)^{-1}\big(f(x_n) + g(x_n)\big), \quad n = 1, 2, \ldots, \quad x_0, x_1 \in X \tag{4}$$

where by $[x, y; g]$ we have denoted the first order divided difference of g at the points $x, y \in X$.

So, the proposed method is obtained by combining the Newton's method with the method of chord. The r-convergence order of this method, denoted by p, is the same as for the method of chord (where $p = \frac{1+\sqrt{5}}{2} \approx 1.618$), which is greater than the r-order of the methods (3) and (3′) (see also the numerical example), but is less than the r-order of Newton's method (where usually $p = 2$).

But, unlike the method of chord, the proposed method has a better rate of convergence, because the use of $f'(x_n)$ instead of $[x_{n-1},\ x_n; f]$, as it is in the method of chord, does not affect the coefficient c_2 from the inequalities of the type:

$$\|x_{n+1} - x_n\| \leq c_1 \|x_n - x_{n-1}\|^2 + c_2 \|x_n - x_{n-1}\| \, \|x_{n-1} - x_{n-2}\| ,$$

which we shall obtain in the following.

2. THE CONVERGENCE OF THE METHOD

We shall use, as in [1] and [2] the known definitions for the divided differences of an operator.

Definition 1. *An operator belonging to the space $\mathcal{L}(X,\ X)$ (the Banach space of the linear and bounded operators from X to X) is called the first order divided difference of the operator $g : X \to X$ at the points $x_0,\ y_0 \in X$ if the following properties hold:*

a) $[x_0, y_0;\ g](y_0 - x_0) = g(y_0) - g(x_0)$, *for* $x_0 \neq y_0$;
b) *if g is Fréchet differentiable at $x_0 \in X$, then*

$$[x_0, x_0;\ g] = g'(x_0).$$

Definition 2. *An operator belonging to the space $\mathcal{L}(X, \mathcal{L}(X, X))$, denoted by $[x_0, y_0, z_0;\ g]$ is called the second order divided difference of the operator $g : X \to X$ at the points $x_0, y_0, z_0 \in X$ if the following properties hold:*

a′) $[x_0, y_0, z_0;\ g]\ (z_0 - x_0) = [y_0, z_0;\ g] - [x_0, y_0;\ g]$ *for the distinct points* $x_0, y_0, z_0 \in X$;
b′) *if g is two times differentiable at $x_0 \in X$, then*

$$[x_0, x_0, x_0;\ g] = \tfrac{1}{2} g''(x_0).$$

We shall denote by $B_r(x_1) = \{x \in X|\ \ \|x - x_1\| < r\}$ the ball having the center at $x_1 \in X$ and the radius $r > 0$.

Concerning the convergence of the iterative process (4) we shall prove the following result.

Theorem 3. *If there exist the elements $x_0, x_1 \in X$ and the positive real numbers r, l, M, K and ε such that the conditions*

i) *the operator f is Fréchet differentiable on $B_r(x_1)$ and f' satisfies*

$$\|f'(x) - f'(y)\| \leq l \|x - y\| , \quad \forall x, y \in B_r(x_1);$$

ii) *the operator g is continuous on $B_r(x_1)$,*
iii) *for any distinct points $x, y \in B_r(x_1)$ there exists the application $(f'(y) + [x, y;\ g])^{-1}$ and the inequality*

$$\left\| (f'(y) + [x, y;\ g])^{-1} \right\| \leq M$$

is true;

iv) *for any distinct points* x, y, $z \in B_r(x_1)$ *we have the inequality*

$$\|[x, y, z;\, g]\| \leq K;$$

v) *the elements* x_0, x_1 *satisfy*

$$\|x_1 - x_0\| \leq M\varepsilon, \quad \textit{where} \ \ \varepsilon = \|f(x_1) + g(x_1)\|\,;$$

vi) *the following relations hold:*

$$\|x_2 - x_1\| \leq \|x_1 - x_0\|\,, \quad \textit{with} \ \ x_2 \ \textit{given by (4) for} \ n = 1,$$

$$q = M^2\varepsilon\left(\tfrac{l}{2} + 2K\right) < 1, \ \textit{and}$$

$$r = \tfrac{M\varepsilon}{q}\sum_{k=1}^{\infty} q^{u_k},$$

where $(u_k)_{k\geq 0}$ *is the Fibonacci's sequence* $u_{k+1} = u_k + u_{k-1}$, $k \geq 1$, $u_0 = u_1 = 1$;

are fulfilled, then the sequence $(x_n)_{n\geq 0}$ *generated by* (4) *is well defined, all its terms belonging to* $B_r(x_1)$.

Moreover, the following properties are true:

j) *the sequence* $(x_n)_{n\geq 0}$ *is convergent;*

jj) *let* $x^* = \lim_{n\to\infty} x_n$. *Then* x^* *is a solution of the equation* (1);

jjj) *we have the a priori error estimates:*

$$\|x^* - x_n\| \leq \frac{M\varepsilon}{q\left(1 - q^{\frac{p^n(p-1)}{\sqrt{5}}}\right)}\left(q^{\frac{1}{\sqrt{5}}}\right)^{p^n}, \qquad n \geq 1, p = \tfrac{1+\sqrt{5}}{2}.$$

Proof. We shall prove first by induction that, for any $n \geq 2$,

$$x_n \in B_r(x_1), \tag{5}$$

$$\|x_n - x_{n-1}\| \leq \|x_{n-1} - x_{n-2}\|\,, \text{ and} \tag{6}$$

$$\|x_n - x_{n-1}\| \leq q^{u_{n-1}-1}M\varepsilon. \tag{7}$$

For $n = 2$, from $v)$ and $vi)$ we infer the above relations.

Let us suppose now that relations (5), (6) and (7) hold for $n = 2, 3, \ldots, k$, where $k \geq 2$. Since $x_k, x_{k-1} \in B_r(x_1)$, we can construct x_{k+1} from (4), whence, using $iii)$, we have

$$\|x_{k+1} - x_k\| = \left\|\left(f'(x_k) + [x_{k-1}, x_k;\, g]\right)^{-1}\left(f(x_k) + g(x_k)\right)\right\| \leq M\,\|f(x_k) + g(x_k)\|\,.$$

For the estimation of $\|f(x_k) + g(x_k)\|$ we shall rely on the equality

$$\begin{aligned} g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1};\, g](x_k - x_{k-1}) &= \\ = [x_{k-2}, x_{k-1}, x_k;\, g](x_k - x_{k-1})(x_k - x_{k-2})& \end{aligned}$$

(easily obtained from Definition 1 and Definition 2), which imply, using $iv)$,

$$\begin{aligned} &\|g(x_k) - g(x_{k-1}) - [x_{k-2}, x_{k-1};\, g](x_k - x_{k-1})\| \leq \\ &\leq K\,\|x_k - x_{k-1}\|\left(\|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\|\right) \end{aligned} \tag{8}$$

and on the inequality

$$\left\| f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) \right\| \leq \tfrac{l}{2} \left\| x_k - x_{k-1} \right\|^2, \tag{9}$$

valid because of the assumptions $i)$ concerning f.

For $n = k-1$, by (4), we get

$$-\left(f'(x_{k-1}) + [x_{k-2}, x_{k-1}; g]\right)(x_k - x_{k-1}) - f(x_{k-1}) - g(x_{k-1}) = 0,$$

whence

$$\begin{aligned} f(x_k) + g(x_k) =& f(x_k) - f(x_{k-1}) - f'(x_{k-1})(x_k - x_{k-1}) + g(x_k) - g(x_{k-1}) - \\ &- [x_{k-2}, x_{k-1}; g](x_k - x_{k-1}). \end{aligned}$$

The above relation, together with (8), (9) and (6) for $n = k$ imply

$$\begin{aligned} &\left\| x_{k+1} - x_k \right\| \leq \\ &\leq M \left\| f(x_k) + g(x_k) \right\| \\ &\leq \tfrac{Ml}{2} \left\| x_k - x_{k-1} \right\|^2 + MK \left\| x_k - x_{k-1} \right\| \left(\left\| x_k - x_{k-1} \right\| + \left\| x_{k-1} - x_{k-2} \right\| \right) \\ &\leq M \left\| x_k - x_{k-1} \right\| \left(\tfrac{l}{2} \left\| x_{k-1} - x_{k-2} \right\| + 2K \left\| x_{k-1} - x_{k-2} \right\| \right) \\ &= M \left(\tfrac{l}{2} + 2K \right) \left\| x_k - x_{k-1} \right\| \left\| x_{k-1} - x_{k-2} \right\|. \end{aligned}$$

From the hypothesis of the induction we have on one hand that

$$\begin{aligned} \left\| x_{k+1} - x_k \right\| \leq& M \left(\tfrac{l}{2} + 2K \right) q^{u_{k-2}-1} M\varepsilon \left\| x_k - x_{k-1} \right\| \\ =& q^{u_{k-2}} \left\| x_k - x_{k-1} \right\| \\ <& \left\| x_k - x_{k-1} \right\|, \end{aligned}$$

that is, (6) for $n = k+1$, and, on the other hand

$$\left\| x_{k+1} - x_k \right\| \leq q^{u_{k-2}} \left\| x_k - x_{k-1} \right\| \leq q^{u_{k-2}} q^{u_{k-1}} M\varepsilon = q^{u_k} M\varepsilon,$$

that is, (7) for $n = k+1$.

The fact that $x_{k+1} \in B_r(x_1)$ results from:

$$\begin{aligned} \left\| x_{k+1} - x_1 \right\| &\leq \left\| x_2 - x_1 \right\| + \left\| x_3 - x_2 \right\| + \cdots + \left\| x_{k+1} - x_k \right\| \\ &\leq \tfrac{M\varepsilon}{q} (q^{u_1} + q^{u_2} + \cdots + q^{u_k}) < r. \end{aligned}$$

Now we shall prove that $(x_n)_{n \geq 0}$ is a Cauchy sequence, whence $j)$ follows.

It is obvious that

$$u_k = \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\tfrac{1-\sqrt{5}}{2}\right)^{k+1} \right) \geq \tfrac{1}{\sqrt{5}} \left(\tfrac{1+\sqrt{5}}{2}\right)^k = \tfrac{p^k}{\sqrt{5}},$$

for $k \geq 1$.

So, for any $k \geq 1$, $m \geq 1$ we have

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \|x_{k+1} - x_k\| + \|x_{k+2} - x_{k+1}\| + \cdots + \|x_{k+m} - x_{k+m-1}\| \\ &\leq \tfrac{M\varepsilon}{q}(q^{u_k} + q^{u_{k+1}} + \cdots + q^{u_{k+m-1}}) \\ &\leq \tfrac{M\varepsilon}{q}\Big(q^{\frac{p^k}{\sqrt{5}}} + q^{\frac{p^{k+1}}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-1}}{\sqrt{5}}}\Big).\end{aligned}$$

Using Bernoulli's inequality, it follows

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^{k+1}-p^k}{\sqrt{5}}} + q^{\frac{p^{k+2}-p^k}{\sqrt{5}}} + \cdots + q^{\frac{p^{k+m-1}-p^k}{\sqrt{5}}}\Big) \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(p^2-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(p^{m-1}-1)}{\sqrt{5}}}\Big) \\ &\leq \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big(1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + q^{\frac{p^k(1+2(p-1)-1)}{\sqrt{5}}} + \cdots + q^{\frac{p^k(1+(m-1)(p-1)-1)}{\sqrt{5}}}\Big) \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}}\Big[1 + q^{\frac{p^k(p-1)}{\sqrt{5}}} + \Big(q^{\frac{p^k(p-1)}{\sqrt{5}}}\Big)^2 + \cdots + \Big(q^{\frac{p^k(p-1)}{\sqrt{5}}}\Big)^{m-1}\Big] \\ &= \tfrac{M\varepsilon}{q} q^{\frac{p^k}{\sqrt{5}}} \frac{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}}{1 - q^{\frac{p^k(p-1)}{\sqrt{5}}}}.\end{aligned}$$

Hence

$$\|x_{k+m} - x_k\| \leq \frac{M\varepsilon q^{\frac{p^k}{\sqrt{5}}}\Big(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}m}\Big)}{q(1 - q^{\frac{p^k(p-1)}{\sqrt{5}}})}, \quad k \geq 1,$$

and $(x_n)_{n\geq 0}$ is a Cauchy sequence.

It follows that $(x_n)_{n\geq 0}$ is convergent, and let $x^* = \lim_{n\to\infty} x_n$. For $n \to \infty$ in (4) we get that x^* is a solution of (1). For $m \to \infty$ in the above n equality we obtain the very relation *jjj*).

The theorem is proved. □

3. NUMERICAL EXAMPLE

Given the system

$$\begin{cases} 3x^2y + y^2 - 1 + |x-1| = 0 \\ x^4 + xy^3 - 1 + |y| = 0, \end{cases}$$

we shall consider $X + (\mathbb{R}^2, \|\cdot\|_\infty)$, $\|x\|_\infty = \|(x', x'')\|_\infty = \max\{|x'|, |x''|\}$, $f = (f_1, f_2)$, $g = (g_1, g_2)$. For $x = (x', x'') \in \mathbb{R}^2$ we take $f_1(x', x'') = 3(x')^2x'' + (x'')^2 - 1$, $f_2(x', x'') = (x')^4 + x'(x'')^3 - 1$, $g_1(x', x'') = |x' - 1|$, $g_2(x', x'') = |x''|$.

We shall take $[x, y;\, g] \in M_{2\times 2}(\mathbb{R})$ as

$$\begin{aligned}[x, y;\, g]_{i,1} &= \tfrac{g_i(y', y'') - g_i(x', y'')}{y' - x'}, \\ [x, y;\, g]_{i,2} &= \tfrac{g_i(x', y'') - g_i(x', x'')}{y'' - x''}, \quad i = 1, 2.\end{aligned}$$

Using method (3) with $x_0 = (1, 0)$ we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	1	0	
1	1	0.333 333 333 333 333	$3.333 \cdot 10^{-1}$
2	0.906 550 218 340 611	0.354 002 911 208 151	$9.344 \cdot 10^{-2}$
3	0.885 328 400 663 412	0.338 027 276 361 332	$2.122 \cdot 10^{-2}$
4	0.891 329 556 832 800	0.326 613 976 593 566	$1.141 \cdot 10^{-2}$
5	0.895 238 815 463 844	0.326 406 852 843 625	$3.909 \cdot 10^{-3}$
6	0.895 154 671 372 635	0.327 730 334 045 043	$1.323 \cdot 10^{-3}$
7	0.894 673 743 471 137	0.327 979 154 372 032	$4.809 \cdot 10^{-4}$
8	0.894 598 908 977 448	0.327 865 059 348 755	$1.140 \cdot 10^{-4}$
9	0.894 643 228 355 865	0.327 815 039 208 286	$5.002 \cdot 10^{-5}$
10	0.894 659 993 615 645	0.327 819 889 264 891	$1.676 \cdot 10^{-5}$
11	0.894 657 640 195 329	0.327 826 728 208 560	$6.838 \cdot 10^{-6}$
12	0.894 655 219 565 091	0.327 827 351 826 856	$2.420 \cdot 10^{-6}$
13	0.894 655 074 977 661	0.327 826 643 198 819	$7.086 \cdot 10^{-7}$
...			
39	0.894 655 373 334 687	0.327 826 521 746 298	$5.149 \cdot 10^{-19}$

Using the method of chord with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	$5.000 \cdot 10^{+00}$
2	0.989 800 874 210 782	0.012 627 489 072 365	$1.262 \cdot 10^{-02}$
3	0.921 814 765 493 287	0.307 939 916 152 262	$2.953 \cdot 10^{-01}$
4	0.900 073 765 669 214	0.325 927 010 697 792	$2.174 \cdot 10^{-02}$
5	0.894 939 851 624 105	0.327 725 437 396 226	$5.133 \cdot 10^{-03}$
6	0.894 658 420 586 013	0.327 825 363 500 783	$2.814 \cdot 10^{-04}$
7	0.894 655 375 077 418	0.327 826 521 051 833	$3.045 \cdot 10^{-06}$
8	0.894 655 373 334 698	0.327 826 521 746 293	$1.742 \cdot 10^{-09}$
9	0.894 655 373 334 687	0.327 826 521 746 298	$1.076 \cdot 10^{-14}$
10	0.894 655 373 334 687	0.327 826 521 746 298	$5.421 \cdot 10^{-20}$

Using method (4) with $x_0 = (5, 5)$, $x_1 = (1, 0)$, we obtain

n	$x_n^{(1)}$	$x_n^{(2)}$	$\|x_n - x_{n-1}\|$
0	5	5	
1	1	0	$5.000 \cdot 10^{+00}$
2	0.909090909090909	0.363636363636364	$3.636 \cdot 10^{-01}$
3	0.894886945874111	0.329098638203090	$3.453 \cdot 10^{-02}$
4	0.894655531991499	0.327827544745569	$1.271 \cdot 10^{-03}$
5	0.894655373334793	0.327826521746906	$1.022 \cdot 10^{-06}$
6	0.894655373334687	0.327826521746298	$6.089 \cdot 10^{-13}$
7	0.894655373334687	0.327826521746298	$2.710 \cdot 10^{-20}$

It can be easily seen that, given these data, method (4) is converging faster than (3) and than the method of chord.

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