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# On the chord method

In the paper [1], I.K. Argyros considers as divided difference of the mapping $f : X_1 \to X_2$, where $X_1$ and $X_2$ are Banach spaces, a linear mapping $[x, y; f] \in \mathcal{L}(X_1, X_2)$ which fulfils the following conditions:

(a) $[x, y; f](y - x) = f(y) - f(x)$, for every $x, y \in D$, where $D \subseteq X_1$ is a subset of $X_1$;

(b) there exist the real constants $l_1 \geq 0$, $l_2 \geq 0$, $l_3 \geq 0$ and $p \in (0, 1]$ such that for every $x, y, u \in D$ the following inequality holds:

$$\| [y, u; f] - [x, y; f] \| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_3 \|y - u\|^p .$$

In [1] the hypothesis that the equation:

$$f(x) = 0 \tag{1}$$

admits a simple solution $x^*$ in adopted, and conditions for the convergence of the sequence $(x_n)_{n\geq 0}$ generated by the chord method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \qquad n = 1, 2, \ldots,\ x_0, x_1 \in D_0 \tag{2}$$

are given.

In a recent paper [2] there is shown that, with the hypotheses considered in [1], the convergence speed of the sequence generated by (2) and the error estimation are featured by the inequality:

$$\|x^* - x_{n+1}\| \leq \alpha\, d_0^{t_1^{n+1}}, \tag{3}$$

where $\alpha$ is a precised constant, $0 < d_0 < 1$ and $t_1$ is the positive root of the equation $t^2 - t - p$.

We shall admit further down that the divided difference operator fulfils the conditions (a) and (b), and search for supplementary conditions in order to make equation (1) admit a solution $x^*$ into a precised domain $D_0$ and the sequence $(x_n)_{n\geq 0}$ generated by (2) converge to this solution.

Observe firstly that the identity:

$$x_n - [x_{n-1}, x_n; f]^{-1} f(x_n) = x_{n-1} - [x_{n-1}, x_n; f]^{-1} f(x_{n-1}) \tag{4}$$

holds for every $n = 1, 2, \ldots$ with the hypothesis that the linear mapping $[x_{n-1}, x_n; f]$ admits an inverse mapping.

The following identity

$$\begin{aligned} f(x_{n+1}) =& f(x_n) + [x_{n-1}, x_n; f](x_{n+1} - x_n) \\ &+ ([x_n, x_{n+1}; f] - [x_{n-1}, x_n; f])(x_{n+1} - x_n), \qquad n = 1, 2, \ldots \end{aligned} \tag{5}$$

Let $B > 0, \alpha > 0, 0 < d_0 < 1$, and $x_0, x \in X_1$. Consider the sphere

$$U = \left\{ x \in X_1 : \|x - x_0\| \leq \tfrac{B\alpha d_0}{1 - d_0^{t_1 - 1}} \right\} \tag{6}$$

where $t_1 = \frac{1+\sqrt{1+4p}}{2}$ that is, the positive root, the equation:

$$t^2 - t - p = 0 \tag{7}$$

The following theorem holds:

**Theorem 1.** *If the divided difference* $[x, y; f]$ *fulfils the conditions* (a) *and* (b) *for every* $x, y \in U$ *and the following hypotheses:*

(1) *the mapping* $[x, y; f]$ *admits a bounded inverse mapping for every* $x, y \in U$*, namely there exists a constant* $B > 0$ *such that* $\left\| [x, y; f]^{-1} \right\| \leq B$

(ii)

$$\alpha = \frac{1}{B^{(1+p)/p} (l_1 + l_2 + l_3)^{1/p}};$$

(iii)

$$\|x_1 - x_0\| \leq B\alpha d_0, \quad \|f(x_0)\| \leq \alpha d_0, \quad \|f(x_1)\| \leq \alpha d_0^{t_1}$$

*are also fulfilled, then equation* (1) *has at least one solution* $x^* \in U$ *and the sequence* $(x_n)_{n \geq 0}$ *generated by* (2) *converges to* $x^*$*, the convergence speed and the error estimation being featured by the inequality:*

$$\|x^* - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n (t_1 - 1)}}.$$

*Proof.* From (2) for $n = 1$ we deduce:

$$\|x_2 - x_1\| \leq B \|f(x_1)\| \leq B\alpha d_0^{t_1}$$

from which, taking also into account iii. it follows

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq B\alpha d_0^{t_1} + B\alpha d_0 \\ &\leq B\alpha d_0 \left(1 + d_0^{t_1 - 1}\right) \\ &< \frac{B\alpha d_0}{1 - d_0^{t_1 - 1}} \end{aligned}$$

from which it results that $x_2 \in U$.

Using the fact that $x_2 \in U$, the identities (4) and (5), and the inequality a), we obtain

$$\begin{aligned} \|f(x_2)\| &\leq \|x_2 - x_1\| \left(l_1 \|x_2 - x_0\|^p + l_2 \|x_1 - x_0\|^p + l_3 \|x_2 - x_1\|^p\right) \\ &\leq B \|f(x_1)\| \left(l_1 B^p \|f(x_0)\|^p + l_2 \|x_1 - x_0\|^p + l_3 B^p \|f(x_1)\|^p\right) \\ &\leq B\alpha d_0^{t_1} \left(l_1 B^p \alpha^p d_0^p + l_2 B^p \alpha^p d_0^p + l_3 B^p \alpha^p d_0^{t_1}\right) \\ &\leq B^{p+1} \alpha^{p+1} d_0^{t_1 + p} \left(l_1 + l_2 + l_3 d_0^{p(t_1 - 1)}\right) \\ &= B^{p+1} \alpha^{p+1} \left(l_1 + l_2 + l_3 d_0^{p(t_1 - 1)}\right) d_0^{t_1^2 - p} \leq \alpha d_0^{t_1^2} \end{aligned}$$

since $\alpha^p B^{p+1} \left(l_1 + l_2 + l_3 d_0^{p(t_1 - 1)}\right) \leq \alpha^p B^{p+1} (l_1 + l_2 + l_3) < 1$.

From the above inequality follows therefore:

$$\|f(x_2)\| \leq \alpha d_0^{t_1^2}$$

Suppose by induction that:

(a') $x_i \in U,\ i = 0, 1, \ldots, k;$

(b') $\|f(x_i)\| \le \alpha d_0^{t_1^i},\ i = 1, 2, \ldots, k.$

Then, for $x_{k+1}$we have:

$$\begin{aligned}
&\|x_{k+1} - x_0\| \le \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \ldots + \|x_1 - x_0\| \\
&\le B\|f(x_k)\| + B\|f(x_{k-1})\| + \ldots + B\alpha d_0 \\
&\le B\alpha d_0^{t_1^k} + B\alpha d_0^{t_1^{k-1}} + \ldots + B\alpha d_0 \\
&= B\alpha d_0 \left(1 + d_0^{t_1 - 1} + d_0^{t_1^2 - 1} + \ldots + d_0^{t_1^k - 1}\right) \\
&\le B\alpha d_0 \left(1 + d_0^{t_1 - 1} + d_0^{2(t_1-1)} + \ldots + d_0^{k(t_1-1)}\right) \le \tfrac{B\alpha d_0}{1 - d_0^{t_1 - 1}}
\end{aligned}$$

from which follows that $x_{k+1} \in U$. Proceeding now for $x_{k+1}$, as in the case of $x_2$, we obtain:

$$\begin{aligned}
\|f(x_{k+1})\| &\le B^{p+1}\alpha^{p+1}\left(l_1 + l_2 + l_3 d_o^{p t_1^{k-1}(t_1-1)}\right) d_0^{t_1^{k-1}(t_1+p)} \\
&\le B^{p+1}\alpha^{p+1}(l_1 + l_2 + l_3)\, d_0^{t_1^{k+1}} \le \alpha d_0^{t_1^{k+1}}
\end{aligned}$$

It results therefore that the relations (a') and (b') hold for every $i \in \mathbb{N}$.

Now we shall show that the sequence $(x_n)_{n\ge 0}$ is fundamental.

Indeed, for every $n, s \in \mathbb{N}$ we have:

$$\begin{aligned}
\|x_{n+s} - x_n\| &\le \sum_{k=n}^{n+s-1} \|x_{k+1} - x_k\| \le \sum_{k=n}^{n+s-1} B\|f(x_k)\| \le B\alpha \sum_{k=n}^{n+s-1} d_0^{t_1^k} \\
&= B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^k - t_1^n} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n-s-1} d_0^{t_1^n\left(t_1^{k-n} - 1\right)} \\
&\le B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^n (k-n)(t_1-1)} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} \left(d_0^{t_1^n(t_1-1)}\right)^{k-n} \\
&\le \tfrac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n(t_1-1)}}.
\end{aligned}$$

By the last inequality and the fact that $0 < d_0 < 1$ and $t_1 > 1$ follows that the sequence $(x_n)_{n\ge 2}$ is fundamental. For $s \to \infty$, from the inequality:

$$\|x_{n+s} - x_n\| \le \tfrac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n(t_1-1)}}$$

follows the inequality:

$$\|x^* - x_n\| \le \tfrac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n(t_1-1)}}.$$

In [1] Argyros showed that if the divided difference $[x, y; f]$ fulfils the conditions (a) and (b) then $f$ is Fréchet differentiable and $[x, x; f] = f'(x)$. From this fact follows that the mapping $f$ is continuous on $B$: hence at limit for $n \to \infty$ in the inequality:

$$\|f(x_n)\| \le \alpha d_0^{t_1^n},$$

one obtains:

$$\|f(x^*)\| \leq 0$$

from which results $f(x^*) = 0$. With this the theorem is entirely proved $\square$

**Remark 2.** *In* [5], [6] *Schmidt imposes in the divided difference conditions similar to the conditions (a) and (b) given by Argyros in* [1], *but for* $p = 1$. *The same conditions are reproduced in* [2], *too.*

## References

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Institutul de Calcul
Academia Română
Cluj-Napoca