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ON THE CONVERGENCY OF A STEFFENSEN - TYPE METHOD

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1. In the paper [1] I.K.Argyros adopts for the divided difference of the mapping $f: X_1 \to X_2$, where X_1 and X_2 are Banach spaces, the following definition:

Definition. One calls *divided difference* of the application f on the points $x,y \in X_1$ a linear application $[x,y;f] \in \mathcal{L}(X_1,X_2)$ which fulfils the following conditions:

(a) $[x,y;f](y - x) = f(y) - f(x)$ for every $x,y \in D \subseteq X_1$;

(b) there exist the real constants $l_1 > 0$, $l_2 > 0$, $l_3 > 0$, $p \in (0,1]$ such that for every $x,y,u \in D$ the following inequality holds:

$$\|[y,u;f] - [x,y;f]\| \le l_1\|x - u\|^p + l_2\|x - y\|^p + l_3\|u - y\|^p .$$

In [4] there are obtained refinements of Argyros results concerning the secant method applied to the solution of the equation:

$$f(x) = 0, \tag{1}$$

where $f: X_1 \to X_2$.

2. We shall study further down the convergency of Steffensen's method for the solution of equation (1), namely the convergency of the sequence $(x_n)_{n\ge0}$ generated by means of the following procedure:

(2) $x_{n+1} = x_n - [x_n, g(x_n); f]^{-1} f(x_n), \quad x_0 \in X_1, \quad n = 0,1,\ldots,$

where $g: X_1 \to X_1$ is an operator having at least one fixed point which coincides with the solution of equation (1).

Obviously, the sequence $(x_n)_{n\geq 0}$ can be generated by means of the procedure (2) if at each iteration step there exists the mapping $[x_n, g(x_n); f]^{-1}$.

For our purpose observe firstly that the following identities:

(3) $x_n - [x_n, g(x_n); f]^{-1} f(x_n) = g(x_n) - [x_n, g(x_n); f]^{-1} f(g(x_n)),$

(4) $f(x_{n+1}) = f(g(x_n)) + [x_n, g(x_n); f]\,(x_{n+1} - g(x_n)) +$

$+ ([g(x_n), x_{n+1}; f] - [x_n, g(x_n); f])\,(x_{n+1} - g(x_n))$

hold for every $n = 0,1,\ldots$

Let $x_0 \in X_1$ be an element, and consider the nonnegative real numbers: B, ϵ_0, ρ_0, $p \in (0,1]$, α, β, $q \geq 1$, l_1, l_2 and l_3, where

$$\rho_0 = B\alpha(l_1 B^p + l_2 \beta^p + l_3 B^p \alpha^p \|f(x_0)\|^{p(q-1)})$$

and

$$\epsilon_0 = \rho^{1/(p+q-1)} \|f(x_0)\| .$$

Denote $r = \max\{B, \beta\}$ and suppose that $S \subseteq D$, where:

$$S = \left\{ x \in X_1 \mid \|x - x_0\| \leq \frac{r\epsilon_0}{\rho_0^{1/(p+q-1)}(1 - \epsilon_0^{p+q-1})} \right\}$$

The following theorem holds:

THEOREM. *If the constants* B, ϵ_0, ρ_0, p, α, β, q, l_1, l_2, l_3, *the mapping* f *and* g, *and the initial element* $x_0 \in X_1$, *as well, fulfil the conditions:*

(I) *for every* $x,y \in S$ *there exists* $[x,y;f]^{-1}$, *and* $\|[x,y;f]^{-1}\| \leq B$

(II) *for every* $x \in S$, $\|f(g(x))\| \leq \alpha \|f(x)\|^q$;

(III) *for every* $x \in S$, $\|x - g(x)\| \leq \beta \|f(x)\|$;

(IV) *the divided difference of the mapping* f *fulfils the*

conditions (a) and (b) specified in the definition given in Section 1;

(V) $\epsilon_0 < 1$,

then the sequence $(x_n)_{n\geq 0}$ *generated by the procedure (2) is convergent, and, if we denote* $\bar{x} = \lim x_n$, *then* $f(\bar{x}) = 0$ *and the following delimitation holds:*

$$\|\bar{x} - x_n\| \leq \frac{r\rho_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1 - \epsilon_0^{p+q-1})} .$$

Proof. Consider $x_0 \in X_1$ for which the condition (V) is fulfiled. Taking into account the condition (b) and the procedure (2), from the identities (3) and (4) it results:

$$\|x_1 - x_0\| \leq B\|f(x_0)\| \leq \frac{B\epsilon_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}} \|f(x_0)\| \leq \frac{r\epsilon_0}{\rho_0^{1/(p+q-1)}(1-\epsilon_0^{p+q-1})} .$$

from which follows that $x_1 \in S$.

Here was used the inequality:

$$\|g(x_0) - x_0\| \leq \beta\|f(x_0)\| \leq \frac{r\epsilon_0}{\rho_0^{1/(p+q-1)}(1 - \epsilon_0^{p+q-1})} ,$$

from which follows that $g(x_0) \in S$.

Now, considering the above results, we have:

$$\|f(x_1)\| \leq \|[g(x_0), x_1; f] - [x_0, g(x_0); f]\| \cdot \|x_1 - g(x_0)\| \leq$$

$$\leq \beta\alpha[l_1 B^p + l_2 \beta^p + l_3 B^p \alpha^p \|f(x_0)\|^{p(q-1)}]\|f(x_0)\|^{p+q} = \rho_0 \|f(x_0)\|^{p+q}$$

This inequality leads to:

$$\rho_0^{1/(p+q-1)} \|f(x_1)\| \leq (\rho_0^{1/(p+q-1)} \|f(x_0)\|)^{p+q}$$

or, using the notation $\epsilon_1 = \rho_0^{1/(p+q-1)} \|f(x_1)\|$:

$$\epsilon_1 \leq \epsilon_0^{p+q}$$

From this inequality follows that $\|f(x_1)\| \leq \|f(x_0)\|$, and if

$$\rho_1 = B\alpha(l_1 B^p + l_2 \beta^p + l_3 B^p \alpha^p \|f(x_1)\|^{p(q-1)}) \text{ then } \rho_1 \leq \rho_0 .$$

Suppose now that the following properties hold:

(α) $x_p \in S$;

(β) $\|f(x_p)\| \leq \|f(x_{p-1})\|$;

(γ) $\varepsilon_p \leq \varepsilon_0^{(p+q)^p}$, $\varepsilon_p = \rho_0^{1/(p+q-1)} \|f(x_p)\|$, $p = 1,2,\dots,k$.

From (2) for $n = k$ we obtain:

$$\|x_{k+1} - x_k\| \leq B\|f(x_k)\| \leq \frac{r\varepsilon_k}{\rho_0^{1/(p+q-1)}} \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}} ,$$

which leads to:

$$\|x_{k+1} - x_0\| \leq \frac{r}{\rho_0^{1/(p+q-1)}} (\varepsilon_0 + \varepsilon_0^{p+q} + \varepsilon_0^{(p+q)^2} + \dots + \varepsilon_0^{(p+q)^k}) \leq$$

$$\leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1 - \varepsilon_0^{p+q-1})} ,$$

namely $x_{k+1} \in S$.

Here was used the inequality:

$$\|g(x_k) - x_k\| \leq \beta\|f(x_k)\| \leq \frac{r\rho_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}} \|f(x_k)\| \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}} ,$$

from which follows immediately:

$$\|g(x_k) - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1 - \varepsilon_0^{p+q-1})} ,$$

that is, $g(x_k) \in S$.

As to $\|f(x_{k+1})\|$ we have:

$$\|f(x_{k+1})\| \leq B\alpha(l_1B^p + l_2\beta^p + l_3\alpha^pB^p \|f(x_k)\|^{p(q-1)})\|f(x_k)\|^{p+q}$$

namely

$$\|f(x_{k+1})\| \leq \rho_0 \|f x_k)\|^{p+q} ,$$

which yields:

$$\varepsilon_{k+1} \leq \varepsilon_k^{p+q} \leq \varepsilon_0^{(p+q)^{k+1}} .$$

By virtue of the above proved results follows that the properties (α) - (γ) hold for every $p \in N$.

We prove further down that the sequence $(x_n)_{n\geq 0}$ is a

fundamental sequence. Indeed, we have:

$$\|x_{n+s} - x_n\| \leq \|x_{n+s} - x_{n+s-1}\| + \|x_{n+s-1} - x_{n+s-2}\| + \dots + \|x_{n+1} - x_n\| \leq$$

$$\leq B(\|f(x_n)\| + \|f(x_{n+1})\| + \dots + \|f(x_{n+s-1})\|) \leq$$

$$\leq \frac{B}{\rho_0^{1/(p+q-1)}} (\varepsilon_0^{(p+q)^n} + \varepsilon_0^{(p+q)^{n+1}} + \dots + \varepsilon_0^{(p+q)^{n+s-1}}) \leq$$

$$\leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}} (1 + \varepsilon^{p+q-1} + \varepsilon^{(p+q)^2-1} + \varepsilon^{(p+q)^{s-1}}) \leq$$

$$\leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1 - \varepsilon_0^{p+q-1})} ,$$

that is, for every $s,n \in N$ the following inequality holds:

$$\|x_{n+s} - x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1 - \varepsilon_0^{p+q-1})} ,$$

from which, since $\varepsilon_0 < 1$, it results that the sequence $(x_n)_{n\geq 0}$ is fundamental. Since X_1 is a Banach space, there exists $\lim_{n\to\infty} x_n = \bar{x}$, and

$$\|\bar{x} - x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1 - \varepsilon_0^{p+q-1})} ,$$

which leads, for $n = 0$, to $\bar{x} \in S$.

From the inequality $\varepsilon_n \leq \varepsilon_0^{(p+q)^n}$, for $n \to \infty$, we obtain:

$$f(\bar{x}) = \lim_{n\to\infty} f(x_n) = 0 ,$$

and one sees that $\bar{x}$ is the solution of the equation (1).

R E F E R E N C E S

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