

ON THE CONVERGENCY OF A STEFFENSEN-TYPE METHOD

ION PĂVĂLOIU

1. In the paper [1] I.K. Argyros adopts for the divided difference of the mapping $f : X_1 \rightarrow X_2$, where X_1 and X_2 are Banach spaces, the following definition:

Definition 1. One calls divided difference of the application f at the points $x, y, \in X_1$ a linear application $[x, y; f] \in \mathcal{L}(X_1, X_2)$ which fulfils the following conditions:

- (a) $[x, y; f](y - x) = f(y) - f(x)$ for every $x, y \in D \subseteq X_1$;
- (b) there exist the real constants $l_1 > 0$, $l_2 > 0$, $l_3 > 0$, $p \in (0, 1]$ such that for every $x, y, u \in D$ the following inequality holds:

$$\|[y, u; f] - [x, y; f]\| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_3 \|u - y\|^p.$$

In [4] there are obtained refinements of Argyros results concerning the secant method applied to the solution of the equation:

$$(1) \quad f(x) = 0$$

where $f : X_1 \rightarrow X_2$

2. We shall study further down the convergence of Steffensen's method for the solution of equation (1), namely the convergence of the sequence $(x_n)_{n \geq 0}$ generated by means of the following procedure:

$$(2) \quad x_{n+1} = x_n - [x_n, g(x_n); f]^{-1} f(x_n), \quad x_0 \in X_1, \quad n = 0, 1, \dots,$$

where $g : X_1 \rightarrow X_1$ is an operator having at least one fixed point which coincides with the solution of equation (1).

Obviously, the sequence $(x_n)_{n \geq 0}$ can be generated by means of the procedure (2) if at each iteration step there exists the mapping $[x_n, g(x_n); f]^{-1}$.

For our purpose observe firstly that the following identities:

$$(3) \quad x_n - [x_n, g(x_n); f]^{-1} f(x_n) =$$

$$(4) \quad = g(x_n) - [x_n, g(x_n); f]^{-1} f(g(x_n)) f(x_{n+1})$$

$$= f(g(x_n)) + [x_n, g(x_n); f](x_{n+1} - g(x_n))$$

$$+ ([g(x_n), x_{n+1}; f] - [x_n, g(x_n); f])(x_{n+1} - g(x_n))$$

hold for every $n = 0, 1, \dots$

Let $x_0 \in X_1$ be an element, and consider the nonnegative real numbers: $B, \varepsilon_0, \rho_0, p \in (0, 1]$, $\alpha, \beta, q \geq 1$, l_1, l_2 and l_3 , where

$$\rho_0 = \beta \alpha (l_1 B^p + l_2 B^p + l_3 B^p \alpha^p \|f(x_0)\|^{p(q-1)})$$

and

$$\varepsilon_0 = \rho_0^{1/(p+q-1)} \|f(x_0)\|.$$

Denote $r = \max\{B, \beta\}$ and suppose that $S \subseteq D$, where:

$$S = \left\{ x \in X_1 : \|x - x_0\| \leq \frac{r \varepsilon_0}{\rho_0^{1/(p+q-1)} (1 - \varepsilon_0^{p+q-1})} \right\}.$$

The following theorem holds:

Theorem 1. *If the constants $B, \varepsilon_0, \rho_0, p, \alpha, \beta, q, l_1, l_2, l_3$, the mapping f and g , and the initial element $x_0 \in X_1$, as well, fulfil the conditions:*

- (I) *for every $x, y \in S$ there exists $[x, y; f]^{-1}$, and $\|[x, y; f]^{-1}\| \leq B$*
- (II) *for every $x \in S$, $\|f(g(x))\| \leq \alpha \|f(x)\|^q$;*
- (III) *for every $x \in S$, $\|x - g(x)\| \leq \beta \|f(x)\|$;*
- (IV) *the divided difference of the mapping f fulfils the conditions (a) and (b) specified in the definition given in Section 1;*

(V) $\varepsilon_0 < 1$,

then the sequence $(x_n)_{n \geq 0}$ generated by the procedure (2) is convergent, and, if we denote $\bar{x} = \lim x_n$, then $f(\bar{x}) = 0$ and the following delimitation holds:

$$\|x - x_n\| \leq \frac{r\rho_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})}.$$

Proof. Consider $x_0 \in X_1$ for which the condition (V) is fulfilled. Taking into account the condition (b) and the procedure (2), from the identities (3) and (4) it results:

$$\|x_1 - x_0\| \leq B \|f(x_0)\| \leq \frac{B\varepsilon_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}} \|f(x_0)\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},$$

from which follows $x_1 \in S$.

Here was used the inequality:

$$\|g(x_0) - x_0\| \leq \beta \|f(x_0)\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},$$

from which follows that $g(x_0) \in S$.

Now, considering the above results, we have:

$$\begin{aligned} \|f(x_1)\| &\leq \|[g(x_0), x_1; f] - [x_0, g(x_0); f]\| \cdot \|x_1 - g(x_0)\| \\ &\leq \beta\alpha \left[l_1 B^p + l_2 B^p + l_3 B^p \alpha^p \|f(x_0)\|^{p(q-1)} \right] \|f(x_0)\|^{p+q} \\ &= \rho_0 \|f(x_0)\|^{p+q} \end{aligned}$$

This inequality leads to:

$$\rho_0^{1/(p+q-1)} \|f(x_1)\| \leq \rho_0^{1/(p+q-1)} \|f(x_0)\|^{p+q}$$

or, using the notation $\varepsilon_1 = \rho_0^{1/(p+q-1)} \|f(x_1)\|$:

$$\varepsilon_1 \leq \varepsilon_0^{p+q}$$

From this inequality follows that $\|f(x_1)\| \leq \|f(x_0)\|$, and if

$$\rho_1 = \beta\alpha \left(l_1 B^p + l_2 B^p + l_3 B^p \alpha^p \|f(x_1)\|^{p(q-1)} \right)$$

then $\rho_1 \leq \rho_0$.

Suppose now that the following properties hold:

- (α) $x_p \in S$;
- (β) $\|f(x_p)\| \leq \|f(x_{p-1})\|$;
- (γ) $\varepsilon_p \leq \varepsilon_0^{(p+q)^p}$, $\varepsilon_p = \rho_0^{1/(p+q-1)} \|f(x_p)\|$, $p = 1, 2, \dots, k$

From (2) for $n = k$ we obtain:

$$\|x_{k+1} - x_k\| \leq B \|f(x_k)\| \leq \frac{r\varepsilon_k}{\rho_0^{1/(p+q-1)}} \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}},$$

which leads to:

$$\begin{aligned} \|x_{k+1} - x_0\| &\leq \frac{r}{\rho_0^{1/(p+q-1)}} \left(\varepsilon_0 + \varepsilon_0^{p+q} + \varepsilon_0^{(p+q)^2} + \dots + \varepsilon_0^{(p+q)^k} \right) \\ &\leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})}, \end{aligned}$$

namely $x_{k+1} \in S$.

Here was used the inequality:

$$\|g(x_k) - x_k\| \leq B \|f(x_k)\| \leq \frac{r\rho_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}} \|f(x_k)\| \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}},$$

from which follows immediately:

$$\|g(x_k) - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},$$

that is, $g(x_k) \in S$.

As to $\|f(x_{k+1})\|$ we have:

$$\|f(x_{k+1})\| \leq \beta\alpha \left(l_1 B^p + l_2 B^p + l_3 \alpha^p B^p \|f(x_k)\|^{p(q-1)} \right) \|f(x_k)\|^{p+q}$$

namely

$$\|f(x_{k+1})\| \leq \rho_0 \|f(x_k)\|^{p+q},$$

which yields:

$$\varepsilon_{k+1} \leq \varepsilon_k^{p+q} \leq \varepsilon_0^{(p+q)^{k+1}}.$$

By virtue of the above proved results follows that the properties (α)–(γ) hold for every $p \in \mathbb{N}$.

We prove further down that the sequence $(x_n)_{n \geq 0}$ is a fundamental sequence. Indeed, we have:

$$\begin{aligned}
\|x_{n+s} - x_n\| &\leq \|x_{n+s} - x_{n+s-1}\| + \|x_{n+s-1} - x_{n+s-2}\| + \dots + \|x_{n+1} - x_n\| \\
&\leq B (\|f(x_n)\| + \|f(x_{n+1})\| + \dots + \|f(x_{n+s-1})\|) \\
&\leq \frac{B}{\rho_0^{1/(p+q-1)}} \left(\varepsilon_0^{(p+q)^n} + \varepsilon_0^{(p+q)^{n+1}} + \dots + \varepsilon_0^{(p+q)^{n+s-1}} \right) \\
&\leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}} \left(1 + \varepsilon_0^{p+q-1} + \varepsilon_0^{(p+q)^2-1} + \varepsilon_0^{(p+q)^{s-1}-1} \right) \\
&\leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},
\end{aligned}$$

that is, for every $s, n \in \mathbb{N}$ the following inequality holds:

$$\|x_{n+s} - x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},$$

from which, since $\varepsilon_0 < 1$, it results that the sequence $(x_n)_{n \geq 0}$ is fundamental. Since X_1 is a Banach space, there exists $\lim_{n \rightarrow \infty} x_n = \bar{x}$,

and

$$\|\bar{x} - x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})},$$

which leads, for $n = 0$, to $\bar{x} \in S$.

From the inequality $\varepsilon_n \leq \varepsilon_0^{(p+q)^n}$, for $n \rightarrow \infty$, we obtain:

$$f(\bar{x}) = \lim_{n \rightarrow \infty} f(x_n) = 0,$$

and one sees that \bar{x} is the solution of the equation (1). □

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Institutul de Calcul
Oficiul Poștal 1
C.P. 68
3400 Cluj-Napoca
Romania

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