"Babeş-Bolyai" University
Faculty of Mathematics and Physics
Research Seminars
Seminar on Functional Analysis and Numerical Methods
Preprint nr.1, 1986, pp.133-136

SUR L'ESTIMATION DES ERREURS EN CONVERGENCE NUMÉRIQUE DE CERTAINES MÉTHODES D'ITÉRATION

(English translation)

ON THE ESTIMATION OF ERRORS IN NUMERICAL CONVERGENCE OF CERTAIN ITERATION METHODS

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Let E be a Banach space and

$$x = \lambda D(x) + y, \qquad D(\theta) = \theta, \tag{1}$$

an operatorial equation, where $\lambda \in \mathbb{R}$, $D : E \to E$, $x, y \in E$, and θ is the null element of the space E.

In order to solve equation (1) we consider the following iterative process:

$$x_{n+1} = \lambda D(x_n) + y, \qquad n = 0, 1, \ldots, x_0 = y. \tag{2}$$

We denote by $S = \{x \in E : \|x\| \leq \rho\}$ the ball of radius ρ and center θ.

Regarding the convergence of iterations (2) we shall consider the following theorem

Theorem 1. *If the application D and the element y from equation* (1) *verify the following conditions:*

i. $\|D(x_1) - D(x_2)\| \leq C(\rho)\|x_1 - x_2\|$ *for all* $x_1, x_2 \in S(\theta)$, *where* $C : (0, +\infty) \to (0, +\infty)$ *is a functional;*
ii. $\gamma = |\lambda| C(\rho) < 1$;
iii. $\|y\| \leq (1 - \gamma)\rho$,

then equation (1) *admits a unique solution* $\bar{x} \in S(\theta, \rho)$.

This solution is obtained as the limit of the sequence $(x_n)_{n=0}^{\infty}$ *generated by method* (2), *and the following estimation holds:*

$$\|\bar{x} - x_n\| \leq \tfrac{\gamma^{n+1}}{1-\gamma}\|y\|, \qquad n = 0, 1, \dots \tag{3}$$

Let $D_\varepsilon : E \to E$ be an application which verifies the conditions:

i$_1$. $\|D(x) - D_\varepsilon(x)\| \leq \eta_1(\varepsilon, \rho), \forall x \in B(\theta, \rho)$, where $\eta_1 : [0, +\infty) \times [0, +\infty)$ and $\lim\limits_{\varepsilon_1 \to 0} \eta_1(\varepsilon, \rho) = 0,\ \forall \rho > 0$
ii$_1$. $\|D_\varepsilon(x_1) - D_\varepsilon(x_2)\| \leq C_\varepsilon(\rho)\|x_1 - x_2\|$ for every $x_1, x_2 \in S(\theta, \rho)$ where $C_\varepsilon : (0, +\infty) \to (0, +\infty)$;
iii$_1$. $|C(\rho) - C_\varepsilon(\rho)| < \eta_2(\varepsilon)$ where $\eta_2 : [0, +\infty) \to [0, +\infty)$ and $\lim\limits_{\varepsilon \to 0} \eta_2(\varepsilon) = 0$;
iv$_1$. We consider an element for which $\|y - y_\varepsilon\| \leq \eta_3(\varepsilon)$ where $\eta_3 : [0, +\infty) \to [0, +\infty)$ and $\lim \eta_3(\varepsilon) = 0$.

In order to solve equation (1) we consider, instead of iterative method (2) the following iterative procedure:

$$\xi_{n+1} = \lambda D_\varepsilon(\xi_n) + y_\varepsilon, \qquad n = 0, 1, \dots, \xi_0 = y_\varepsilon. \tag{4}$$

Regarding the convergence of method (4) we obtain the following theorem:

Theorem 2. *If the conditions of Theorem* 1 *are fulfilled, the operator* D_ε *and the element* y_ε *verify conditions* i_1–iv_1, *and if*

$$(5) \qquad \delta = (1-\gamma)\,\rho - y\,\|y\| > 0,$$

then there exists an $\bar{\varepsilon} > 0$, *so that for all* $\varepsilon < \bar{\varepsilon}$ *we have*

$$(6) \qquad \gamma_\varepsilon = |\lambda|\, C_\varepsilon\,(\rho) \le \gamma + |\lambda|\,\eta_2\,(\varepsilon) < 1;$$

and

$$(7) \qquad \|y_\varepsilon\| \le (1-\gamma_\varepsilon)\,\rho.$$

Proof. Indeed, from the fact that $\gamma = |\lambda|\,C\,(\rho) < 1$ and $\lim_{\varepsilon\to 0} \eta_2\,(\varepsilon) = 0$ it follows that there exists a number $\bar{\varepsilon}_1 > 0$ so that for $\varepsilon < \bar{\varepsilon}_1$ we have

$$\gamma_\varepsilon = |\lambda|\,C_\varepsilon\,(\rho) \le |\lambda|\,C\,(\rho) + |\lambda|\,\eta_2\,(\varepsilon) < 1$$

and

$$\begin{aligned}\|y_\varepsilon\| &\le \|y\| + \eta_3\,(\varepsilon) \le (1-\gamma)\,\rho - \delta + \eta_3\,(\varepsilon)\\ &\le (1-\gamma_\varepsilon)\,\rho + |\lambda|\,\eta_2\,(\varepsilon)\,\rho + \eta_3\,(\varepsilon) - \delta \le (1-\gamma_\varepsilon)\,\rho\end{aligned}$$

for $\varepsilon < \bar{\varepsilon}_2$ because $\eta_3\,(\varepsilon) \to 0$ for $\varepsilon \to 0$ and $\eta_2\,(\varepsilon) \to 0$ as $\varepsilon \to 0$.

If we take now

$$\bar{\varepsilon} = \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\}$$

then the theorem is proved. □

Relations (6) and (7) assure the convergence of sequence $(\xi_n)_{n=0}^{\infty}$ determined by method (4).

We show now that in the conditions of Theorem 2 we have the following estimation:

$$(8) \qquad \|\bar{x} - \xi_n\| \le \frac{|\lambda|\,C\,(\rho)\,\|\xi_n - \xi_{n-1}\| + |\lambda|\,\eta_1\,(\varepsilon,\rho) + \eta_3\,(\varepsilon)}{1-\gamma}$$

Indeed, we have

$$\begin{aligned}\|\bar{x}-\xi_n\| &\leq |\lambda| \, \|D(\bar{x}) - D_\varepsilon(\xi_{n-1})\| + \|y - y_\varepsilon\| \\ &\leq |\lambda| \, C(\rho) \, \|\bar{x}-\xi_n\| + |\lambda| \, C(\rho) \, \|\xi_n - \xi_{n-1}\| + |\lambda| \, \eta_1(\varepsilon,\rho) + \eta_3(\varepsilon)\end{aligned}$$

so inequality (8) follows. From (8) it results

$$\left\|\bar{x}-\overline{\xi}\right\| \leq \frac{|\lambda| \, \eta_1(\varepsilon,\rho) + \eta_3(\varepsilon)}{1-\gamma}, \qquad \text{as } n \to \infty,$$

where $\overline{\xi} = \lim_{n\to\infty} \xi_n$.

References

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