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EXTENSIONS OF SOME MEAN VALUE THEOREMS
Ioan Muntean

1. Introduction. The classical mean value theorems for real functions, usually attributed to M. Rolle (1691), J. L. Lagrange (1797) and A.-L. Cauchy (1823), together with the famous rule of G. F. l'Hôpital (1696), constitute today the central part of real analysis and its applications. In the last century and more, numerous researches and extensions of these theorems have been appearing. They concern functions: 1) taking complex values or values in a normed space (even in a topological vector space), 2) possessing a derivative in an extended sense (infinite, lateral, Dini, approximate), 3) deprived of any derivative (even in extended senses) at some points. Other investigations refer to mean value theorems for divided differences or to various converses of the classical mean value theorems.

In this paper we limit oneself to real functions of a real variable, and we propose to attenuate the continuity hypothesis, which is present in almost all mean value theorems, until the primitivebility or even the Darboux condition for involved functions. So attenuated, these requirements are combined in
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Institutul de Calcul
Oficiu Postal 1
C.P. 68
3400 Cluj - Napoca
Romania

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"BARES-BOLYAI" UNIVERSITY
Faculty of Mathematics
Research Seminars
Seminar on Mathematical Analysis

REMARKS ON THE SECANT METHOD FOR THE SOLUTION OF NONLINEAR OPERATORIAL EQUATIONS
Ico Păvăloiu

This note has for purpose some refinements of the convergency conditions and error delimitations obtained by I.K. Argyros in [1] with respect to the secant method for the solution of the equation:

\[ f(x) = 0, \]

where \( f : X \rightarrow X \) is a nonlinear operator, while \( X \) and \( X \) are Banach spaces.

If we denote by \( [x,y;f] \) the divided difference of the mapping \( f \) on the point \( x \) and \( y \), then for fixed \( x,y \) we have \( [x,y;f] \in \mathbb{X}(X,X) \). It is known that in certain conditions the sequence \( (x_n)_{n=0}^\infty \) generated by the secant method:

\[ x_{n+1} = x_n - [x_{n-1},x_n;f]^{-1} f(x_n), \quad n=1,2,\ldots \]

converges to the solution \( x^* \) of equation (1).

1. Generalizing a result on J.E. Denis [2], I.K. Argyros [1] studies the convergency of the method (2) with the assumptions that the operator \( f \) is Fréchet differentiable, while the derivative \( f'(x) \) fulfils a Hölder - like condition on a set \( D(f) \), namely there exist a constant \( C > 0 \) and number \( p \in (0,1] \) such that the inequality:

\[ |f(x) - f(y)| \leq C|x - y|^p, \quad x, y \in D(f) \]
holds for every \( x, y \in D \). In this case we shall say that \( f'(+ \cdot) \in H_0(C, p) \).

In the quoted paper I.K. Argyros defines the divided difference operator \([x, y; f] \) as a linear operator which fulfills the conditions:

\( [x, y; f] (y-x) = f(y) - f(x), \ \forall x, y \in D, \)

and, in addition, for every \( x, y, u \in D \) the following inequality holds:

\( \| [x, y; f] - [y, u; f] \| \leq l_1 |x-u|^p + l_2 |x-y|^p + l_2 |y-u|^p, \)

where \( l_1, l_2, l_3 \geq 0 \) are constants which do not depend on \( x, y \) and \( u \), while \( p \in (0, 1) \).

Let \( x^* \) be a simple solution of (1). We mean by that the mapping \( f \) \( x^* \) admits a bounded inverse mapping, and if \( [x^*, x^*; f] = f' x^* \) then \( [x^*, x^*; f] \) admits a bounded inverse mapping.

Thus the continuousness of the mapping \([x, y; f] \) with respect to the variable \( x \) and \( y \) ensures the existence of a number \( \varepsilon > 0 \) such that the mapping \([x, y; f] \) admits a bounded inverse mapping for every \( x, y \in U(x^*, \varepsilon) \), where \( U(x^*, \varepsilon) = \{ x \in D | |x - x^*| < \varepsilon \} \), that is, the set \( B(x, y) = \{ [x, y; f] \} \) is uniformly bounded in \( \overline{U}(x^*, \varepsilon) = \{ x \in D | |x - x^*| < \varepsilon \} \).

THEOREM 1.1. Let \( f: X_1 \rightarrow X_1 \) and let \( D \subseteq X_1 \) an open set.

The following conditions are fulfilled:

(a) \( x^* \in D \) is a simple solution of the equation (1);

(b) there exist \( \varepsilon_1 > 0, b > 0 \) such that \( [x, y; f] \) is bounded for every \( x, y \in D \);

(c) there exists a convex set \( B = B_0(C, p) \) such that \( x^* \in B_0 \), and there exists \( b > 0 \), with \( D \subseteq \epsilon B_0(C, p) \) for every \( x, y \in D \).

Let \( \varepsilon > 0 \) such that:

\( \varepsilon < \min \{ \varepsilon_1, (q(p))^{1/p} \} \)

where:

\( q(p) = \frac{b}{P+1} \left[ 2^p (l_1 + l_2) (1 + p) + C. \right] \)

Then, if \( x_0, x_1, \ldots, x_n \), the iterates \( x_n, n = 2, 3, \ldots \), generated by (2) are well defined and belong to the set \( U(x^*, r) \), while the sequence \( (x_n) \) converges to the unique solution \( x^* \) of equation (1).

Moreover, the following estimation:

\( \| x_{n+1} - x^* \| = \| y_{n+1} \| - \| x^* \| + \| y_n - x^* \| = \| y_n - x^* \| + \| y_0 - y_1 \| + \ldots + \| y_n - x^* \|^{p+1} \)

holds for sufficiently large \( n \), where:

\( y_1 = b(l_1 + l_2) \rho \),

\( y_2 = \frac{bC}{1+\rho} \)

while \( l_1, l_2 \) and \( p \) were defined by the relation (5).

In order to prove this theorem the author uses the following two lemmas:

**Lemma 1.** Let \( f: X_1 \rightarrow X_1 \) and \( D \subseteq X_1 \). Suppose that \( D \) is an open set and \( f(+ \cdot) \) does exist in every point of \( D \). If, for a convex set \( D \subseteq D \), \( f(+ \cdot) \subseteq H_0(C, p) \), then for every \( x, y \in D \), the following inequality holds:

\( |f(x) - f(y) - f'(x)(y-x)| \leq \frac{C}{|x-y|^{1+\rho}}. \)

**Lemma 2.** If \( [x, y; f] \) fulfills the conditions (4) and (5), then the following relations hold:

(a) \( [x, x; f] = f'(x) \) for every \( x \in D \).
from the proof of Theorem 1 follows, for the error estimation and for the convergence speeds of the sequence \( \{x_n\} \) the inequality:

\[
|x_{n+1} - x'| \leq (N(r))^{n+1} |x_n - x'|
\]

where one shows that \( N(r) \in (0, 1) \).

2. We shall make further down some remarks upon the above exposed results, showing that the hypotheses imposed in [1] can lead to more rich conclusions with respect to both the convergence order of the secant method and the error estimation.

Suppose that \( x_0 \) and \( x_1 \) fulfill the conditions:

(a') \( |x' - x_0| \leq \alpha d_0 \),

(b') \( |x' - x_1| \leq \min\{\alpha d_0, |x' - x_0|\} \)

where \( 0 < d_0 < 1 \), \( \alpha = (q(p))^{-1/p} \), while \( t_1 \) is the positive root of the equation:

\[
t^2 - t - p = 0 \quad 14(1+4p)^{1/2} \]

namely \( t_1 = \frac{1}{2} \frac{1+14p}{2} \).

Using the conditions (4) and (5), Lemmas 1 and 2, and the hypotheses of Theorem (1), it result easily from (2), for \( n=1 \), the inequality [1]:

\[
|x_{n+1} - x'| \leq \alpha d_0 \quad |x_{n+1} - x'| \leq \alpha d_0 \quad t_1(1+p) \quad t_1(1+p)
\]

\[
|X_2 - X'| \leq \alpha d_0 \quad \gamma d_0 \quad t_1(1+p) \quad t_1(1+p)
\]

\[
= \alpha d_0 \quad \gamma d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]

But

\[
|x_{n+1} - x'| \leq \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]

\[
then the following inequality holds:
\[
|x_n - x'| \leq \beta t_1 \quad d_0
\]

We prove now that \( |x_n - x'| \leq \beta t_1 \). From the inequality

\[
|x_{n+1} - x'| \leq \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]

since \( d_0 < 1 \) and, as we saw above, \( \alpha d_0 < 1 \). Assume now that for \( n \geq 2 \), the following relations hold:

\[
\text{(a'')} \quad |x_n - x'| \leq \beta t_1 \quad d_0
\]

\[
\text{(b'')} \quad |x_n - x'| \leq \min\{\beta t_1 \quad d_0, |x_n-1 - x'|\}
\]

Proceeding as in the case of \( x_2 \), taking into account

\[
|X_{n+1} - x'| \leq \beta t_1 \quad d_0 \quad \gamma d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]

\[
then, as previously, it is easy to show that:
\[
|X_{n+1} - x'| \leq \beta t_1 \quad d_0 \quad \gamma d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]

In order to complete the proof, we shall show that:

\[
|x_{n+1} - x'| \leq \beta t_1 \quad d_0
\]

Indeed, from (7') we deduce:

\[
|x_{n+1} - x'| \leq \beta t_1 \quad d_0 \quad \gamma d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0 \quad \gamma d_0 \quad \alpha d_0
\]
But \( d_0 = 1 \) and \( a > 0 \), \( \gamma_1 + \gamma_2 \), \( d_0 \), therefore:
\[
\{x_{n+1} - x' \} \leq \{x_n - x' \}.
\]

We proved in this way the following theorem:

**Theorem 2.** If the conditions of Theorem 1 are fulfilled, with the difference that \( x_0 \) and \( x_1 \) are chosen in such a manner to verify the relations \( a' = \gamma_1 (b')^a \), where \( a = (q(p))^{-1/p} \) and \( d_n(0,1) \), then, for every \( n \in \mathbb{N} \), \( x_{n+1} = (x_n)_{n \in \mathbb{N}} \{x - x' \} \leq \epsilon \) and the following inequality holds:
\[
\sum_{n=1}^{\infty} \{x_{n+1} - x' \} \leq \epsilon d_0^{n+1}, \quad n=0,1,\ldots
\]

**Remark.** The inequality (13) contains in its right-hand side a number substantially smaller than that yielded by relation (10).

**References**


*Institutul de Calcul*
*Oficiul Postal 1*
*C.P. 68*
*3400 Cluj-Napoca*
*Romania*

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