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„BABEŞ—BOLYAI" UNIVERSITY
FACULTY OF MATHEMATICS AND PHYSICS
RESEARCH SEMINARS

SEMINAR ON MATHEMATICAL ANALYSIS

Preprint Nr. 7, 1991

CLUJ-NAPOCA

ROMANIA

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"BABEŞ-BOLYAI" UNIVERSITY
Faculty of Mathematics
Research Seminars
Seminar on Mathematical Analysis
Preprint Nr. 7, 1991, pp.7-24

EXTENSIONS OF SOME MEAN VALUE THEOREMS

Ioan Muntean

1. Introduction. The classical mean value theorems for real functions, usually attributed to M. Rolle (1691), J. L. Lagrange (1797) and A.-L. Cauchy (1823), together with the famous rule of G. F. l'Hôpital (1696), constitute today the central part of real analysis and its applications. In the last century and more, numerous researches and extensions of these theorems have been appearing. They concern functions: 1) taking complex values or values in a normed space (even in a topological vector space), 2) possessing a derivative in an extended sense (infinite, lateral, Dini, approximate), 3) deprived of any derivative (even in extended senses) at some points. Other investigations refer to mean value theorems for divided differences or to various converses of the classical mean value theorems.

In this paper we limit oneself to real functions of a real variable, and we propose to attenuate the continuity hypothesis, which is present in almost all mean value theorems, until the primitivability or even the Darboux condition for involved functions. So attenuated, these requirements are combined in

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This paper is in final form and no version of it is or will be submitted for publication elsewhere.

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Preprint Nr.7, 1991,pp.127 - 132

REMARKS ON THE SECANT METHOD FOR THE SOLUTION OF NONLINEAR OPERATORIAL EQUATIONS

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This note has for purpose some refinements of the convergency conditions and error delimitations obtained by I.K. Argyros in [1] with respect to the secant method for the solution of the equation:

(1) $f(x) = 0,$

where $f: X_1 \to X_2$ is a nonlinear operator, while X_1 and X_2 are Banach spaces.

If we denote by $[x,y;f]$ the divided difference of the mapping f on the point x and y, then for fixed x,y we have $[x,y;f]\in\mathcal{L}(X_1,X_2)$. It is known that in certain conditions the sequence $(x_n)_{n\geq o}$ generated by the secant method:

(2) $x_{n+1} = x_n - [x_{n-1},x_n;f]^{-1} fx_n), \ x_o,x_1\in X_1, \ n=1,2,...$

converges to the solution x^* of equation (1).

1. Generalizing a result on J.E.Denis [2], I.K. Argyros [1] studies the convergency of the method (2) with the assumptions that the operator f is Fréchet differentiable, while the derivative $f'(x)$ fulfils a Hölder - like condition on a set $D\subset X_1$, namely there exist a constant $c>0$ and number $p\in(0,1]$ such that the inequality:

(3) $\| f'(x) - f'(y) \| \le C \| x-y \|^p$

holds for every $x,y \in D$. In this case we shall say that $f'(\cdot) \in H_D(C,p)$.

In the quoted paper I.K. Argyros defines the divided difference operator $[x,y;f]$ as a linear operator wich fulfils the conditions:

(4) $[x,y;f]\ (y-x) = f(y) - f(x), \quad \forall x,y \in D,$

and, in addition, for every $x,y,u \in D$ the following inequality holds:

(5) $\| [x,y;f] - [y,u;f] \| \le l_1 \| x-u \|^p + l_2 \| x-y \|^p + l_2 \| y-u \|^p,$

where $l_1 \ge 0$, $l_2 \ge 0$ are constants which do not depend on x,y and u, while $p \in (0,1]$.

Let x^* be a simple solution of (1). We mean by that the mapping $f'(x^*)$ admits a bounded inverse mapping, and if $[x^*,x^*;f]=f'(x^*)$ then $[x^*,x^*;f]$ admits a bounded inverse mapping. Thus the continuousness of the mapping $[x,y;f]$ with respect to the variable x and y ensures the existence of a number $\varepsilon>0$ such that the mapping $[x,y;f]$ admits a bounded inverse mapping for every $x,y \in U(x^*,\varepsilon)$, where $U(x^*,\varepsilon) = \{x \in X_1 \mid \| x-x^* \| < \varepsilon\}$ that is, the set $B(x,y) = \| [x,y;f]^{-1} \|$ is uniformly bounded in $\overline{U}(x^*,\varepsilon) = \{x \in X_1 \mid \| x-x^* \| \le \varepsilon\}$.

THEOREM 1.[1] *Let $f: X_1 \to X_2$ and let $D \subset X_1$ an open set. The following conditions are fulfilled:*

(a) *$x^* \in D$ is a simple solution of the equation (1);*

(b) *there exist $\varepsilon>0$, $b>0$ such that $\| [x,y;f]^{-1} \| \le b$ for every $x,y \in U(x^*,\varepsilon)$;*

(c) *there exists a convex set $D_o \subset D$ such that $x^* \in D_o$, and there exists $\varepsilon_1>0$, with $0<\varepsilon_1<\varepsilon$ such that $f'(\cdot) \in H_{D_o}(C,p)$ for every $x,y \in D_o$ and $U(x^*,\varepsilon_1) \subset D_o$.*

Let $r>0$ such that:

(6) $0<r<\min\{\varepsilon_1, (q(p))^{-1/p}\}$

where:

(7) $$q(p) = \frac{b}{p+1}\left[2^p (l_1+l_2)(1+p) + C\right].$$

Then, if $x_o x_1 \in \overline{U}(x^*,r)$, the iterates x_n, $n=2,3,\ldots$, generated by (2) are well defined and belong to the set $\overline{U}(x^*,r)$, while the sequence $(x_n)_{n\ge o}$ converges to the unique solution x^* of equation (1).

Moreover, the following estimation:

(7') $\| x_{n+1} - x^* \| \le \gamma_1 \| x_{n-1} - x^* \|^p . \| x_n - x^* \| + \gamma_2 \| x_n - x^* \|^{p+1}$

holds for sufficiently great n, where:

(8) $\gamma_1 = b(l_1 + l_2)\ 2^p,$

(9) $$\gamma_2 = \frac{bC}{1+p}$$

while l_1, l_2 and p where precised by the relation (5).

In order to prove this theorem the author uses the following two lemmas:

LEMMA 1. [1]. *Let $f: X_1 \to X_2$ and $D \subset X_1$. Suppose that D is an open set and $f'(\cdot)$ does exist in every point of D. If, for a convex set $D_o \subseteq D$, $f'(\cdot) \in H_{D_o}(C,p)$, then for every $x,y \in D_o$ the following inequality holds:*

$$\| f(x) - f(y) - f'(x)(y-x) \| \le \frac{C}{1+p} \| x-y \|^{1+p}.$$

LEMMA 2. [1]. *If $[x,y;f]$ fulfils the conditions (4) and (5), then the following relations hold:*

(a) *$[x,x;f] = f'(x)$ for every $x \in D_o$;*

(b) $f'(\cdot)\in H_{D_o}(2(l_1 + l_2), p)$.

From the proof of Theorem 1 follows, for the eror estimation and for the convergency spees of the sequence $(x_n)_{n\geq 0}$, the inequality:

(10) $\|x_{n+1} - x^*\| \leq (M(r))^{n+1}\|x_o - x^*\|$

where one shows that $M(r)\in(0,1)$.

2. We shall make further down some remarks upon the above exposed results, showing that the hypotheses imposed in [1] can lead to more rich conclusions with respect to both the convergency order of the secant method and the arror estimation.

Suppose that x_o and x_1 fulfil the conditions:

(a') $\|x^* - x_o\| \leq \alpha d_o$;

(b') $\|x^* - x_1\| \leq \min \{\alpha d_o^{t_1}, \|x^* - x_o\|\}$

where $0<d_o<1$, $\alpha = (q(p))^{-1/p}$, while t_1 is the positive root of the equation:

(11) $t^2 - t - p = 0$

namely $t_1 = \dfrac{1+(1+4p)^{1/2}}{2}$.

Using the conditions (4) and (5), Lemmas 1 and 2, and the hypotheses of Theorem (1), it result easily from (2), for n=1, the inequality [1]:

(12) $\|x_2 - x^*\| \leq \gamma_1 \|x_o - x^*\|^p \|x_1 - x^*\| + \gamma_2\|x_1 - x^*\|^{p+1}$

from which, using (a') and (b') and the fact that t_1 is a root of equation (11), we obtain:

$$\|x_2-x^*\| \leq \gamma_1\alpha^p d_o^p \alpha d_o^{t_1} + \gamma_2\alpha^{1+p} d_o^{t_1(1+p)} = \alpha^{1+p}(\gamma_1 d_o^{t_1+p} + \gamma_2 d_o^{t_1(1+p)}) =$$

$$= \alpha^{1+p} d_o^{t_1+p} (\gamma_1+\gamma_2 d_o^{p(t_1-1)}) = \alpha d_o^{t_1^2} (\gamma_1+\gamma_2 d_o^{p(t_1-1)})\alpha^p.$$

But

$$(\gamma_1 + \gamma_2 d_o^{p(t_1-1)})\alpha^p = \frac{\gamma_1 + \gamma_2 d_o^{p(t_1-1)}}{\gamma_1 + \gamma_2} < 1 ,$$

then the following inequality holds

$$\|x_2 - x^*\| \leq \alpha d_o^{t_1^2} .$$

We prove now that $\|x_2 - x^*\| \leq \|x_1 - x^*\|$. From the inequality (12) we obtain:

$$\|x_2 - x^*\| (\gamma_1\alpha^p d_o^p + \gamma_2\alpha^p d_o^{t_1 p}) \|x_1 - x^*\| \leq$$

$$\leq \alpha^p d_o^p (\gamma_1 + \gamma_2 d_o^{p(t_1-1)}) \|x_1 - x^*\| < \|x_1 - x^*\|$$

since $d_o^p < 1$ and, as we saw above, $\alpha^p (\gamma_1 + \gamma_2 d_o^{p(t_1-1)}) < 1$.

Assume now that for $n \in N$, $n\geq 2$, the following relations hold:

(a") $\|x_{n-1} - x^*\| \leq \alpha d_o^{t_1^{n-1}}$;

(b") $\|x_n - x^*\| \leq \min \{\alpha d_o^{t_1^n}, \|x_{n-1} - x^*\|\}$

Proceeding as in the case of x_2, and taking into account (a"), (b") and (7'), we obtain:

$$\|x_{n+1} - x^*\| \leq \alpha^{1+p} d_o^{t_1^{n+1}} \cdot (\gamma_1 + \gamma_2 d_o^{pt_1^{n-1}(t_1+1)}) =$$

$$= \alpha d_o^{t_1^{n+1}} \cdot \alpha^p (\gamma_1 + \gamma_2 d_o^{pt_1^{n-1}(t_1-1)}) \leq \alpha d_o^{t_1^{n+1}} ,$$

since, as previously, it is easy to show that:

$$\alpha^p (\gamma_1 + \gamma_2 d_o^{pt_1^{n-1}(t_1-1)}) < 1$$

In order to complete the proof, we shall show that:

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$$

Indeed, from (7') we deduce:

$$\|x_{n+1} - x^*\| \leq \gamma_1\alpha^p d_o^{pt_1^{n-1}} + \gamma_2\alpha^p d_o^{pt_1^n})\|x_n - x^*\|$$

But $d_o < 1$ and $\alpha^p (\gamma_1 + \gamma_2 d_o^{p(t_1-1)}) < 1$, therefore:

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|.$$

We proved in this way the following theorem:

THEOREM 2. *If the conditions of Theorem 1 are fulfilled, with the difference that x_o and x_1 are chosen in such a maner to verify the relations (a') and (b'), where $\alpha = (q(p))^{-1/p}$ and $d_o \in (0,1)$, then, for every $n \in N$, $x_n \in U = \{x \in X_1 | \|x - x^*\| < \alpha\}$ and the following inequality holds:*

$$(13) \quad \|x_{n+1} - x^*\| \le \alpha d_o^{t_1^{n+1}}, \quad n=0,1,\ldots$$

Remark. The inequality (13) contains in its right-hand side a number substantially smaller than that yielded by relation (10).

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Preprint Nr.7, 1991, pp.133 - 150

SUR L'APPROXIMATION DES SOLUTIONS DES PROBLÈMES POLYLOCAUX POUR DES ÉQUATIONS DIFFÉRENTIELLES NON - LINÉAIRES

Adrian Diaconu

Le but de ce travail est d'appliquer certains résultats établis dans les travaux [3], [4], [5] qui se réfèrent à la convergence d'un procédé itératif pour la résolution d'une équation opérationnelle non-linéaire. Ce procédé qui dérive de la bienconnue méthode de Newton-Kantorovitch se différencie de cette méthode par l'approximation des solutions des problèmes polylocaux pour des équations différentielles non-linéarires ordinaires.

1.Notions et résultats préliminaires.

Considérons l'équation opérationnelle:

$$(1.1) \quad P(x) = \theta$$

où $P : X \to Y$ est une application non - linéaire; X et Y étant des espaces linéaires normés et θ représentant l'élément nul de l'espace Y.

La bien-connue méthode de Newton - Kantorovitch pour cette équation consiste dans la construction d'une suite $(x_n)_{n\in N}$ par la relation de récurrence suivante:

$$(1.2) \quad x_{n+1} = x_n - [P'(x_n)]^{-1} P(x_n)$$