REMARKS ON THE SECANT METHOD FOR THE SOLUTION OF NONLINEAR OPERATORIAL EQUATIONS

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This note has for purpose some refinements of the convergence conditions and error delimitations obtained by I.K. Argyros in [1] with respect to the secant method for the solution of the equation:

\[ f(x) = 0, \]

where \( f : X_1 \to X_2 \) is a nonlinear operator, while \( X_1 \) and \( X_2 \) are Banach spaces.

If we denote by \([x, y; f]\) the divided difference of the mapping \( f \) on the point \( x \) and \( y \), then for fixed \( x, y \) we have \([x, y; f] \in \mathcal{L}(X_1, X_2)\). It is known that in certain conditions the sequence \((x_n)_{n \geq 0}\) generated by the secant method:

\[ x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad x_0, x_1 \in X_1, \quad n = 1, 2, \ldots \]

converges to the solution \( x^* \) of equation (1).

1. Generalizing a result on J.E. Dennis [2], I.K. Argyros [1] studies the convergence of the method (2) with the assumptions that the operator \( f \) is Fréchet differentiable, while the derivative \( f'(x) \) fulfills a Hölder-like condition on a set \( D \subset X_1 \), namely there exist a constant \( C > 0 \) and number \( p \in (0, 1] \) such that the inequality:

\[ \|f'(x) - f'(y)\| \leq C \|x - y\|^p \]
holds for every $x, y \in D$. In this case we shall say that $f'(\cdot) \in H_{D}(C, p)$.

In the quoted paper I.K. Argyros defines the divided difference operator $[x, y; f]$ as a linear operator which fulfils the conditions:

\[(4) \quad [x, y; f](y - x) = f(y) - f(x), \quad \forall x, y \in D,\]

and, in addition, for every $x, y, u \in D$ the following inequality holds:

\[(5) \quad \|[x, y; f] - [y, u; f]\| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_2 \|y - u\|^p,\]

where $l_1 \geq 0$, $l_2 \geq 0$ are constants which do not depend on $x, y$ and $u$, while $p \in (0, 1]$.

Let $x^*$ be a simple solution of (1). We mean by that the mapping $f'(x^*)$ admits a bounded inverse mapping, and if $[x^*, x^*; f] = f'(x^*)$ then $[x^*, x^*; f]$ admits a bounded inverse mapping. Thus the continuity of the mapping $[x, y; f]$ with respect to the variable $x$ and $y$ ensures the existence of a number $\varepsilon > 0$ such that the mapping $[x, y; f]$ admits a bounded inverse mapping for every $x, y \in U(x^*, \varepsilon)$, where $U(x^*, \varepsilon) = \{x \in X_1 : \|x - x^*\| < \varepsilon\}$ that is, the set $B(x, y) = \|[x, y; f]^{-1}\|$ is uniformly bounded in $U(x^*, \varepsilon) = \{x \in X_1 : \|x - x^*\| \leq \varepsilon\}$.

**Theorem 1.** [1] Let $f : X_1 \to X_2$ and let $D \subset X_1$ an open set. The following conditions are fulfilled:

(a) $x^* \in D$ is a simple solution of the equation [1];
(b) there exist $\varepsilon \in 0$, $b > 0$ such that $\|[x, y; f]^{-1}\| \leq b$ for every $x, y \in U(x^*, \varepsilon)$;
(c) there exists a convex set $D_0 \subset D$ such that $x^* \in D_0$, and there exists $\varepsilon_1 > 0$, with $0 < \varepsilon_1 < \varepsilon$ such that $f'(\cdot) \in H_{D_0}(C, p)$ for every $x, y \in D_0$ and $U(x^*, \varepsilon_1) \subset D_0$. 
Let \( r > 0 \) such that:

\[
0 < r < \min\{\varepsilon_1, (q(p))^{-1/p}\}
\]

where:

\[
q(p) = \frac{b}{p+1} \left[ 2^p (l_1 + l_2) (1 + p) + C \right].
\]

Then, if \( x_0, x_1 \in \bar{U}(x^*, r) \), the iterates \( x_n, n = 2, 3, \ldots \), generated by (2) are well defined and belong to the set \( \bar{U}(x^*, r) \), while the sequence \( (x_n)_{n \geq 0} \) converges to the unique solution \( x^* \) of equation (1).

Moreover, the following estimation:

\[
\|x_{n+1} - x^*\| \leq \gamma_1 \|x_{n-1} - x^*\|^p \cdot \|x_n - x^*\| + \gamma_2 \|x_n - x^*\|^{p+1}
\]

holds for sufficiently large \( n \), where:

\[
\gamma_1 = b (l_1 + l_2)^2 p,
\]

\[
\gamma_2 = \frac{bC}{1+p}
\]

while \( l_1, l_2 \) and \( p \) were precised by the relation (5).

In order to prove this theorem the author uses the following two lemmas:

**Lemma 1.** Let \( f : X_1 \to X_2 \) and \( D \subset X_1 \). Suppose that \( D \) is an open set and \( f'(\cdot) \) does exist in every point of \( D \). If, for a convex set \( D_0 \subset D, f'(\cdot) \in H_{D_0}(C, p) \), then for every \( x, y \in D_0 \) the following inequality holds:

\[
\|f(x) - f(y) - f'(x)(y - x)\| \leq \frac{C}{1+p} \|x - y\|^{1+p}.
\]

**Lemma 2.** If \( [x, y; f] \) fulfils the conditions (4) and (5), the following relations hold:

(a) \( [x, x; f] = f'(x) \) for every \( x \in D_0 \);
(b) \( f'(\cdot) \in H_{D_0}(2(l_1 + l_2), p) \).
From the proof of Theorem 1 follows, for the error estimation and for the convergence speeds of the sequence \((x_n)_{n \geq 0}\), the inequality:

\[
\|x_{n+1} - x^*\| \leq (M(r))^{n+1} \|x_0 - x^*\|
\]

where one shows that \(M(r) \in (0,1)\).

2. We shall make further down some remarks upon the above exposed results, showing that the hypotheses imposed in [1] can lead to more rich conclusions with respect to both the convergency order of the secant method and the error estimation.

Suppose that \(x_0\) and \(x_1\) fulfil the conditions:

(a’) \(\|x^* - x_0\| \leq \alpha d_0\);

(b’) \(\|x^* - x_1\| \leq \min\{\alpha d_1, \|x^* - x_0\|\}\)

where \(0 < d_0 < 1\), \(\alpha = (q(p))^{-1}\), while \(t_1\) is the positive root of the equation:

\[
t^2 - t - p = 0
\]

namely

\[
t_1 = \frac{1+(1+4p)1/2}{2}.
\]

Using the condition (4) and (5), Lemmas 1 and 2 and the hypotheses of 1, it results easily from (2), for \(n = 1\), the inequality [1]:

\[
\|x_2 - x^*\| \leq \gamma_1 \|x_0 - x^*\|^p \|x_1 - x^*\| + \gamma_2 \|x_1 - x^*\|^{p+1}
\]

from which, using (a’) and (b’) and the fact that \(t_1\) is a root of equation (12), we obtain:

\[
\|x_2 - x^*\| \leq \gamma_1 \alpha^p d_0^p \alpha d_0^{d_1} + \gamma_2 \alpha^{1+p} d_0^{d_1(1+p)}
\]

\[
= \alpha^{1+p} \left( \gamma_1 d_0^{d_1(1+p)} + \gamma_2 d_0^{d_1(1+p)} \right)
\]

\[
= \alpha^{1+p} d_0^{d_1(1+p)} \left( \gamma_1 + \gamma_2 d_0^{d_1(t_1-1)} \right)
\]

\[
= \alpha d_0^{d_1^2} \left( \gamma_1 + \gamma_2 d_0^{d_1(t_1-1)} \right) \alpha^p.
\]
But
\[
\left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \alpha^p = \frac{\gamma_1 + \gamma_2 d_0^{p(t_1-1)}}{\gamma_1 + \gamma_2} < 1,
\]
then the following inequality holds
\[
\|x_2 - x^*\| \leq \alpha d_0^2.
\]
We prove now that \(\|x_2 - x^*\| \leq \|x_1 - x^*\|\). From the inequality \((13)\) we obtain:
\[
\|x_2 - x^*\| \left(\gamma_1 \alpha^p d_0^p + \gamma_2 \alpha^p d_0^{p(1_p)}\right) \|x_1 - x^*\| \leq \alpha^p d_0^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \|x_1 - x^*\| < \|x_1 - x^*\|
\]
since \(d_0^p < 1\) and, as we saw above, \(\alpha^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) < 1\).

Assume now that for \(n \in \mathbb{N}, n \geq 2\), the following relations hold:
\[(a'') \|x_{n-1} - x^*\| \leq \alpha d_0^{n-1} ;
\]
\[(b'') \|x_n - x^*\| \leq \min\{\alpha d_0^2, \|x_{n-1} - x^*\|\} \]
Proceeding as in the case of \(x_2\), and taking into account \((a''), (b'')\) and \((8)\), we obtain:
\[
\|x_{n+1} - x^*\| \leq \alpha^{1+p} d_0^{n+1} \cdot \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) = \alpha d_0^{n+1} \cdot \alpha^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \leq \alpha d_0^{n+1},
\]
since, as previously, it is easy to show that:
\[
\alpha^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) < 1
\]
In order to complete the proof, we shall show that:
\[
\|x_{n+1} - x^*\| \leq \|x_n - x^*\|
\]
Indeed, from \((8)\) we deduce:
\[
\|x_{n+1} - x^*\| \leq \left(\gamma_1 \alpha^p d_0^{p(t_1-1)} + \gamma_2 \alpha^p d_0^{p(t_1-1)}\right) \|x_n - x^*\|.
\]
But $d_0 < 1$ and $\alpha^p(\gamma_1 + \gamma_2 d_0^{(t_1-1)}) < 1$, therefore:

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$  

We proved in this way the following theorem:

**Theorem 2.** If the conditions of Theorem 1 are fulfilled, with the difference that $x_0$ and $x_1$ are chosen in such a manner to verify the relations (a') and (b'), where $\alpha = (q(p))^{-1/p}$ and $d_0 \in (0, 1)$, then, for every $n \in \mathbb{N}$, $x_n \in U = \{x \in X_1 | \|x - x^*\| < \alpha\}$ and the following inequality holds:

$$\|x_{n+1} - x^*\| \leq \alpha d_0^{\frac{n+1}{t_1}}, \quad n = 0, 1, \ldots$$

**Remark.** The inequality (14) contains in its right-hand side a number substantially smaller than that yielded by relation (11).

**References**


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