"Babeş-Bolyai" University
Faculty of Mathematics and Physics
Research Seminars
Seminar on Mathematical Analysis
Preprint Nr.7, 1991, pp.127-132

REMARKS ON THE SECANT METHOD FOR THE SOLUTION OF NONLINEAR OPERATORIAL EQUATIONS

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This note has for purpose some refinements of the convergence conditions and error delimitations obtained by I.K. Argyros in [1] with respect to the secant method for the solution of the equation:

$$f(x) = 0, \tag{1}$$

where $f : X_1 \to X_2$ is a nonlinear operator, while X_1 and X_2 are Banach spaces.

If we denote by $[x, y; f]$ the divided difference of the mapping f on the point x and y, then for fixed x, y we have $[x, y; f] \in \mathcal{L}(X_1, X_2)$. It is known that in certain conditions the sequence $(x_n)_{n\geq 0}$ generated by the secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n),\ x_0, x_1 \in X_1,\ n = 1, 2, \ldots \tag{2}$$

converges to the solution x^* of equation (1).

1. Generalizing a result on J.E. Dennis [2], I.K. Argyros [1] studies the convergence of the method (2) with the assumptions that the operator f is Fréchet differentiable, while the derivative $f'(x)$ fulfils a Hölder-like condition on a set $D \subset X_1$,namely there exist a constant $C > 0$ and number $p \in (0, 1]$ such that the inequality:

$$\left\|f'(x) - f'(y)\right\| \leq C \left\|x - y\right\|^p \tag{3}$$

holds for every $x, y \in D$. In this case we shall say that $f'(\cdot) \in H_D(C,p)$.

In the quoted paper I.K. Argyros defines the divided difference operator $[x,y;f]$ as a linear operator which fulfils the conditions:

$$[x,y;f](y-x) = f(y) - f(x), \qquad \forall x, y \in D, \tag{4}$$

and, in addition, for every $x, y, u \in D$ the following inequality holds:

$$\|[x,y;f] - [y,u;f]\| \le l_1 \|x-u\|^p + l_2 \|x-y\|^p + l_2 \|y-u\|^p, \tag{5}$$

where $l_1 \ge 0,\ l_2 \ge 0$ are constants which do not depend on x, y and u, while $p \in (0,1]$.

Let x^* be a simple solution of (1). We mean by that the mapping $f'(x^*)$ admits a bounded inverse mapping, and if $[x^*,x^*;f] = f'(x^*)$ then $[x^*,x^*;f]$ admits a bounded inverse mapping. Thus the continuity of the mapping $[x,y;f]$ with respect to the variable x and y ensures the existence of a number $\varepsilon > 0$ such that the mapping $[x,y;f]$ admits a bounded inverse mapping for every $x, y \in U(x^*,\varepsilon)$, where $U(x^*,\varepsilon) = \{x \in X_1 : \|x - x^*\| < \varepsilon\}$ that is, the set $B(x,y) = \|[x,y;f]^{-1}\|$ is uniformly bounded in $U(x^*,\varepsilon) = \{x \in X_1 : \|x-x^*\| \le \varepsilon\}$.

Theorem 1. [1] *Let $f : X_1 \to X_2$ and let $D \subset X_1$ an open set. The following conditions are fulfilled:*

(a) *$x^* \in D$ is a simple solution of the equation* (1);
(b) *there exist $\varepsilon \in 0,\ b > 0$ such that $\|[x,y;f]^{-1}\| \le b$ for every $x, y \in U(x^*,\varepsilon)$;*
(c) *there exists a convex set $D_0 \subset D$ such that $x^* \in D_0$, and there exists $\varepsilon_1 > 0$, with $0 < \varepsilon_1 < \varepsilon$ such that $f'(\cdot) \in H_{D_0}(C,p)$ for every $x, y \in D_0$ and $U(x^*,\varepsilon_1) \subset D_0$.*

Let $r>0$ *such that:*

(6) $$0<r<\min\{\varepsilon_1,(q(p))^{-1/p}\}$$

where:

(7) $$q(p)=\frac{b}{p+1}\left[2^p(l_1+l_2)(1+p)+C\right].$$

Then, if $x_0x_1\in\bar{U}(x^*,r)$*, the iterates* $x_n,\ n=2,3,\ldots,$ *generated by* (2) *are well defined and belong to the set* $\bar{U}(x^*,r)$*, while the sequence* $(x_n)_{n\geq 0}$ *converges to the unique solution* x^* *of equation* (1).

Moreover, the following estimation:

(8) $$\|x_{n+1}-x^*\|\leq\gamma_1\|x_{n-1}-x^*\|^p\cdot\|x_n-x^*\|+\gamma_2\|x_n-x^*\|^{p+1}$$

holds for sufficiently great n*, where:*

(9) $$\gamma_1=b(l_1+l_2)2^p,$$

(10) $$\gamma_2=\frac{bC}{1+p}$$

while l_1,l_2 *and* p *were precised by the relation* (5).

In order to prove this theorem the author uses the following two lemmas:

Lemma 1. [1]. *Let* $f:X_1\to X_2$ *and* $D\subset X_1$. *Suppose that* D *is an open set and* $f'(\cdot)$ *does exist in every point of* D. *If, for a convex set* $D_0\subseteq D, f'(\cdot)\in H_{D_0}(C,p)$*, then for every* $x,y\in D_0$ *the following inequality holds:*

$$\left\|f(x)-f(y)-f'(x)(y-x)\right\|\leq\frac{C}{1+p}\|x-y\|^{1+p}.$$

Lemma 2. [1]. *If* $[x,y;f]$ *fulfils the conditions (4) and (5), the following relations hold:*

(a) $[x,x;f]=f'(x)$ *for every* $x\in D_0$;
(b) $f'(\cdot)\in H_{D_0}(2(l_1+l_2),p)$.

From the proof of Theorem 1 follows, for the error estimation and for the convergence speeds of the sequence $(x_n)_{n\geq 0}$, the inequality:

$$\|x_{n+1} - x^*\| \leq (M(r))^{n+1} \|x_0 - x^*\| \tag{11}$$

where one shows that $M(r) \in (0,1)$.

2. We shall make further down some remarks upon the above exposed results, showing that the hypotheses imposed in [1] can lead to more rich conclusions with respect to both the convergency order of the secant method and the error estimation.

Suppose that x_0 and x_1 fulfil the conditions:

(a') $\|x^* - x_0\| \leq \alpha d_0$;
(b') $\|x^* - x_1\| \leq \min\{\alpha d_0^{t_1}, \|x^* - x_0\|\}$

where $0 < d_0 < 1,\ \alpha = (q(p))^{-1}$, while t_1 is the positive root of the equation:

$$t^2 - t - p = 0 \tag{12}$$

$$\text{namely } t_1 = \tfrac{1+(1+4p)^{1/2}}{2}.$$

Using the condition (4) and (5), Lemmas 1 and 2, and the hypotheses of 1, it results easily from (2),for $n = 1$, the inequality [1]:

$$\|x_2 - x^*\| \leq \gamma_1 \|x_0 - x^*\|^p \|x_1 - x^*\| + \gamma_2 \|x_1 - x^*\|^{p+1} \tag{13}$$

from which, using (a') and (b') and the fact that t_1 is a root of equation (12), we obtain:

$$\begin{aligned}
\|x_2 - x^*\| &\leq \gamma_1 \alpha^p d_0^p \alpha d_0^{t_1} + \gamma_2 \alpha^{1+p} d_0^{t_1(1+p)} \\
&= \alpha^{1+p} \left(\gamma_1 d_0^{t_1+p} + \gamma_2 d_0^{t_1(1+p)}\right) \\
&= \alpha^{1+p} d_0^{t_1+p} \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \\
&= \alpha d_0^{t_1^2} \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \alpha^p.
\end{aligned}$$

But

$$\left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right)\alpha^p = \frac{\gamma_1 + \gamma_2 d_0^{p(t_1-1)}}{\gamma_1 + \gamma_2} < 1,$$

then the following inequality holds

$$\|x_2 - x^*\| \le \alpha d_0^{t_1^2}.$$

We prove now that $\|x_2 - x^*\| \le \|x_1 - x^*\|$. From the inequality (13) we obtain:

$$\|x_2 - x^*\| \left(\gamma_1 \alpha^p d_0^p + \gamma_2 \alpha^p d_0^{t_1 p}\right)\|x_1 - x^*\| \le$$
$$\le \alpha^p d_0^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right)\|x_1 - x^*\| < \|x_1 - x^*\|$$

since $d_0^p < 1$ and, as we saw above, $\alpha^p \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) < 1$.

Assume now that for $n \in \mathbb{N}$, $n \ge 2$, the following relations hold:

(a”) $\|x_{n-1} - x^*\| \le \alpha d_0^{t_1^{n-1}}$;

(b”) $\|x_n - x^*\| \le \min\{\alpha d_0^{t_1^n}, \|x_{n-1} - x^*\|\}$

Proceeding as in the case of x_2, and taking into account (a”), (b”) and (8), we obtain:

$$\|x_{n+1} - x^*\| \le \alpha^{1+p} d_0^{t_1^{n+1}} \cdot \left(\gamma_1 + \gamma_2 d_0^{pt_1^{n-1}(t_1+1)}\right) =$$
$$= \alpha d_0^{t_1^{n+1}} \cdot \alpha^p \left(\gamma_1 + \gamma_2 d_0^{pt_1^{n-1}(t_1-1)}\right) \le \alpha d_0^{t_1^{n+1}},$$

since, as previously, it is easy to show that:

$$\alpha^p \left(\gamma_1 + \gamma_2 d_0^{pt_1^{n-1}(t_1-1)}\right) < 1$$

In order to complete the proof, we shall show that:

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|$$

Indeed, form (8) we deduce:

$$\|x_{n+1} - x^*\| \le (\gamma_1 \alpha^p d_0^{pt_1^{n-1}} + \gamma_2 \alpha^p d_0^{pt_1^n})\|x_n - x^*\|.$$

But $d_0 < 1$ and $\alpha^p\left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) < 1$, therefore:

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

We proved in this way the following theorem:

Theorem 2. *If the conditions of Theorem* 1 *are fulfilled, with the difference that* x_0 *and* x_1 *are chosen in such a manner to verify the relations* (a') *and* (b'), *where* $\alpha = (q(p))^{-1/p}$ *and* $d_0 \in (0,1)$, *then, for every* $n \in \mathbb{N}$, $x_n \in U = \{x \in X_1 | \, \|x - x^*\| < \alpha\}$ *and the following inequality holds:*

$$\|x_{n+1} - x^*\| \leq \alpha d_0^{t_1^{n+1}}, \qquad n = 0, 1, \ldots \tag{14}$$

Remark. The inequality (14) contains in its right-hand side a number substantially smaller than that yielded by relation (11).

References

[1] Argyros, I.K., *The secant method and fixed points of nonlinear operators*, Mh. Math. 106, 85–94 (1988).

[2] Dennis, J.E., *Toward a unified convergence theory for Newton like methods*, Nonlinear Functional analysis and Applications (Ed. by L.B. Rall), pp. 425–472, New York, John Wiley (1986).

clickable → [3] Păvăloiu, I., *Introduction to the Theory of Approximation of Equations Solutions*, Dacia Ed., Cluj-Napoca, 1976 (in Romanian).

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This paper is in final form and no version of it is or will be submitted for publication elsewhere.