On a robust Aitken–Newton method based on the Hermite polynomial

Ion Păvăloiu, Emil Cătinaş*

"T. Popoviciu" Institute of Numerical Analysis (Romanian Academy), P.O. Box 68-1, Cluj-Napoca, Romania

A R T I C L E I N F O

MSC 2010:
65H05

Keywords:
Nonlinear equations in $\mathbb{R}$
Newton-type iterative methods
Inverse interpolation
(Sided) convergence domains
Monotone convergence

A B S T R A C T

We introduce an Aitken–Newton iterative method for nonlinear equations, which is obtained by using the Hermite inverse interpolation polynomial of degree 2, with two nodes given by the Newton method.

The local convergence of these iterates is shown to be 8, and the efficiency index is $\sqrt[5]{8} \approx 1.51$, which is not optimal in the sense of Kung and Traub. However, we show that under supplementary conditions (sometimes easy to verify) the inner and outer iterates converge monotonically to the solution. This aspect allows an improved control of the iteration stopping (avoiding divisions by zero) and offer an alternative way to the estimation of radius of attraction balls in ensuring the convergence of the iterates. Numerical examples show that this method may become competitive and in certain circumstances even more robust than certain optimal methods of same convergence order.

1. Introduction

In this paper, we study certain iterative methods for solving nonlinear equations

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$. The Aitken and Steffensen-type methods consider two more equations, equivalent to the above one:

$$x - g_i(x) = 0, \quad i = 1, 2, \tag{2}$$

where $g_i : [a, b] \to [a, b]$.

We admit that

(A) Eq. (1) has a solution $x^* \in]a, b[$.

The most known methods of Steffensen, Aitken, or Aitken–Steffensen type are obtained from inverse interpolation polynomials of degree 1, having the knots determined with the aid of different functions g_1, g_2 (see, e.g., [1–5,13]).

More general methods of this type have been obtained by using inverse interpolation polynomials of degree 2 [11]; they have led to iterative methods of q-order at least 3, and with efficiency indexes usually higher than for the case of interpolation polynomials of degree 1 [15]. Obviously, the convergence order and the efficiency index of the resulted methods depend on the functions g_i.

Here, we study a method based on the Hermite inverse interpolation polynomial of degree 2, with the nodes generated with the aid of the Newton iteration function, $x - f(x)/f'(x)$, in (2).

We need the following assumption on f:

(B) f is three times derivable on $[a, b]$ and $f'(x) \neq 0$, $\forall x \in [a, b]$.

This assumption implies that function f is monotone and continuous on $[a, b]$, therefore $\exists f^{-1} : J \to [a, b]$, $J := f([a, b])$, and by (B) we have

$$x^* = f^{-1}(0).$$

In order to approximate the solution x^* we shall consider the Hermite inverse interpolation polynomial of degree 2.

Let $a_1, a_2 \in [a, b]$, denote $b_1 = f(a_1)$, $b_2 = f(a_2)$ so that

$$f^{-1}(b_1) = a_1, \qquad f^{-1}(b_2) = a_2,$$

and

$$f^{-1}(\cdot)'|_{y=b_2} = \tfrac{1}{f'(a_2)}.$$

The polynomial of degree 2 which satisfies

$$P(b_1) = a_1$$
$$P(b_2) = a_2$$
$$P'(b_2) = \tfrac{1}{f'(a_2)}$$

is the Hermite polynomial, given by

$$P(y) = a_1 + [b_1, b_2; f^{-1}](y - b_1) + [b_1, b_2, b_2; f^{-1}](y - b_1)(y - b_2), \tag{3}$$

having the remainder

$$f^{-1}(y) - P(y) = [y, b_1, b_2, b_2; f^{-1}](y - b_1)(y - b_2)^2, \quad \forall y \in J,$$

where $[u, v; h]$, $[u, v, w; h]$ denote the first, respectively the second order divided difference of the function h at the nodes u, v resp. u, v, w.

Setting $y = 0$ in (3) we are led to another approximation for x^*:

$$x^* \approx P(0) = a_1 - [b_1, b_2; f^{-1}]b_1 + [b_1, b_2, b_2; f^{-1}]b_1 b_2, \tag{4}$$

having the error

$$x^* - P(0) = -[0, b_1, b_2, b_2; f^{-1}]b_1 b_2^2. \tag{5}$$

Formula (4) will be used in an iterative fashion, by setting $a_3 = P(0)$, and so on. However, instead of (4) and (5) we need formulas which do not need the evaluation of the inverse function.

It is known that the divided differences satisfy the following relations (see, e.g., [12]):

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]}$$

$$[b_1, b_2, b_2; f^{-1}] = -\frac{[a_1, a_2, a_2; f]}{[a_1, a_2; f]^2 f'(a_2)}$$

which lead us to a formula for a_3 based only on the values of f and its derivatives:

$$x^* \approx a_3 = a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_2; f] f(a_1) f(a_2)}{[a_1, a_2; f]^2 f'(a_2)}. \tag{6}$$

Taking into account (B) and the Mean Value Theorem on the divided differences, the error in (5) becomes

$$[0, b_1, b_2, b_2; f^{-1}] = \frac{f^{-1}(\eta)'''}{6}, \quad \eta \in \operatorname{int} J.$$

Since (see, e.g., [12])

$$f^{-1}(y)''' = \frac{3\big(f''(x)\big)^2 - f'(x)f'''(x)}{f'(x)^5}, \quad \text{with } y = f(x),$$

and f is one-to-one, it follows that $\exists \xi \in]a, b[$ with $\eta = f(\xi)$ such that

$$[0, b_1, b_2, b_2; f^{-1}] = \frac{3\big(f''(\xi)\big)^2 - f'(\xi)f'''(\xi)}{6(f'(\xi))^5}. \tag{7}$$

$$ \tag{8} $$

the error in (7) is written as

$$x^* - a_3 = -\frac{E_f(\xi)}{6(f'(\xi))^5} b_1 b_2^2. \tag{9}$$

As we have mentioned, the functions g_1 and g_2 in (2) are taken, with the aid of the Newton iteration function, as:

$$g_1(x) = x - \frac{f(x)}{f'(x)};$$

$$g_2(x) = g_1(g_1(x)),$$

so that, setting $a_1 = z_n$ and $a_2 = y_n$ in (6), we are led to the following iterations:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{[z_n, y_n; f]} - \frac{[z_n, y_n, y_n; f] f(z_n) f(y_n)}{[y_n, z_n; f]^2 f'(y_n)}, \quad n = 0, 1, \dots \end{aligned} \tag{10}$$

We refer to this as the Aitken–Newton method.

We shall show in Section 2 that under some very simple conditions, this method converges locally with q-order 8. Since, at each iteration step we need to compute 5 function evaluations (i.e., $f(x_n), f'(x_n), f(y_n), f'(y_n)$ and $f(z_n)$), the efficiency index is $\sqrt[5]{8} \approx 1.51$, which is not optimal in the sense of Kung and Traub.

There are a lot of optimal methods (see, e.g., [7,8,10,16]) but, as we shall see in the numerical examples section, the optimal efficiency index is not a paramount by itself, since there exist situations when some optimal methods may have very small attraction sets for certain solutions.

In Section 3 we show that if there exists a set $D \subseteq [a, b]$ containing a solution x^* such that on this set, f has monotone and convex properties, E_f is positive, and the Fourier condition is verified, then for any initial approximation in D, the method converges monotonically to x^*:

$$x^* < x_{n+1} < z_n < y_n < x_n, \quad (\text{or} \quad x_n < y_n < z_n < x_{n+1} < x^*), \quad n = 0, 1, \dots$$

It is important to note that the convergence holds as large as the domain D is, and this domain may extend to the left or to the right of the solution much more than ensured by (even sharp) estimates for the radius of an attraction ball.

The above inequalities show another advantage of this method, since the stopping criterions (such as $|x_{n-1} - x_n| < tol_1$ or $|f(x_n)| < tol_2$) may be considered for the extended (inner) iterates x_n, y_n, z_n, x_{n+1} (resp. for the values of f at them), avoiding divisions by zeros.

These aspects, as well as other ones, are treated in Section 4, regarding numerical examples.

2. Local convergence

The following result holds.

Theorem 1. *Under assumptions (A) and (B), the Aitken–Newton method (10) converges locally, with q-convergence order 8:*

$$x^* - x_{n+1} = A_n (x^* - x_n)^8, \tag{11}$$

where

$$A_n = -\frac{E_f(\xi_n) f''(\theta_n) (f''(\delta_n))^4 f'(\mu_n) (f'(\nu_n))^2}{192 (f'(\xi_n))^5 (f'(x_n))^4 f'(y_n)},$$

for certain $\xi_n, \theta_n, \delta_n, \mu_n, \nu_n \in]a, b[$. *The asymptotic constant is given by*

$$A = \lim_{n \to \infty} A_n = -\frac{E_f(x^*) (f''(x^*))^5}{192 (f'(x^*))^7}.$$

Proof. We assume in the beginning that the elements of the sequences $(x_n)_{n \geq 0}$, $(y_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ remain in $[a, b]$.

Assumption (B) and the first relation in (4) imply the existence of $\delta_n \in]a, b[$ such that

$$x^* - y_n = -\frac{f''(\delta_n)}{2 f'(x_n)} (x^* - x_n)^2, \quad n = 0, 1, \dots \tag{12}$$

Analogously, the second relation in (4) attracts

$$x^* - z_n = -\frac{f''(\theta_n)}{2 f'(y_n)} (x^* - y_n)^2, \quad n = 0, 1, \dots, \tag{13}$$

with $\theta_n \in]a, b[$.

Considering (9), with $a_3 = x_{n+1}$, $b_1 = f(z_n)$ and $b_2 = f(y_n)$, we get

$$x^* - x_{n+1} = -\frac{E_f(\xi_n)}{6(f'(\xi_n))^5} f(z_n)(f(y_n))^2, \quad n = 0, 1, \dots$$

The values $f(z_n)$ şi $f(y_n)$ are easily expressed by

$$f(z_n) = f'(\mu_n)(z_n - x^*),$$

$$f(y_n) = f'(\nu_n)(y_n - x^*), \tag{14}$$

with $\mu_n, \nu_n \in]a, b[$.

Relations (12)–(14) imply (11).

The induction holds if x_0 is chosen sufficiently close to x^*. □

3. Monotone convergence

We consider the following supplementary conditions on f

(C) Function E_f given by (8) is strictly positive on $[a, b]$:

$$E_f(x) = 3(f''(x))^2 - f'(x)f'''(x) > 0, \quad \forall x \in]a, b[;$$

(D) The initial approximation $x_0 \in [a, b]$ satisfies the Fourier condition:

$$f(x_0)f''(x_0) > 0.$$

Remark 1. As we shall see in the numerical examples, condition (C) may sometimes be easily verified (e.g, when f' and f'' have different signs, etc.).

We obtain the following results:

Theorem 2. *If f and x_0 satisfy assumptions (A)–(D) and, moreover:*

(i_1) $f'(x) > 0$, $\forall x \in [a, b]$;
(ii_1) $f''(x) > 0$, $\forall x \in [a, b]$;

then the iterates of the Aitken–Newton method (10) satisfy:

(j_1)$x^* < x_{n+1} < z_n < y_n < x_n$, $n = 0, 1, \dots$
(jj_1) $\lim x_n = \lim y_n = \lim z_n = x^*$.

Proof. We proceed by induction.

Let $x_n \in [a, b]$ satisfy the Fourier condition. By (ii_1) we have that $f(x_n) > 0$, therefore $x_n > x^*$, by (i_1). Relations (12) and (13) attract $y_n > x^*$ and $z_n > x^*$. The first relation in (10), together with $f(x_n) > 0$ imply that $y_n < x_n$. Analogously, $z_n < y_n$. The third relation in (10) and the hypotheses imply that $z_n > x_{n+1} > x^*$. Statement (j_1) is proved.

It follows that sequences (x_n), (y_n), (z_n) are convergent to the same limit ℓ, and (10) shows that $f(\ell) = 0$, i.e., $\ell = x^*$. □

Theorem 3. *If f and x_0 satisfy (A)–(D) and moreover,*

(i_2) $f'(x) < 0$, $\forall x \in [a, b]$;
(ii_2) $f''(x) < 0$, $\forall x \in [a, b]$,

then the same conclusions as those in Theorem 2 hold.

The proof is easily obtained applying Theorem 2 to the function $h = -f$.

The following two theorems have similar proofs.

Theorem 4. *If f and x_0 satisfy conditions (A)–(D) and moreover,*

(i_3) $f'(x) < 0$, $\forall x \in [a, b]$;
(ii_3) $f''(x) > 0$, $\forall x \in [a, b]$;

then the iterates of the Aitken–Newton method (10) satisfy

(j_3) $x_n < y_n < z_n < x_{n+1} < x^*$, $n = 0, 1, \dots,$
(jj_3) $\lim x_n = \lim y_n = \lim z_n = x^*$.

Theorem 5. *If f and x_0 satisfy conditions (A)–(D) and moreover,*

Table 1
Aitken–Newton iterates (10), $f(x) = e^{2x} + \sin x - 2$.

n	x_n	y_n	z_n	$f(x_n)$
0	1	5.932655378778493e−1	3.446691220304792e−1	6.2e+0
1	2.781136458347832e−1	2.739285803512798e−1	2.739153432766920e−1	1.8e−2
2	2.739153431449791e−1			0

Table 2
Aitken–Newton iterates (10), $f(x) = e^x - 4x^2$.

n	x_n	y_n	z_n	$f(x_n)$
0	1	7.573293140767846e−1	7.161639906789638e−1	−1.2e+00
1	7.148090008114115e−1	7.148059123705082e−1	7.148059123627778e−1	−1.1e−05
2	7.148059123627779e−1			−4.4e−16

(i_4) $f'(x) > 0$, $\forall x \in [a, b]$;
(ii_4) $f''(x) < 0$, $\forall x \in [a, b]$,

then the iterates of (10) satisfy the conclusion of Theorem 4.

Remark 2. As we shall see in the numerical examples, even if the hypotheses of the above monotone convergence results are not fulfilled, the iterates may converge (due to the local convergence Theorem 1) and sometimes even monotonically (Theorems 2–5 offer sufficient but not also necessary conditions).

4. Numerical examples

In this section we shall consider some examples for which we run programs in Matlab, in double precision floating point arithmetic (this is the setting most encountered in practice). As we shall see, the high convergence orders and the monotone properties are properly reflected in this setting, and we do not need higher precision.

There are a lot of optimal iterative methods having the convergence order 8. We shall present only a few of them, for which we have found a better convergence of the Aitken–Newton iterates in certain situations.

We begin with two examples, which show the rapid convergence of the Aitken–Newton iterates (10).

Example 1. Let

$$f(x) = e^{2x} + \sin x - 2, \quad x \in [0, 1].$$

We have: $f(0) = -1$, $f(1) = e^2 + \sin 1 - 2 > 0$, $f'(x) = 2e^{2x} + \cos x > 0$, $x \in [0, 1]$, $f''(x) = 4e^{2x} - \sin x > 0$, $f'''(x) = 8e^{2x} - \cos x$ and

$$E_f(x) > 32e^{4x} - 30e^{2x} + 1 > 0, \qquad \forall x > 0.$$

Taking $x_0 = 1$, Theorem 2 applies and we obtain decreasing sequences x_n, y_n, z_n, as can be seen in Table 1.

Example 2. Consider

$$f(x) = e^x - 4x^2, \qquad x \in [\tfrac{1}{2}, 1].$$

We have $f\left(\frac{1}{2}\right) = \sqrt{e} - 1 > 0$, $f(1) = e - 4 < 0$, $f'(x) = e^x - 8x < 0$, $f''(x) = e^x - 8 < 0$, $f'''(x) = e^x$ and $E_f(x) = 2e^{2x} - 48e^x + 8xe^x + 192 > 0$, $x \in [\frac{1}{2}, 1]$.

Taking $x_0 = 1$, Theorem 2 applies again and we obtain the results presented in Table 2.

Next, we present two examples and we compare the studied method to other methods. In order to obtain smaller tables, we used the `format short` command in Matlab. It is worth mentioning that such choice may lead to results that appear integers, while they are not (e.g., the value of y_4 in Table 6 is shown to be 2, while $f(y_4)$ should be 0); the explanation resides in the rounding made in the conversion process.

Example 3. Consider the following equation (see, e.g., [6])

$$f(x) = e^x \sin x + \ln(x^2 + 1), \qquad x^* = 0.$$

The largest interval to study the monotone convergence of our method by Theorems 2–5 is $[a, b] := [x^*, 1.54\ldots]$, since f'' vanishes at b (being positive on $[a, b]$). The Fourier condition (D) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$, while the derivatives f', f'' are positive on this interval. The conclusions of Theorem 2 apply.

The Aitken–Newton method leads to the following results, presented in Table 3.

Table 3
Aitken–Newton iterates, $f(x) = e^x \sin x + \ln(x^2 + 1)$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	1.54	5.8778	0.51233	1.0513	0.17152	0.2316
1	0.048016	0.052662	0.0039166	0.0039473	3.0245e−05	3.0246e−05
2	3.4821e−09	3.4821e−09	3.6375e−17	3.6375e−17	0	0

Table 4
Cordero–Torregrosa–Vassileva iterates, $f(x) = e^x \sin x + \ln(x^2 + 1)$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	1.49	5.592	0.51	1.0442	0.21795	0.31529
1	−0.36451	−0.12284	−0.87167	0.24508	−0.66891	0.052125
2	−0.57963	−0.017122	−0.60388	0.00048485	−0.60323	5.9483e−07
3	−0.60323	−1.9295e−12	−0.60323	0	−0.60323	0
0	1.48	5.535	0.50911	1.0414	0.21622	0.31202
1	−0.32703	−0.13002	−1.258	0.67839	−0.83324	0.20558
2	0.10514	0.12758	0.015884	0.01639	4.5216e−4	4.5257e−4
3	−6.6097e−07	−6.6097e−07	8.7366e−13	8.7366e−13	−2.1176e−22	−2.1176e−22

Table 5
Kou–Wang iterates, $f(x) = e^x \sin x + \ln(x^2 + 1)$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	1.49	5.592	0.51	1.0442	0.21795	0.31529
1	−0.26375	−0.13301	2.4987	9.2742	1.1273	3.6088
2	−189.6439	10.4903	805.0965	Inf	NaN	NaN
0	1.48	5.535	0.50911	1.0414	0.21622	0.31202
1	−0.23421	−0.13021	0.68334	1.6336	0.24215	0.36247
2	1.1187	3.5648	0.41752	0.7763	0.14701	0.19107
3	−0.015069	−0.014616	4.8068e−4	4.8114e−4	4.2097e−07	4.2097e−07
4	7.7382e−13	7.7382e−13	1.7964e−24	1.7964e−24	−2.7804e−36	−2.7804e−36

The convergence may be not very fast for initial approximations away from the solution.

It is worth noting that the method converges for $-0.3 \le x_0 < x_1^*$ too (local convergence near $x^* = 0$ assured by Theorem 1), despite the Fourier condition does not hold. For $x_0 = -0.3$ one obtains $x_1 \approx -0.25 < 0$, then $y_1 \approx 1.7 > 0$ and the rest of the iterates remain positive, converging monotonically to x_1^*. For $x_0 = -0.4$, the method converges to another solution of the equation, $x_2^* = -0.603\ldots$

The optimal method introduced by Cordero et al. [8] has a smaller convergence domain for $x_0 > x_1^*$, since the iterates converge for $x_0 = 1.48$ ($x_4 = 1.3741e-32$), while for $x_0 = 1.49$ the iterates jump over x_1^* and converge to x_2^* as can be seen in Table 4. By checking all the initial approximations between 0 and 1.48 with step 0.001, we found out that the convergence domain is even smaller, since taking 1.442 as initial approximation does not lead to convergence.

The Kou–Wang method (formula (25) in [9]) converges for $x_0 = 1.48$ ($x_4 = 0$) and diverges for $x_0 = 1.49$, as can be seen in Table 5.

Example 4. Consider the following equation (see, e.g., [14])

$$f(x) = (x-2)(x^{10} + x + 1)e^{-x-1}, \qquad x^* = 2.$$

The largest interval to study the monotone convergence of our method by Theorems 2–5 is $[a, b] := [x^*, 7.9\ldots]$, since f'' vanishes at b (being positive on $[a, b]$).

The Fourier condition (D) holds on $[a, b]$ (and does not hold for $x < a$), $E_f(x) > 0$ on $[a, b]$, while both the derivatives f', f'' are positive on this interval. The conclusions of Theorem 2 apply.

It is interesting to note that in [14, Remark 6] Petković observed that the methods studied there have a small convergence domain to the left of the solution: the choice of $x_0 = 1.8$ caused a bad convergence behavior of those iterates. We believe that this behavior may be explained by the fact that the derivative of f vanishes at $x = 1.78\ldots$

The Aitken–Newton method leads to the results presented in Table 6. The iterates converge even for $x_0 > 7.9$ (and to the left of the solution as well, but for x_0 higher than 1.72). Of course the convergence may be not very fast when the initial approximations are away from the solution.

The optimal method introduced by Cordero et al. [8] converges to x^* for $x_0 = 6.46$ and it does not for $x_0 = 6.47$, as shown in Table 7.

The optimal method introduced by Liu and Wang (formula (18) in [10]) converges to $x^* = 2$ for $x_0 = 2.359$ (it needs 5 iterates) but for $x_0 = 2.36$ it converges to another solution, $x_1^* = 1512.626\ldots$. The results are presented in Table 8.

Table 6
Aitken–Newton iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	7.9	761907.1334	5.6028	148982.786	4.6615	44837.6641
1	4.0818	16594.4155	3.5637	5385.3696	3.1548	1769.5473
2	2.8568	655.665	2.5841	215.3342	2.3658	69.4249
3	2.2125	24.0727	2.0909	6.6087	2.0232	1.3004
4	2.0026	0.13254	2	0.0013264	2	1.3712e−07
5	2	−1.1353e−14	2	0		

Table 7
Cordero, Torregrosa and Vassileva iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	6.46	324955.0041	5.165	89878.025	4.3634	27700.9456
1	1.8472	−4.1246	2.3095	48.8543	2.0877	6.3062
2	1.9112	−3.155	2.053	3.3344	2.0049	0.25362
3	1.9998	−0.0091403	2	6.5281e−06	2	1.9074e−12
4	2	0				
0	6.47	327469.2878	5.1701	90456.4068	4.3678	27912.268
1	1.8136	−4.336	2.9391	879.3346	2.3777	74.4975
⋮						
7	−56.4878	−2.424e+43				

Table 8
Liu–Wang iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

n	x_n	$f(x_n)$	y_n	$f(y_n)$	z_n	$f(z_n)$
0	2.359	66.6580	2.1929	20.3973	2.0620	4.0392
1	1.8237	−4.2901	2.6406	277.0720	2.2353	28.8601
2	2.7206	387.8421	2.4745	126.5702	2.2432	30.6641
3	2.1785	17.9263	2.0697	4.6753	2.0103	0.5496
4	2.0009	0.0449	2.0000	1.5555e−04	2.0000	1.0885e−09
5	2	0				
0	2.36	67.0603	2.1937	20.5289	2.0624	4.0706
1	1.7919	−4.3898	5.5436	1.3980e+05	3.6678	6.9018e+03
2	1.5126e+03	0				

Table 9
Petković–Euler-like iterates, $f(x) = (x-2)(x^{10}+x+1)e^{-x-1}$.

n	x_n	$f(x_n)$
0	2.15	13.5861
1	2.0016	0.082608
2	2	0
0	2.16	15.0282
1	2.0053−0.0052421i	0.27199−0.2795i
2	2+1.7364e−15i	1.0672e−12+8.8785e−14i
3	2	0

Among the optimal methods in [14] (the methods with convergence orders higher than 8 were corrected in a subsequent paper), the modified Ostrowski and Maheshwari methods behave very well for this example (we have studied the convergence only to the right of the solution). The modified Euler-like method has a small domain of convergence (in $\mathbb{R}$), since it converges to x^* for $x_0 = 2.15$, while for $x_0 = 2.16$ it generates square roots of negative numbers. Matlab has the feature of implicitly dealing with complex numbers, and the iterates finally converge (in $\mathbb{C}$) to the solution (see Table 9).

Conclusions

The Aitken–Newton method studied in this paper present, under certain circumstances, some advantages over the optimal methods; the obtained sufficient conditions for guaranteed convergence may theoretically lead to larger convergence domains (especially sided convergence intervals) than from estimates of attraction balls, while a few examples shown that the attraction domain of the method is larger than for some optimal methods.

References

[1] S. Amat, J. Blanda, S. Busquier, A Steffensen type method with modified functions, Riv. Mat. Univ. Parma 7 (2007) 125–133.
[2] S. Amat, S. Busquier, J.A. Ezquerro, M.A. Hernández-Verón, A Steffensen type method of two steps in banach spaces with applications, J. Comput. Appl. Math. 291 (2016) 317–331.
[3] I.K. Argyros, A new convergence theorem for the Steffensen method in Banach space and applications, Rev. Anal. Numér. Théor. Approx. 29 (1) (2000) 119–127.
[4] I.K. Argyros, S. Hilout, An improved local convergence analysis for Newton–Steffensen-type method, J. Appl. Math. Comp. 32 (1) (2010) 111–118.
[5] E. Cătinaş, On some Steffensen-type iterative method for a class of nonlinear equations, Rev. Anal. Numér. Théor. Approx. 24 (1-2) (1995) 37–43.
[6] C. Chun, Certain improvements of Chebyshev–Halley methods with accelerated fourth-order convergence, Appl. Math. Comput. 189 (1) (2007) 597–601.
[7] A. Cordero, J.L. Hueso, E. Martinez, J.R. Torregrosa, Generating optimal derivative free iterative methods for nonlinear equations by using polynomial interpolations, Math. Comput. Model. 57 (7-8) (2013) 1950–1956.
[8] A. Cordero, J.R. Torregrosa, M.P. Vassileva, A family of modified Ostrowski's methods with optimal eighth order of convergence, Appl. Math. Lett. 24 (12) (2011) 2082–2086.
[9] J. Kou, X. Wang, Some improvements of Ostrowski's method, Appl. Math. Lett. 23 (1) (2010) 92–96.
[10] L. Liu, X. Wang, Eighth-order methods with high efficiency index for solving nonlinear equations, Appl. Math. Comput. 215 (9) (2010) 3449–3454.
[11] I. Păvăloiu, Approximation of the root of equations by Aitken–Steffensen-type monotonic sequences, Calcolo 32 (1-2) (1995) 69–82.
[12] I. Păvăloiu, E. Cătinaş, Bilateral approximation for some Aitken–Steffensen–Hermite type method of order three, Appl. Math. Comput. 217 (12) (2011) 5838–5846.
[13] I. Păvăloiu, E. Cătinaş, On an Aitken–Newton type method, Numer. Algorithm 62 (2) (2013) 253–260.
[14] M.S. Petković, On a general class of multipoint root-finding methods of high computational efficiency, SIAM J. Numer. Anal. 47 (6) (2010) 4402–4414.
[15] J.R. Sharma, A composite third order Newton–Steffensen method for solving nonlinear equations, Appl. Math. Comput. 169 (1) (2005) 242–246.
[16] F. Soleymani, S.K. Vanani, Optimal Steffensen-type method with eighth order of convergence, Comput. Math. Appl. 62 (12) (2011) 4619–4626.