Spectral methods in linear stability. Applications to thermal convection with variable gravity field

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Abstract

The onset of convection in a horizontal layer of fluid heated from below in the presence of a gravity field varying across the layer is numerically investigated. The eigenvalue problem governing the linear stability of the mechanical equilibria of the fluid, in the case of free boundaries, is a sixth order differential equation with Dirichlet and hinged boundary conditions. It is transformed into a system of second order differential equations supplied only with Dirichlet boundary conditions. Then it is solved using two distinct classes of spectral methods namely, weighted residuals (Galerkin type) methods and a collocation (pseudospectral) method, both based on Chebyshev polynomials. The methods provide a fairly accurate approximation of the lower part of the spectrum without any scale resolution restriction. The Viola's eigenvalue problem is considered as a benchmark one. A conjecture is stated for the first eigenvalue of this problem.

1. Introduction

It is a matter of evidence that the hydrodynamic stability problems become more complex due to incorporation of technologically important effects. Consequently, the order of the differential equations increases rapidly.

Generally the discretization methods applied to such problems lead to matrices which become less conditioned as both the order of approximation and the order of differentiation increase. This fact is particularly true for spectral methods. Roughly, for $N = 2^9$ (N stands for the spectral parameter), the condition number for the second order Chebyshev differentiation matrix is of order $O(10^{18})$, that of the fourth order differentiation matrix attains something of order $O(10^{24})$ and eventually that of the eighth order differentiation matrix raises to something of order $O(10^{30})$.

In this context, the aim of this paper is two-fold. First, we introduce a new method to solve numerically high order and even two-point eigenvalue (boundary value) problems with Dirichlet and hinged boundary conditions. It reduces the problem to a system of second order differential equations supplied only with Dirichlet boundary conditions.

Further, accurate spectral methods are used in order to solve that Dirichlet boundary value problem numerically. Second, the method is applied to solve a hydrodynamic stability problem of Bénard type.

It is well known that, the classical Orr–Sommerfeld eigenvalue problem, which is encountered in the linear hydrodynamic stability analysis of some basic flows, was first solved using a direct (tau) spectral method by S. Orszag in 1971 in his seminal paper [20]. He determined the critical Reynolds number associated with the most unstable shear mode corresponding to Poiseuille flow, i.e., $Re = 5772.22$.

Two other important papers, of Dongarra et al. [5] and that of Melenk et al. [19], consider in a general mathematical framework spectral methods for hydrodynamic stability problems. They analyze the Orr–Sommerfeld equation supplied with homogeneous boundary conditions which contain derivative up to the first order. The pseudospectra and the numerical range of this Orr–Sommerfeld operator is also investigated by Reddy, Schmid and Henningson in their paper [24].

The high accuracy of tau, Galerkin and collocation spectral methods was subsequently, intensively exploited in solving second order as well as higher order (especially fourth order) differential eigenvalue problems (see also [16]). The main difficulty with the later problems consists in imposing boundary conditions which involve derivatives of order higher than one. From the mathematical point of view this reduces to a Hermite interpolation problem. In a quite general setting it was solved by Huang and Sloan in their paper [17]. Unfortunately, in practical computations, the resulting trial and test functions lead to tedious computations with significant rounding errors.

Up to our knowledge there are two main strategies to avoid these difficulties. The first one requires a special attention in choosing the spaces of test and trial functions. It is important to design spaces of functions which satisfy as many as possible boundary conditions. In our previous paper [11] we built up a Chebyshev tau method in order to solve a hydrodynamic stability problem connected with the Marangoni–Plateau–Gibbs convection. The eigenvalue problem was a non-standard one, i.e., two out of the four boundary conditions attached to the Orr–Sommerfeld equation contained derivatives of order two and three as well as the eigenparameter. Our numerical results were lately confirmed by Greenberg and Marleta [13]. A more recent paper of Blyth and Pozrikidis [1], on the same topic, deals with analogue eigenproblems. The main drawback of this approach consists in the fact that it depends strictly on the problem at hand.

A much more flexible and general strategy is considered in this paper. Mainly it transforms the original problem into one containing lower order derivatives in the differential equation as well as in the boundary conditions. Sometimes this means to rewrite a problem as a first order system of differential equations. The shooting method and the compound matrix method are examples of such method. Instead we suggest a method which takes into account the particularities of the problem (parity of order of differentiation, symmetries, etc.). It reduces a fourth or sixth order eigenvalue problem (generally, an even order problem) with Dirichlet and hinged boundary conditions to a system of second order equations supplied exclusively with Dirichlet boundary conditions. These boundary conditions are much more simple to handle than the hinged one. Further, this procedure simplifies considerably the construction of test and trial functions in tau and Galerkin methods as well as the formulation of differentiation matrices in the collocation (pseudospectral) method. Thus, it is applicable to any differential boundary value problem which contains only even order of differentiation supplied with Dirichlet and hinged boundary conditions. This is our main theoretical result.

Actually, we used the Galerkin and Petrov–Galerkin methods as well as a collocation method in order to approximate the lower part of the spectrum. It is important to observe that the computations were carried out without any scale resolution restriction. The Chebyshev collocation was the most accurate, the fastest and the most flexible with respect to the implementation process.

The Viola's eigenvalue problem is considered as a benchmark one. This is a fourth order singularly perturbed differential eigenvalue problem. We use two strategies in order to solve it. First, we solve the problem as a second order one supplied only with Dirichlet boundary conditions ("D^2" strategy) and observe that the differential singularity deteriorates considerably the conditioning of the discretization matrices. In spite of this inconvenience, our Chebyshev collocation method provides numerical results which improves those reported by Straughan in the Appendix of his monograph [27]. Second, we alternatively handle the problem as a fourth order one using in our collocation method shape functions which fulfill all boundary conditions. Unfortunately, in this case, even for moderate values of N, i.e., $N = 128$, the algorithm becomes unstable showing that the "D^2" strategy is plainly superior with respect to accuracy. One drawback of this strategy is the superior cost of the computation of algebraic eigenvalues. As we work with the qz algorithm in solving the algebraic eigenvalue problem, the cost is eight times larger than that of solving the fourth

order problem. In spite of this aspect, even for the hydrodynamic stability problem, where the cost is 27 times larger, the computing time and the memory requirements remain reasonable.

Eventually, the above "D^2" strategy was implemented making use of some weighted residuals (Chebyshev Galerkin type) methods as well as of a Chebyshev collocation method in order to solve a hydrodynamic stability problem of Bénard type involving variable gravity field.

2. The hydrodynamic stability problem

Variations in the gravity field occurring in, on and above the Earth's surface have their importance in some physical problems such as the crystal growth, e.g., [29], some convection problems from porous media, e.g., [21], and other convective flows from biomechanics, chemistry or geophysics. Consequently, their effects on the stability bounds must be investigated.

In the present paper we are concerned with a problem governing the convection-conduction in a horizontal layer bounded by two impermeable walls of a fluid heated from below. Across the layer the gravity field will vary according to some specific laws. The heat conducting viscous fluid is contained in a layer situated between the planes $z = 0$ and $z = h$. For $t > 0$ the conduction and convective motion is governed by the conservation equations of momentum, mass and internal energy ([27], p. 93)

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})\mathbf{v} = -\dfrac{1}{\rho}\,\mathrm{grad}\,p + \nu\Delta\mathbf{v} + \mathbf{g}(z)\alpha T, \\ \mathrm{div}\,\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + (\mathbf{v} \cdot \mathrm{grad})T = k\Delta T, \end{cases} \tag{1}$$

where, as usual, ν is the coefficient of kinematic viscosity, ρ is the density, α the thermal expansion coefficient, k the thermal diffusivity, p the pressure, T the temperature, and $\mathbf{v}$ the velocity field. $\mathbf{g}(z) := g(1 + \varepsilon h(z))\mathbf{k}$ stands for the variable gravity field, with g constant, $\mathbf{k}$ the unit vector in the z-direction, and ε the scale parameter. Consequently, $\varepsilon h(z)$ represents the gravity variation.

The boundary conditions appropriate to fixed surfaces and specified constant temperature, i.e., T_U on $z = h$, T_L on $z = 0$ are (see [27])

$$\mathbf{v} \equiv \mathbf{0}, \quad z = 0, h, \qquad T = T_L, \quad z = 0; \qquad T = T_U, \quad z = h. \tag{2}$$

We assume that $T_L > T_U$.

The stationary solution of Eq. (1) has the form

$$\mathbf{v} \equiv \mathbf{0}, \qquad T = -\beta z + T_0, \tag{3}$$

where β stands for the pressure gradient and T_0 is considered as a reference temperature. In order to investigate the linear stability of (3), we undertake a usual normal mode analysis, i.e., for instance, for the first velocity component u, we write $u := \exp(\sigma t + imx + iny)W(z)$. Consequently, the following two-point boundary value problem governing the linear stability of this conduction stationary solution results (see [27] for technical details)

$$\begin{cases} (D^2 - a^2)^2 W = R(1 + \varepsilon h(z))a^2\Theta, \\ (D^2 - a^2)\Theta = -RW, \end{cases} \tag{4}$$

$$W = DW = \Theta = 0 \quad \text{at } z = 0, 1. \tag{5}$$

Throughout this paper we use the notation $D(\cdot) := d(\cdot)/dz$. The nondimensional quantity $Ra := R^2 = \alpha g\beta h^4/k\nu$ is the Rayleigh number and it represents the eigenparameter of the problem (4)–(5), where a is the wavenumber, and W and Θ are the amplitudes of the vertical velocity and temperature perturbation respectively. They form the eigenfunction of the eigenvalue problem (4)–(5). This case is considered in detail in the monograph of Straughan [27].

The boundary conditions on a free surface are that the perturbations of the stress components are zero. This implies that a free surface behaves as a rigid one with tangential slip but without any tangential stress [10]. In our case, for normal modes perturbations, the boundary conditions reduce to

$$W = D^2W = \Theta = 0 \quad \text{at } z = 0, 1. \tag{6}$$

We will treat here only the free boundaries case, so the corresponding eigenvalue problem is (4)–(6). In the paper of Herron [15], it was proven that in this situation the principle of exchange of stabilities is valid.

B. Straughan provided in the above quoted monograph numerical evaluation of the Rayleigh number in the case of rigid boundaries by using the energy method. In our paper [7], for linearly decreasing gravity fields, we obtained two stability criteria using isoperimetric inequalities and showing that in this case the domain of stability enlarges. Stability bounds were found by us in the case of both free and rigid boundaries [6] using spectral methods based on trigonometric series. In [8] and [9] the eigenvalue problem (4)–(6) was studied using expansion series based on polynomials (Legendre and Chebyshev polynomials).

Taking into account the order of differentiation in (4)–(6) we introduce the new variable

$$\Psi := (D^2 - a^2)W. \tag{7}$$

Thus, the two-point boundary value problem (4)–(6) can be rewritten as the second order system

$$\begin{cases} (D^2 - a^2)W - \Psi = 0, \\ (D^2 - a^2)\Psi - R(1 + \varepsilon h(z))a^2 \Theta = 0, \\ (D^2 - a^2)\Theta + RW = 0, \end{cases} \tag{8}$$

with the homogeneous Dirichlet boundary conditions

$$W = \Psi = \Theta = 0 \quad \text{at } z = 0, 1. \tag{9}$$

This means a generalized eigenvalue problem.

It is worth noting at this moment that the eigenvalue problem (4)–(6) can be rewritten more compactly as a sixth order differential equation

$$(D^2 - a^2)^3 W = -Ra(1 + \varepsilon h(z))a^2 W, \tag{10}$$

supplied with the Dirichlet and hinged boundary conditions, namely

$$W = D^2 W = 0 \quad \text{at } z = 0, 1. \tag{11}$$

The new variable Ψ introduced by the transformation (7) reduced this problem to a Dirichlet problem, i.e., (8)–(9). The mathematical formalism requires one more boundary condition to be added to the boundary conditions (11) in order to close the eigenvalue problem (10)–(11). We introduce the condition $DW = 0$ at $z = 0, 1$ and observe that with usual variational arguments it is simply to show that Ra is indeed positive.

It is fairly important to underline why we integrate numerically the problem (8)–(9) instead of the problem (10)–(11). The main reason is the rapid worsening (increasing) of the condition number of the discretization matrices as both the order of differentiation and the spectral parameter N (number of terms in the spectral series) get larger. For the Chebyshev differentiation matrices, the fact is illustrated in Fig. 1.

3. The Viola's eigenvalue problem

As we do not have theoretical estimations for the spectrum of our hydrodynamic stability problem we will use the following eigenvalue problem as a benchmark one. More than that we try to make more explicit the reason we prefer to transform high order eigenvalue problems in systems of differential equations containing only second order derivatives, i.e., "D^2" strategy.

The Viola eigenvalue problem reads (see Straughan [27] p. 218 and the works quoted there)

$$\begin{cases} D^2[(1 - \theta x)^3 D^2 u] - \lambda(1 - \theta x)u = 0, & x \in (0, 1), \\ u = D^2 u = 0, & x = 0, 1, \end{cases} \tag{12}$$

where the parameter θ satisfies $0 \leqslant \theta < 1$. The problem becomes singular as $\theta \to 1^-$.

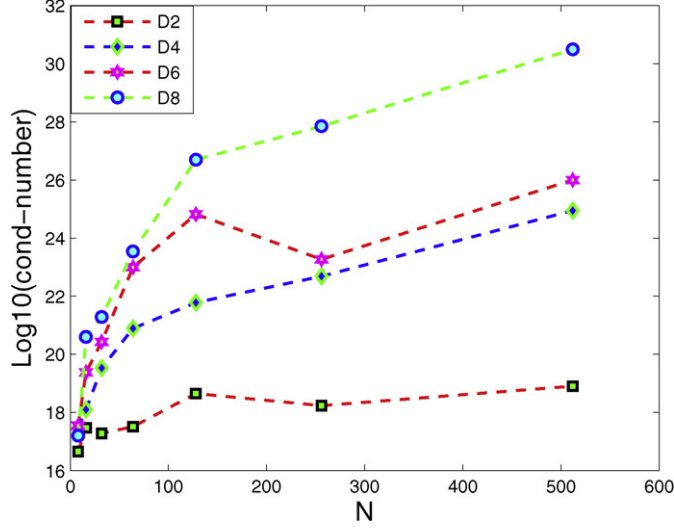With a change of variable analogous to (7), i.e.,

$$v := (1 - \theta x)^3 D^2 u,$$

Fig. 1. Logarithm of the condition number of the Chebyshev differentiation matrices D^2, D^4, D^6 and D^8, versus N.

the problem (12) reduces to an eigenvalue problem for the pair $(u\ v)^T$ containing only Dirichlet boundary conditions, namely

$$\begin{cases} (1-\theta x)^3 D^2 u - v = 0, & x \in (0,1), \\ D^2 v = \lambda(1-\theta x)u, \\ u = v = 0, & x = 0, 1. \end{cases}$$

As we make use of Chebyshev polynomials, the linear transformation of independent variables $z := 2x - 1$ shifts our problem on the close interval $[-1,\ 1]$. To go further with our method we solve this eigenvalue problem by the Chebyshev collocation method. This method transforms the problem into the following generalized algebraic eigenvalue problem

$$\mathbf{A} \cdot \mathbf{w} = \lambda \mathbf{B} \cdot \mathbf{w}, \tag{13}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \begin{pmatrix} (\mathbf{Dx})^3 \cdot \mathbf{D2} & -\mathbf{I} \\ \mathbf{Z} & \mathbf{D2} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Dx} & \mathbf{Z} \end{pmatrix}, \tag{14}$$

and the submatrices $\mathbf{D2}$, $\mathbf{Dx}$, $\mathbf{I}$ and $\mathbf{Z}$ stand respectively for the second order differentiation matrix on the Chebyshev nodes of the second kind $\mathbf{x} := \{x_1, \ldots, x_{N-1}\}$, $\mathbf{diag}(1 - \theta\mathbf{x})$, the unitary matrix of order $N - 1$ and the matrix with all zero entries of order $N - 1$. It is important to underline that in the differentiation matrix $\mathbf{D2}$ the boundary conditions are enforced, thus the rows and columns corresponding to the first node x_0 and the last node x_N are deleted. The vector $\mathbf{w}$ contains the values of the approximations of u and v in the nodes of $\mathbf{x}$, i.e., $\mathbf{w} := (u_1, \ldots, u_{N-1}, v_1, \ldots, v_{N-1})^T$. All the differentiation matrices we have used (`chebdif, cheb2bc`) come from the work of Weideman and Reddy [30]. The problem (13) was solved using the MATLAB code `eig`. Our results confirm up to some point those of Straughan [27]. Apart from those we conjecture that the first eigenvalue λ_1 approaches 1 as $\theta \to 1^-$! It is sustained by the numerical results reported in Fig. 2. They were obtained when $N = 512$. Then, corresponding to $\theta = 1 - (1.0e{-}06)$ we got $\lambda_1 = 1.054343$. More than that, for $N = 600$ and the same θ we obtained $\lambda_1 = 1.035803$. We have to observe that all the computations involved by the collocation method were performed in full precision (64 bit arithmetic).

An inconvenience of our approach consists in the fact that corresponding to all zero rows in the matrix $\mathbf{B}$ the eigenvalues equal infinity (the rank of matrix $\mathbf{B}$ equals $N - 1$). The MATLAB code `eig`, which implements the `qz` algorithm and gives access to intermediate results in the computation of generalized eigenvalues, explains this phenomenon. In fact the left generalized eigenvalues for those zero rows are zero.
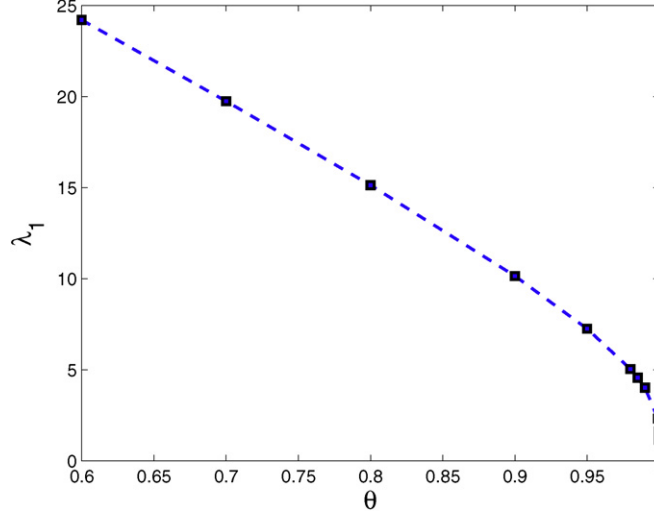
Fig. 2. The behavior of the eigenvalue λ_1 as θ approaches 1^-.

Table 1
The first two eigenvalues and their sensitivities

θ	λ_1	$\mathbf{S}^{(1)}$	$\mathbf{S1}^{(1)}$	λ_2	$\mathbf{S}^{(2)}$	$\mathbf{S1}^{(2)}$
0.5	28.615798	1.88	1.44	517.93	7.66	1.55
$1 - (1.0\text{e}{-}6)$	1.054343	56.42	$16.11\text{e}+05$	130.72	42.54	$16.11\text{e}+05$

As a further check on accuracy, we computed two types of sensitivities defined as

$$S^{(i)} := \frac{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|}{\mathbf{x}_i^H \mathbf{y}_i}, \qquad S1^{(i)} := \frac{\|\mathbf{x}_i\|\,\|\mathbf{y}_i\|}{\sqrt{|\mathbf{y}_i^H \mathbf{A}\mathbf{x}_i|^2 + |\mathbf{y}_i^H \mathbf{B}\mathbf{x}_i|^2}}, \tag{15}$$

for the ith eigenvalue where $\mathbf{x}_i$ and $\mathbf{y}_i$ are the right and left eigenvectors of (13), respectively. They were introduced by Stewart and Sun [26] and used in an analogous problem by Dongarra et al. [5]. Corresponding to Chebyshev collocation method applied to problem (13) with $N = 512$ we summarize the situation in Table 1. It is fairly clear from this table that both measures of sensitivities increase drastically with the parameter θ increasing to 1. In other words, the more singular is the differential problem the more sensitive are the eigenvalues. Since the quantity $\log_{10} S^{(i)}$ is a measure of the number of digits one lose in accuracy this table provide useful information on the accuracy of the actual computations. It is also clear that the first eigenvalue is less sensitive than the second.

A similar conclusion provides Fig. 3. As the curves $\log_{10} \text{cond}(A)$ versus θ, for $N = 256$ and $N = 512$, respectively, are quite parallel and horizontal for $0.6 \leqslant \theta < 0.95$ they sharply increase for $\theta \to 1^-$. An attempt was made to solve the problem (12) as a fourth order one. In order to fulfill the boundary conditions we use this time the more complicated weighted functions

$$(1 - x^2)^3 T_i(x), \quad i = 0, 1, 2, \ldots, N,$$

and in order to obtain the corresponding differentiation matrices we make use of the code `poldif` from the same paper of Weideman and Reddy [30]. In this situation the right-hand side matrix of the generalized eigenvalue problem is a diagonal one and consequently it is easy invertible. This way we are in the situation to solve an usual eigenvalue problem. For values of N around 128, the MATLAB code `qz` as well as the MATLAB code `invshift` (see Quarteroni and Saleri [23] p. 175) which evaluates the eigenvalue of minimum modulus of a matrix by the inverse power method, provide positive values for λ_1. However, they are far larger than those obtained above. For N beyond 128 the situation is even worse. The eigenvalue of minimum modulus becomes negative and also complex eigenvalues appear in the spectrum. This means numerical instability.
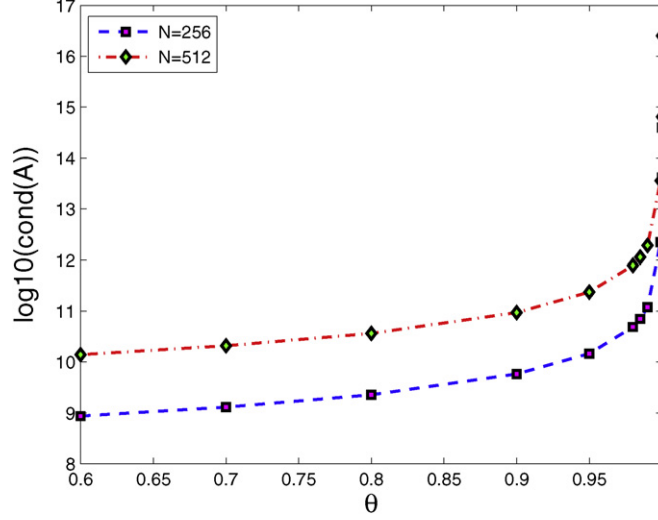
Fig. 3. The conditioning of matrix A with respect to N and θ.

4. Spectral methods for hydrodynamic stability problem (4)–(6)

Actually we look for the smallest eigenvalue Ra (the Rayleigh number) in the problem (8)–(9) defining the neutral manifold.

In order to accomplish that, we reduce the problem to a generalized algebraic eigenvalue one, first by a weighted residual (Galerkin) type method. To this end we formulate the eigenvalue problem in an operatorial framework by introducing the vectorial unknown $\mathbf{U}$ as $\mathbf{U} := (W \ \Psi \ \Theta)^T$, the operator $\mathbf{L} : X \to X$, and the functional space $X := H_w^2(-1, \ 1) \times H_w^2(-1, \ 1) \times H_w^2(-1, \ 1)$ where

$$\mathbf{L} := \mathbf{L}_1 + R \cdot \mathbf{L}_2,$$

and

$$\mathbf{L}_1 := \begin{pmatrix} (4D^2 - a^2) & -I & O \\ O & (4D^2 - a^2) & O \\ O & O & (4D^2 - a^2) \end{pmatrix},$$

$$\mathbf{L}_2 := \begin{pmatrix} O & O & O \\ O & O & -(1 + \varepsilon h(x))a^2 \\ I & O & O \end{pmatrix},$$

I and O stand for the identity and null operator, respectively. The space $H_w^p(-1, \ 1)$, $p \geqslant 0$, is a subspace of the Sobolev space $H^p(-1, \ 1)$. Endowed with the inner product $(\cdot, \cdot)_{p,w}$, defined as usual by

$$(u, v)_{p,w} := \sum_{k=0}^{p} \int_{-1}^{1} \frac{d^k u(x)}{dx^p} \frac{d^k v(x)}{dx^p} w(x) \, dx,$$

with the weight $w(x) := 1/\sqrt{1 - x^2}$, it is a Hilbert space. When $p = 0$ we suppress this index from the definition of the scalar product.

It is clear that $\mathbf{L}_1$ is invertible and $\mathbf{L}_2$ does not have this property.

As in the case of Viola's problem we first translate the problem on the interval $[-1, 1]$.

Consequently, our problem reads:

find Ra and $\mathbf{U} \in X \setminus \{0\}$ such that

$$\begin{cases} \mathbf{L}_1(\mathbf{U}) + R \cdot \mathbf{L}_2(\mathbf{U}) = 0, \\ \mathbf{U} = \mathbf{0} \quad \text{at } x = \pm 1. \end{cases} \tag{16}$$

Various numerical methods can be used in order to solve the problem (16). In the monographs [2], [3], [18] or [28] to quote but a few, details on the spectral methods are provided including the approximation results in the weighted Sobolev spaces $H_w^p(-1, 1)$, $p \geqslant 0$. In our recent monograph [12] we tried to collect some classical and at the same time difficult eigenvalue problems and compare the results obtained by spectral methods with their exact solutions or at least other numerical results (see [22] for such a collection).

Let us consider now the following two finite dimensional spaces X_M, Y_M, both subspaces of X, made up by functions which satisfy the Dirichlet boundary conditions $\mathbf{U} = 0$ at $x = \pm 1$. In our case an approximation $\mathbf{U}_M \in X_M \setminus \{0\}$ of the solution $\mathbf{U} \in X$ has the form $\mathbf{U}_M = (W_N \ \Psi_N \ \Theta_N)^T$, with the N index related to the discretization process. As usual, W_N, Ψ_N, Θ_N are defined as truncated series of orthogonal functions (trial functions)

$$W_N = \sum_{k=0}^{N} \widehat{W}_k e_k^W(x), \qquad \Psi_N = \sum_{k=0}^{N} \widehat{\Psi}_k e_k^\Psi(x), \qquad \Theta_N = \sum_{k=0}^{N} \widehat{\Theta}_k e_k^\Theta(x). \tag{17}$$

In (17) the coefficients $\widehat{W}_k$, $\widehat{\Psi}_k$, $\widehat{\Theta}_k$, $k = 0, 1, 2, \ldots, N$, are unknown and the sets $\{e_k^W(x)\}_{k=0,1,\ldots,N}$, $\{e_k^\Psi(x)\}_{k=0,1,\ldots,N}$, $\{e_k^\Theta(x)\}_{k=0,1,\ldots,N}$, are the sets of trial functions. This implies the equality $M = 3(N + 1)$. The spectral approximation $\mathbf{U}_M$ is obtained by imposing the vanishing of the projection the residual on the finite dimensional space Y_M, subspace of X, so a general weighted residual formulation for the problem (16) reads:

find Ra and $\mathbf{U}_M \in X_M \setminus \{0\}$ such that

$$\left(\mathbf{L}_1(\mathbf{U}_M), \mathbf{V}_M\right)_w + R \cdot \left(\mathbf{L}_2(\mathbf{U}_M), \mathbf{V}_M\right)_w = 0, \quad \forall \mathbf{V}_M \in Y_M. \tag{18}$$

Whenever we perform an integration by parts of D^2 in the above formulation, in order to lower the order of differentiation, the formulation is called week and the method becomes a Galerkin–Petrov one. When the space of trial functions X_M and that of test functions Y_M are identical we come up with the well-known Galerkin method.

In [9] the analytical study of the eigenvalue problem (8)–(9) was performed by using complete sets of orthogonal functions directly on [0, 1], i.e. shifted Chebyshev and Legendre polynomials.

For the particular varying gravity field $h(x) = -(2x - 1)$ we solve the problem (18) using several basis in the test and trial spaces.

(i) **Weighted residuals Galerkin methods (WRG).** We considered two family of functions as basis of the finite dimensional spaces X_M, Y_M. They have the form

$$\{F_j\}_{j=1,2} = \{S_{jk} \times S_{jk} \times S_{jk}\}_{j=1,2; \ k=0,1,\ldots,N}$$

with the Shen's basis $\{S_{1k}\}_{k=0,1,\ldots,N}$, denoted (SB1) and defined by (see [25])

$$S_{1k}(x) := T_k(x) - T_{k+2}(x), \quad k = 0, 1, \ldots, N, \tag{19}$$

and $\{S_{2k}\}_{k=0,1,\ldots,N}$, which is another Shen's basis, denoted this time (SB2) and having the form

$$S_{2k}(x) := T_k(x) - \frac{2(k+2)}{k+3} T_{k+2}(x) + \frac{k+1}{k+3} T_{k+4}(x), \quad k = 0, 1, \ldots, N. \tag{20}$$

The functions from both Shen's basis satisfy boundary conditions of the form

$$S_{jk}(\pm 1) = 0, \quad j = 1, 2; \ k = 0, 1, \ldots, N.$$

Then, in each case, the unknowns W, Ψ, Θ are approximated respectively by

$$W_N = \sum_{k=0}^{N} S_{jk} \widehat{W}_k, \qquad \Psi_N = \sum_{k=0}^{N} S_{jk} \widehat{\Psi}_k, \qquad \Theta_N = \sum_{k=0}^{N} S_{jk} \widehat{\Theta}_k, \quad j = 1, 2.$$

This means that we specify the sets of the trial functions as

$$e_k^W = e_k^\Psi = e_k^\Theta := S_{1k}, \quad k = 0, 1, \ldots, N,$$

for the first variant of the method, and

$$e_k^W = e_k^\Psi = e_k^\Theta := S_{2k}, \quad k = 0, 1, \ldots, N,$$

for the second variant of the method.

Let us introduce now the vector of the unknown coefficients $\widehat{W}_k$, $\widehat{\Psi}_k$, $\widehat{\Theta}_k$ by

$$\mathbf{U}_M^{\text{coef}} := (\widehat{W}_0 \ \widehat{W}_1 \ldots \widehat{W}_N \ \widehat{\Psi}_0 \ldots \widehat{\Psi}_N \ \widehat{\Theta}_0 \ldots \widehat{\Theta}_N)^T.$$

With this notation the equation (18) becomes

$$\mathbf{L} \cdot \mathbf{U}_M^{\text{coef}} = \mathbf{0}, \tag{21}$$

where the matrix $\mathbf{L}$ is defined by

$$\mathbf{L} := (\mathbf{L}_1 S_{jk}, S_{ji})_w + R \cdot (\mathbf{L}_2 S_{jk}, S_{ji})_w, \quad j = 1, 2; \ k, i = 0, 1, \ldots, N,$$

which means a homogeneous linear algebraic system of $3(N+1)$ equations and $3(N+1)$ unknowns.

For the Galerkin method the secular equation has the form

$$\begin{vmatrix} ((4D^2 - a^2)S_{ji}, S_{jk})_w & -\mathbf{I} & \mathbf{Z} \\ \mathbf{Z} & ((4D^2 - a^2)S_{ji}, S_{jk})_w & -R((1 + \varepsilon h(x))S_{ji}, S_{jk})_w \\ R \cdot \mathbf{I} & \mathbf{Z} & ((4D^2 - a^2)S_{ji}, S_{jk})_w \end{vmatrix} = 0,$$

$$i, k = 0, 1, \ldots, N \quad \text{for } j = 1, 2. \tag{22}$$

$\mathbf{I}$ and $\mathbf{Z}$ stand here for the identity and zero matrices, respectively, both of order $N + 1$.

All the scalar products from (22) are based on the properties of Chebyshev polynomials (see for instance [18]), i.e.,

$$(T_k, T_i)_w = \begin{cases} \dfrac{\pi}{2} c_k \delta_{ki}, & \text{if } k = i, \\ 0, & \text{otherwise,} \end{cases}$$

$$(T_k'', T_i)_w = \begin{cases} k(k^2 - i^2)\dfrac{\pi}{2} c_i, & \text{if } k = j + 2, j + 4, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

$$(xT_k, T_i)_w = \begin{cases} \dfrac{\pi}{4} c_i, & \text{if } i = k - 1, k - 3, \\ 0, & \text{otherwise,} \end{cases}$$

with

$$c_i = \begin{cases} 2, & \text{if } i = 0, \\ 1, & \text{if } i \geqslant 1. \end{cases}$$

Eventually, for the Galerkin method the secular equation becomes

$$\begin{vmatrix} ((4D - a^2)S_{ji}, D(S_{jk}))_w & -\mathbf{I} & \mathbf{Z} \\ \mathbf{Z} & ((4D - a^2)S_{ji}, D(S_{jk}))_w & -R((1 + \varepsilon h(x))S_{ji}, S_{jk})_w \\ R \cdot \mathbf{I} & \mathbf{Z} & ((4D - a^2)S_{ji}, D(S_{jk}))_w \end{vmatrix} = 0,$$

$$i, k = 0, 1, \ldots, N \quad \text{for } j = 1, 2. \tag{23}$$

(ii) **Weighted residuals Petrov–Galerkin methods (WRPG).** We considered two variants. At first, in the method (WRPG1), the set of trial functions is of the form $\{\Psi_k \times \Psi_k \times \Psi_k\}_{k=0,1,\ldots,N} \in X_M$ with the Heinrich's basis $\{\Psi_k\}_{k=0,1,\ldots,N}$ [14] defined as

$$\Psi_k(x) := (1 - x^2)T_k(x), \quad k = 0, 1, \ldots, N \tag{24}$$

and the set of test functions is F_1 defined as before. For the second variant (WRPG2) we choose the same set of trial functions as for (WRPG1) and as a set of test functions we choose F_2.

The secular equation is obtained the same way as in the (WRG) method.

The numerical scheme for the generation of the spectrum in the case of weighted residual methods based on trigonometric and polynomial expansions is a Maple code which implements the LU decomposition.

(iii) **Collocation methods.** The Chebyshev collocation method, (CC) for short, reduces the eigenvalue problem (8)–(9) to an algebraic problem of the form (13) where the matrices $\mathbf{A}$ and $\mathbf{B}$ are now

$$\mathbf{A} = \begin{pmatrix} 4\mathbf{D2} - a^2\mathbf{I} & -\mathbf{I} & \mathbf{Z} \\ \mathbf{Z} & 4\mathbf{D2} - a^2\mathbf{I} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & 4\mathbf{D2} - a^2\mathbf{I} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{I} - \varepsilon\mathbf{Dx} \\ -\mathbf{I} & \mathbf{Z} & \mathbf{Z} \end{pmatrix},$$

Table 2
Numerical values for the Rayleigh number for various values of the parameters for $h(x) = -(2x - 1)$ obtained in [6,9]

ε	a^2	Ra: SCP	Ra: SLP	Ra: TS
0.0	4.92	657.512	675.05	657.51
0.01	4.92	660.747	678.45	660.81
0.03	4.92	667.653	685.33	667.52
0.33	4.92	787.363	808.303	787.28
0.2	5.00	730.459	749.95	730.56
0.2	9.00	829.44	846.70	829.39
0.5	7.5	930.982	952.07	930.92
0.5	9.00	994.393	1015.27	994.47
0.75	10.0	1251.178	1276.05	1251.09

Table 3
Numerical values for the Rayleigh number for various values of the parameters for $h(x) = -(2x - 1)$ obtained in this paper

ε	a^2	Ra: WRG(SB1)	Ra: WRG(SB2)	Ra: CC
0.0	4.92	658.54	714.55	657.5133
0.01	4.92	661.88	718.62	660.8173
0.03	4.92	668.53	725.49	667.5254
0.33	4.92	792.54	859.84	787.2880
0.2	5.00	732.95	794.85	730.5647
0.2	9.00	832.09	886.91	829.3918
0.5	7.5	946.05	1013.30	930.9239
0.5	9.00	1010.50	1077.01	994.4721
0.75	10.0	1313.80	1392.61	1251.0924

with the same significance for matrices **D2**, **I** and **Z** as in Section 3. The matrix **Dx** is the diagonal matrix $\mathbf{Dx} = \mathrm{diag}((\mathbf{x} + 1)/2)$.

The numerical results obtained for various values of ε and a^2 with $N = 256$ are reported in the last column of Table 3. No improvement is observed when $N = 512$. They were carried out using the same MATLAB code `eig`.

With respect to the sensitivity defined by (15) we remark that this problem is less sensitive than the singularly perturbed Viola's problem when $\theta \to 1^-$. For the first indicator we obtained $S^{(1)} = 5.789210$ which increases to $S^{(2)} = 15.846905$. This means again that the first eigenvalue is less sensitive than the subsequent ones. The second indicator $S1^{(i)}$ slightly oscillates around 0.59. It is important that they remain practically independent of N.

5. Numerical results

Numerical evaluations of the Rayleigh number for the particular case in which the gravity variation is $\varepsilon h(x) = -\varepsilon(2x - 1)$ are given. For comparison, we present in Table 2, our previous evaluations of the Rayleigh number from [6,9]. These evaluations are obtained by spectral methods based on shifted Chebyshev polynomials (SCP), shifted Legendre polynomials (SLP) and trigonometric series (TS).

The numerical results obtained in this paper by the weighted residual Galerkin methods (WRG) are displayed in Table 3. The third column corresponds to the first Shen's basis (SB1) and the fourth column to the second Shen's basis (SB2). The evaluations obtained by the Chebyshev collocation method (CC) are presented in the last column of the same table. It is fairly clear that the collocation method and the spectral methods reported in Table 2, i.e., (SCP) and (TS) lead to very close results. The worse situation is with Galerkin method based on (SB2). However, the results gathered in this two tables underline the same physical conclusion that a linear decreasing variable gravity field enlarges the domain of stability.

Then numerical evaluations of Ra, obtained for the two variants of the weighted residual Petrov–Galerkin method (WRPG1 and WRPG2), are displayed in Table 4. These numerical results are similar but quite inaccurate when compared with the collocation results. However, the use of the (WRPG) method simplifies the numerical evaluations and reduces the time of computation due to the fact that a larger number of scalar products in the secular equation vanish.

Table 4

Numerical values for the Rayleigh number for various values of the parameters for $h(x) = -(2x - 1)$ obtained by the WRPG method

ε	a^2	Ra: WRPG1	Ra: WRPG2
0.0	4.92	658.59	662.80
0.01	4.92	662.03	666.05
0.03	4.92	668.84	673.19
0.33	4.92	792.70	797.38
0.2	5.00	733.06	737.45
0.2	9.00	831.80	836.02
0.5	7.5	946.05	950.55
0.5	9.00	1010.60	1015.20
0.75	10.0	1314	1319.70

Table 5

Numerical evaluations of the Rayleigh number Ra versus spectral parameter N and CPU time

Ra	N	Time (seconds)
658.692	4	16.66
657.6668	5	41.8
657.5219	7	386.12
657.5129	9	2679.83
657.5127	11	17834.15

All in all, by comparing Tables 2, 3 and 4 it is fairly clear that the Chebyshev collocation method and the Galerkin method based on trigonometric polynomials provide the closest results.

It is also important to compare the two classes of methods with respect to the computational cost. First, for the same N, all the weighted residual methods (WRG and WRPG) lead to larger algebraic problems than the collocation methods, i.e., the two matrices involved in Eq. (23) have each the size $3(N + 1)$ but the matrices A and B of the generalized eigenvalue problem corresponding to (CC) method have each size $3(N + 1)$. Consequently, the qz step is more expensive for the first class of methods. Second, and more important, whenever the tedious computations of the entries in Eq. (23) are done symbolically, the cost of the entire residual method increases more than exponentially with N. This fact is illustrated in Table 5 where the values of the Rayleigh number for the classical case [4], i.e. $\varepsilon = 0$, $a^2 = 4.92$, obtained by (WRG) with the first Shen's basis $\{S_{1k}\}_{k=0,1,\ldots,N}$ are displayed. We can infer from this table that the computation of the matrix entries dominates all other costs.

In spite of this disadvantage, we can conclude that for a relatively small N, i.e., $N = 12$, fairly accurate numerical results are obtained using these methods, i.e., (SCP) and (TS). However, for larger N the computational time highly increases, without leading to a significant improvement of the numerical values (less than 10^{-5}).

A better situation is observed with respect to the collocation method. The spectral order N used is much more large than that used in the weighted residual methods, i.e., N is of order 2^m with $5 \leqslant m \leqslant 10$. Thus, to compare the computational cost of the two methods for large N is useless. Anyway, our numerical experiments showed that the collocation methods are much more rapid (the CPU time increases moderately with N) and more flexible with respect to the implementation process, i.e., quite independent of the problem at hand.

It is also important to notice that, both classes of the methods do not require assumptions on the scale resolution, i.e., that Ra/N^2 is "small", as it is the case with the methods considered by Melenk et al. [19]. Eventually, in Fig. 4 graphical representations of the neutral curves for different values of the scale parameter are given, pointing out its influence on the stability domain.

6. Conclusions

In this paper a numerical study of a linear hydrodynamic stability problem of a convective flow in the presence of a varying gravity field is carried out. The sixth order eigenvalue problem with Dirichlet and hinged boundary conditions is cast into a second order eigenvalue problem supplied only with Dirichlet boundary conditions. It is solved by some weighted residual methods (Galerkin type) methods as well as by a collocation method. The algorithm is fairly
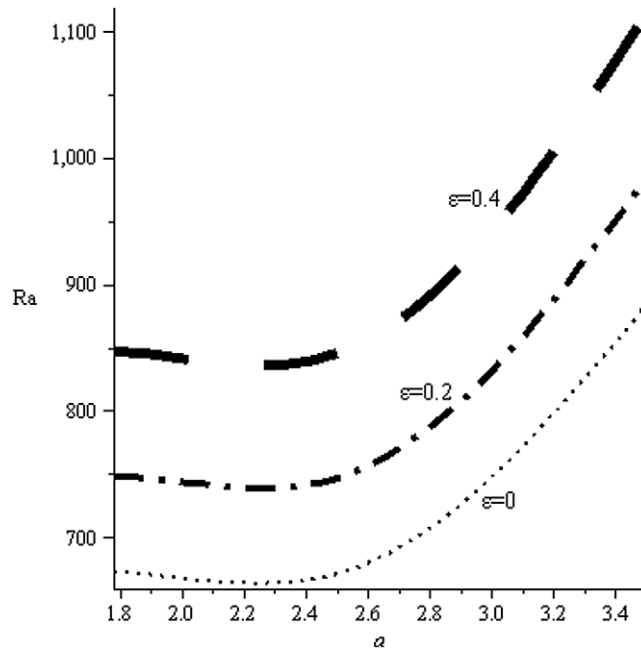
Fig. 4. Neutral curves for various values of the physical parameters.

general and its Chebyshev collocation implementation very fast and flexible. Formally this implementation produces some spurious eigenvalues but this fact is explicable and expected. However, the lower part of the spectrum, which is highly important, is accurately approximated. More than that, no scale resolution restrictions are required.

In spite of the superior cost in computing the generalized algebraic eigenvalues, our "D^2" strategy is plainly superior with respect to accuracy when compared with the same numerical methods applied to the direct formulation.

The Chebyshev collocation version of our method was validated in solving the Viola's eigenvalue problem. It is a singularly perturbed eigenvalue problem with high sensitive eigenvalues. About this problem we conjecture that its first eigenvalue $\lambda_1 \to 1$ as $\theta \to 1^-$.

Acknowledgements

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