A Mixed Variational Formulation of a Contact Problem with Wear

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Abstract We consider a mathematical model which describes the sliding frictional contact between a viscoplastic body and an obstacle, the so-called foundation. The process is quasistatic, the material's behavior is described with a viscoplastic constitutive law with internal state variable and the contact is modelled with normal compliance and unilateral constraint. The wear of the contact surfaces is taken into account, and is modelled with a version of Archard's law. We derive a mixed variational formulation of the problem which involve implicit history-dependent operators. Then, we prove the unique weak solvability of the contact model. The proof is based on a fixed point argument proved in Sofonea et al. (Commun. Pure Appl. Anal. 7:645–658, 2008), combined with a recent abstract existence and uniqueness result for mixed variational problems, obtained in Sofonea and Matei (J. Glob. Optim. 61:591–614, 2014).

Keywords viscoplastic material · Frictional contact · Normal compliance · Unilateral constraint · Wear · Mixed variational formulation · History-dependent operator · Weak solution

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1 Introduction

The mathematical modelling of contact phenomena is rather complex and, usually, leads to strongly nonlinear boundary value problems. The reason arise in the fact that, as shown in [6, 7, 13, 23, 24, 29, 31], accurate mathematical models need to take into consideration the additional phenomena involved in contact processes. These phenomena are the friction, the heat generation, the wear and the adhesion of contacting surfaces, among others. Wear

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is defined as the material loss or change in surface texture occurring when two surfaces of mechanical components contact each other. As the contact process evolves, the contacting surfaces evolve too, via their wear. Wear in sliding systems is often very slow but it is persisting, continuous and cumulative. Its characterization represents one of the basic tasks in the study of machine elements. Indeed, in the process of design of machine elements and tools operating in contact conditions, engineers need to know areas of contact, contact stresses, and they need to predict wear of rubbing elements.

Wear of contact surfaces represents a complex phenomenon. Following [10, 25], it is customary to distinguish among the following wear types: adhesive, abrasive, contact fatigue, freeting, oxidation, corrosion and erosion. In terms of the severity of wear on the wearing surfaces, two broad types of wear phenomena have been mentionned in [1]: severe wear and mild wear. Severe wear is characterized by high wear rates, extensive plastic deformation, transfer of material to the harder counter face, and flake-like metallic wear debris. Mild wear, by contrast, is characterized by low wear rates, minimal plastic deformation, formation of a surface film protecting against metal-to-metal contact, and oxide wear debris.

Due to its crucial role in various technological and biomechanical processes, the wear phenomenon subjects of numerous experimental and theoretical studies. For instance, the evolution of wear gaps in fretting problems was studied numerically in [35], by using the finite element method. Numerical simulations of wear shapes due to pitting phenomena for various operating conditions have been investigated in [9], by using arguments of fracture mechanics. A thermoelastic wheel-rail contact problem with wear has been studied in [4]. Numerical methods for wear problems with application to implanted knee joints has been developed in [28]. An original analytical approach to wear was performed in [10]. General models for frictional contact with wear could be find in [36, 37] as well as in the survey [38]. The mathematical analysis of various models of frictional contact with wear, including existence and uniqueness results of the weak solution, was carried out in [11, 12, 26–29].

A new mathematical model which describes the equilibrium of an elastic body in frictional contact with a moving foundation was recently considered in [34]. There, the contact was modeled with a normal compliance condition with unilateral constraint associated to a sliding version of Coulomb's law of dry friction, and the wear of the foundation was described with a version of Archard's law. A variational formulation of the problem was derived, in a form of the system which couples a time-dependent equation for the stress field, a time-dependent variational inequality for the displacement field and an integral equation for the wear function. The unique weak solvability of the model was proved, by using arguments on time-dependent variational inequalities and fixed point. This result was completed with a convergence result which shows that the solution of a penalized frictional contact problem with wear converges to the solution of the contact model, as the penalization parameter converges to zero.

The current paper represents a continuation of [34] and contains two main novelties. The first one concerns the mathematical model since, in contrast with [34], we consider here that the deformable body is viscoplastic and we model its behavior with a viscoplastic constitutive law with internal state variable. The analysis of this model could be carried out by using arguments similar to those used in [34], with a different choice of spaces and operators. Nevertheless, we choose to present here a different approach, which consists the second novelty of this paper. Thus, in contrast with [34], we derive a mixed variational formulation of the problem in which the unknowns are the stress field, the displacement field, the internal state variable, the wear function and the Lagrange multiplier, then we prove its unique solvability by using a recent abstract existence result in the study of mixed variational problems, proved in [32]. Mixed variational problems involving Lagrange multipliers have been used both in analysis and mechanics, in the study of minimization problems. They provide a useful framework in which a large number of problems involving unilateral constraints can be cast and can be solved numerically. Their study is based on arguments on duality, saddle points theory and fixed point. The literature in the field is extensive, see for instance [3, 8, 14, 18] and the references therein. The analysis of various mixed variational problems associated to contact models can be found in [15, 16, 19–21], for instance.

The rest of the manuscript is structured as follows. In Sect. 2 we present the notation and some preliminary material, including a new abstract result, Thereom 2.2. In Sect. 3 we introduce the model of sliding frictional contact with wear and list the assumption on the data. Then, in Sect. 4 we derive its mixed variational formulation. In Sect. 5 we state and prove our main existence and uniqueness result, Theorem 5.1, which provides the unique solvability of the viscoplastic contact problem with wear. Finally, in Sect. 6 we present relevant particular cases of our contact model and we comment on the corresponding existence and uniqueness results. We also provide a comparison between the fixed point method used in [34] and the Lagrange multiplier method used in the current paper. At the best of our knowledge, these two methods represent the main functional methods used in the study of contact problems with unilateral constraints.

2 Notations and Preliminaries

Everywhere in this paper we use the notation $\mathbb{R}_+$ for the set of positive real numbers and $\mathbb{N}$ for the set of positive integers. Given two sets X and Y we use the notation $X \times Y$ for their cartesian product and, (x, y) will represent a typical point of the set $X \times Y$. All the vector spaces considered below are real vector spaces and, for a vector space X, we use the notation 0_X for the zero element of X. In addition, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces, then $\|\cdot\|_{X\times Y}$ represents the norm of the space $X \times Y$ given by

$$\|z\|_{X\times Y} = \|x\|_X + \|y\|_Y \quad \forall\, z = (x, y) \in X \times Y.$$

We use similar notation for the product of more than two sets or spaces. For a normed space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X and, for a subset $K \subset X$, we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K.

Now, assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and $\mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$. Then, we recall that the operator $\mathcal{S}$ is called a history-dependent operator if the following property holds:

$$\begin{cases} \text{For each } n \in \mathbb{N} \text{ there exists } s_n \geq 0 \text{ such that} \\ \left\|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\right\|_Y \leq s_n \displaystyle\int_0^t \left\|u_1(s) - u_2(s)\right\|_X ds \\ \quad \forall\, u_1, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{cases} \tag{2.1}$$

Note that in (2.1) and everywhere below the notation $\mathcal{S}\eta(t)$ represents the value of the function $\mathcal{S}\eta$ at the point t, i.e. $\mathcal{S}\eta(t) = (\mathcal{S}\eta)(t)$. The notion of history-dependent operator was introduced in [30] and used in a number of papers, see for instance [31] and the references therein. Such kind of operators arise both in Functional Analysis, Theory of Partial Differential Equations and Solid Mechanics, as well. One of their main properties is given by the following fixed point result.

Theorem 2.1 *Let* $(X, \|\cdot\|_X)$ *be a Banach space and let* $\mathcal{S}\colon C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ *be a history-dependent operator. Then the operator* $\mathcal{S}$ *has a unique fixed point* $\eta^* \in C(\mathbb{R}_+; X)$.

Note that Theorem 2.1 represents a particular case of a more general result proved in [33]. Its proof is based on the fact that, if X is a Banach space, then $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms.

We turn now to an abstract result which represents a consequence of Theorem 2.1 and which will be used twice in Sect. 5 of this manuscript. Thus, we assume in what follows that $(X, \|\cdot\|_X)$ is a normed space, $(Y, \|\cdot\|_Y)$ is a Banach space and $A : X \to Y$ and $G : \mathbb{R}_+ \times X \times Y \to Y$ are given operators, which satisfy the following conditions:

$$\left\{\begin{array}{l}\text{There exists } L_A > 0 \text{ such that}\\ \|Ax_1 - Ax_2\|_Y \le L_A\|x_1 - x_2\|_X \quad \forall\, x_1, x_2 \in X.\end{array}\right. \tag{2.2}$$

$$\left\{\begin{array}{l}\text{(a) There exists } L_G > 0 \text{ such that}\\ \qquad \big\|G(t, x_1, y_1) - G(t, x_2, y_2)\big\|_Y \le L_G\big(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y\big)\\ \qquad\quad \forall\, x_1, x_2 \in X,\ \ y_1, y_2 \in Y,\ \ t \in \mathbb{R}_+\\ \text{(b) The mapping } t \mapsto G(t, x, y) \text{ is measurable on } \mathbb{R}_+,\\ \qquad\quad \text{for any } x \in X,\ \ y \in Y.\\ \text{(c) The mapping } t \mapsto G(t, 0_X, 0_Y) \text{ belongs to } L^\infty(\mathbb{R}_+).\end{array}\right. \tag{2.3}$$

Theorem 2.2 *Let* $(X, \|\cdot\|_X)$ *be a normed space,* $(Y, \|\cdot\|_Y)$ *a Banach space and assume that* (2.2)–(2.3) *hold. Then, there exists an operator* $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ *such that for all functions* $x \in C(\mathbb{R}_+; X)$ *and* $y \in C(\mathbb{R}_+; Y)$*, equality*

$$y(t) = Ax(t) + \int_0^t G\big(s, x(s), y(s)\big)\,ds \quad \forall\, t \in \mathbb{R}_+ \tag{2.4}$$

holds if and only if

$$y(t) = Ax(t) + \mathcal{S}x(t) \quad \forall\, t \in \mathbb{R}_+. \tag{2.5}$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; Y)$ *is a history-dependent operator.*

Proof Let $x \in C(\mathbb{R}_+; X)$ and consider the operator $\Lambda : C(\mathbb{R}_+; Y) \to C(\mathbb{R}_+; Y)$ defined by

$$\Lambda\tau(t) = \int_0^t G\big(s, x(s), Ax(s) + \tau(s)\big)\,ds, \tag{2.6}$$

for all $\tau \in C(\mathbb{R}_+; Y)$ and $t \in \mathbb{R}_+$. Note that, using the assumptions (2.2)–(2.3), it follows that the operator Λ is well defined. Moreover, it depends on x but, for simplicity, we do not indicate explicitly this dependence.

Let $\tau_1,\ \tau_2 \in C(\mathbb{R}_+; Y)$ and let $t \in \mathbb{R}_+$. Then, using definition (2.6) and assumption (2.3), we deduce that

$$\begin{aligned}\big\|\Lambda\tau_1(t) - \Lambda\tau_2(t)\big\|_Y &\le \int_0^t \big\|G\big(s, x(s), \tau_1(s) + Ax(s)\big) - G\big(s, x(s), \tau_2(s) + Ax(s)\big)\big\|_Y\,ds\\ &\le L_G \int_0^t \big\|\tau_1(s) - \tau_2(s)\big\|_Y\,ds.\end{aligned} \tag{2.7}$$

This inequality combined with Theorem 2.1 shows that the operator Λ has a unique fixed point in $C(\mathbb{R}_+; Y)$, denoted $\mathcal{S}x$. Moreover, combining (2.6) with equality $\Lambda(\mathcal{S}x) = \mathcal{S}x$ we deduce that $\mathcal{S}x$ is the unique element of the space $C(\mathbb{R}_+; Y)$, which satisfies

$$\mathcal{S}x(t) = \int_0^t G\big(s, x(s), Ax(s) + \mathcal{S}x(s)\big)\, ds \quad \forall\, t \in \mathbb{R}_+. \tag{2.8}$$

This implies the equivalence between equalities (2.4) and (2.5), for all functions $x \in C(\mathbb{R}_+; X)$ and $y \in C(\mathbb{R}_+; Y)$.

To proceed, let $x_1,\, x_2 \in C(\mathbb{R}_+; X)$, $n \in \mathbb{N}$ and $t \in [0, n]$. Then, using (2.8) and taking into account (2.2) and (2.3), we obtain that

$$\begin{aligned}
&\big\|\mathcal{S}x_1(t) - \mathcal{S}x_2(t)\big\|_Y \\
&\quad \le \int_0^t \big\|G\big(s, x_1(s), Ax_1(s) + \mathcal{S}x_1(s)\big) - G\big(s, x_2(s), Ax_2(s) + \mathcal{S}x_2(s)\big)\big\|_Y\, ds \\
&\quad \le L_G(L_A + 1) \int_0^t \big\|x_1(s) - x_2(s)\big\|_X\, ds + L_G \int_0^t \big\|\mathcal{S}x_1(s) - \mathcal{S}x_2(s)\big\|_Y\, ds.
\end{aligned}$$

Using now the Gronwall argument we deduce that

$$\big\|\mathcal{S}x_1(t) - \mathcal{S}x_2(t)\big\|_Y \le L_G(L_A + 1)e^{nL_G} \int_0^t \big\|x_1(s) - x_2(s)\big\|_X\, ds. \tag{2.9}$$

This inequality shows that (2.1) holds with $s_n = L_G(L_A + 1)e^{nL_G}$, which concludes the proof. □

Note that Theorem 2.2 is important since it underlies the history-dependence feature of the solution of the implicit integral equation (2.4). It will be usefull in the study of viscoplastic constitutive laws, as explained in Sect. 5.

Next, we recall an existence and uniqueness result for mixed variational problems. To this end, let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces and we consider two operators $A : X \to X$, $\tilde{\mathcal{S}} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$, a bilinear form $b : X \times Y \to \mathbb{R}$, two functions $f,\, h : \mathbb{R}_+ \to X$ and a set $\Lambda \subset Y$. We assume that the following conditions hold:

$$\left\{\begin{array}{l}
\text{(a) There exists } m_A > 0 \text{ such that} \\
\quad (Au_1 - Au_2, u_1 - u_2)_X \ge m_A\, \|u_1 - u_2\|_X^2 \quad \forall\, u_1, u_2 \in X. \\
\text{(b) There exists } L_A > 0 \text{ such that} \\
\quad \|Au_1 - Au_2\|_X \le L_A \|u_1 - u_2\|_X \quad \forall\, u_1, u_2 \in X.
\end{array}\right. \tag{2.10}$$

$$\left\{\begin{array}{l}
\text{For each } n \in \mathbb{N} \text{ there exists } \tilde{d}_n \ge 0 \text{ and } \tilde{s}_n \ge 0 \text{ such that} \\
\big\|\tilde{\mathcal{S}}u_1(t) - \tilde{\mathcal{S}}u_2(t)\big\|_X \le \tilde{d}_n \big\|u_1(t) - u_2(t)\big\|_X + \tilde{s}_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X\, ds \\
\quad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n].
\end{array}\right. \tag{2.11}$$

$$
\begin{cases}
b : X \times Y \to \mathbb{R} \text{ is a bilinear form such that} \\
\text{(a) There exists } M_b > 0 \text{ such that} \\
\qquad |b(v;\mu)| \le M_b \|v\|_X \|\mu\|_Y \quad \forall\, v \in X,\ \mu \in Y. \\
\text{(b) There exists } \alpha > 0 \text{ such that} \\
\qquad \inf_{\mu \in Y, \mu \neq 0_Y} \ \sup_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \ge \alpha.
\end{cases} \tag{2.12}
$$

$$
f \in C(\mathbb{R}_+; X), \qquad h \in C(\mathbb{R}_+; X). \tag{2.13}
$$

$$
\Lambda \text{ is a closed convex unbounded subset of } Y \text{ that contains } 0_Y. \tag{2.14}
$$

With these data we introduce the following evolutionary problem.

Problem 2.3 *Find the functions $u : \mathbb{R}_+ \to X$ and $\lambda : \mathbb{R}_+ \to \Lambda$ such that*

$$
\big(Au(t), v\big)_X + \big(\tilde{S}u(t), v\big)_X + b\big(v, \lambda(t)\big) = \big(f(t), v\big)_X \quad \forall\, v \in X, \tag{2.15}
$$

$$
b\big(u(t), \mu - \lambda(t)\big) \le b\big(h(t), \mu - \lambda(t)\big) \quad \forall\, \mu \in \Lambda, \tag{2.16}
$$

for all $t \in \mathbb{R}_+$

The unique solvability of Problem 2.3 is provided in the next theorem.

Theorem 2.4 *Assume* (2.10)–(2.14). *There exists $d_0 > 0$ which depends only on A and b such that, if $\tilde{d}_n < d_0$ for all positive integers n, then Problem* 2.3 *has a unique solution (u, λ). Moreover, the solution satisfies $u \in C(\mathbb{R}_+; X)$ and $\lambda \in C(\mathbb{R}_+; \Lambda)$.*

Theorem 2.4 was obtained in [32]. Its proof is based on results on generalized saddle point problems and various estimates, combined with a fixed point argument. The smalness assumption $\tilde{d}_n < d_0$ in the statement of Theorem 2.4 is needed when using the fixed point argument which follows from the Banach contractions principle.

We end this section with further notation and preliminaries related to the contact model we are interested in. We denote by $\mathbb{S}^d$ $(d = 1, 2, 3)$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$
\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,
$$

$$
\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.
$$

Also, we use the notation $\|\boldsymbol{\kappa}\|$ for the Euclidean norm of the element $\boldsymbol{\kappa} \in \mathbb{R}^m$, where $m \in \mathbb{N}$, and $\boldsymbol{0}$ for the zero element of the spaces $\mathbb{R}^d$, $\mathbb{S}^d$ and $\mathbb{R}^m$.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with Lipschitz continuous boundary Γ and let Γ_1, Γ_2 and Γ_3 be three measurable parts of Γ such that $\text{meas}(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Also, we use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the spaces

$$
V = \big\{\boldsymbol{v} = (v_i) \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\big\},
$$

$$
Q = \big\{\boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} : \tau_{ij} = \tau_{ji}\big\}.
$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx,\qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\quad \varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}\,(v_{i,j}+v_{j,i})\ \forall\,\boldsymbol{v}\in H^1(\Omega)^d.$$

Completeness of the space $(V,\|\cdot\|_V)$ follows from the assumption meas$(\Gamma_1)>0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v}\in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d}\le c_0\|\boldsymbol{v}\|_V\quad\forall\,\boldsymbol{v}\in V. \tag{2.17}$$

As in [32] we consider the space

$$W=\{\boldsymbol{z}=\boldsymbol{v}|_{\Gamma_3}:\boldsymbol{v}\in V\},$$

where $\boldsymbol{v}|_{\Gamma_3}$ denotes the restriction of the trace of the element $\boldsymbol{v}\in V$ to Γ_3. We recall that $W\subset H^{1/2}(\Gamma_3;\mathbb{R}^d)$, where $H^{1/2}(\Gamma_3;\mathbb{R}^d)$ is the space of the restriction on Γ_3 of traces on Γ of functions of $H^1(\Omega)^d$. We denote by D the dual of the space W, and by $\langle\cdot,\cdot\rangle_{\Gamma_3}$ the duality pairing between D and W. Nevertheless, for simplicity, we write $\langle\boldsymbol{\mu},\boldsymbol{v}\rangle_{\Gamma_3}$ instead of $\langle\boldsymbol{\mu},\boldsymbol{v}|_{\Gamma_3}\rangle_{\Gamma_3}$, when $\boldsymbol{\mu}\in D$ and $\boldsymbol{v}\in V$.

For a regular function $\boldsymbol{\sigma}\in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\mathrm{Div}\,\boldsymbol{\sigma}=(\sigma_{ij,j})$ and, finally, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx+\int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx=\int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,da\quad\forall\,\boldsymbol{v}\in V. \tag{2.18}$$

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty=\big\{\mathcal{E}=(\mathcal{E}_{ijkl}):\mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d\big\},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty}=\max_{1\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q\le d\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q\quad\forall\,\mathcal{E}\in\mathbf{Q}_\infty,\ \boldsymbol{\tau}\in Q. \tag{2.19}$$

This inequality will be used in several places, in Sect. 5.

3 Problem Statement

The physical setting is similar to that considered in [34] and can be resumed as follows. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3 such that meas$(\Gamma_1) > 0$ and, in addition, Γ_3 is plane. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. It is fixed on Γ_1 and time-dependent surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in sliding frictional contact with a moving obstacle, the so-called foundation, which is made of a hard material covered by a layer of soft material of thickness g. The friction implies the wear of the foundation that we model it with a surface variable, the wear function. Then, the classical formulation of the contact problem is the following.

Problem $\mathcal{P}$ *Find a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$, a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, an internal state variable $\boldsymbol{\kappa} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ and a wear function $w : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\dot{\boldsymbol{u}}(t)\big) + \mathcal{G}\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \boldsymbol{\kappa}(t)\big) \quad in\ \Omega, \tag{3.1}$$

$$\dot{\boldsymbol{\kappa}}(t) = \boldsymbol{G}\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \boldsymbol{\kappa}(t)\big) \quad in\ \Omega, \tag{3.2}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad in\ \Omega, \tag{3.3}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad on\ \Gamma_1, \tag{3.4}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad on\ \Gamma_2, \tag{3.5}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \qquad \sigma_\nu(t) + p\big(u_\nu(t) - w(t)\big) \le 0, \\ \big(u_\nu(t) - g\big)\big(\sigma_\nu(t) + p\big(u_\nu(t) - w(t)\big)\big) = 0 \end{array}\right\} \quad on\ \Gamma_3, \tag{3.6}$$

$$-\boldsymbol{\sigma}_\tau(t) = \eta\, p\big(u_\nu(t) - w(t)\big)\boldsymbol{n}^*(t) \quad on\ \Gamma_3, \tag{3.7}$$

$$\dot{w}(t) = \alpha(t)\, p\big(u_\nu(t) - w(t)\big) \quad on\ \Gamma_3, \tag{3.8}$$

$$w(0) = 0 \quad in\ \Gamma_3, \tag{3.9}$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{\kappa}(0) = \boldsymbol{\kappa}_0 \quad in\ \Omega. \tag{3.10}$$

We now provide a brief description of the equations and conditions in Problem $\mathcal{P}$. Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$.

First, (3.1) and (3.2) represent the rate-type viscoplastic constitutive law with internal state variable in which we assume that elasticity tensor $\mathcal{E}$ and the constitutive functions $\mathcal{G}$ and $\boldsymbol{G}$ satisfy the following conditions:

$$\begin{cases} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i, j, k, l \le d. \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2\ \forall \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega. \end{cases} \tag{3.11}$$

$$
\begin{cases}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
\quad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{\kappa}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{\kappa}_2)\| \\
\qquad \leq L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2\|) \\
\qquad\quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \mathbb{R}^m, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \text{ is measurable on } \Omega, \\
\quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \boldsymbol{\kappa} \in \mathbb{R}^m. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ belongs to } Q.
\end{cases}
\tag{3.12}
$$

$$
\begin{cases}
\text{(a) } \boldsymbol{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \to \mathbb{R}^m. \\
\text{(b) There exists } L_G > 0 \text{ such that} \\
\quad \|\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \boldsymbol{\kappa}_1) - \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \boldsymbol{\kappa}_2)\| \\
\qquad \leq L_G(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2\|) \\
\qquad\quad \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \mathbb{R}^m, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}) \text{ is measurable on } \Omega, \\
\quad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d \text{ and } \boldsymbol{\kappa} \in \mathbb{R}^m. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \boldsymbol{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, \mathbf{0}) \text{ belongs to } L^2(\Omega)^m.
\end{cases}
\tag{3.13}
$$

Constitutive equations of the form (3.1)–(3.2) could describe both elasticity, plasticity, creep, relaxation, hardening and softening phenomena. For this reason they have been considered in the literature in order to model the behavior of real materials like rubbers, metals, pastes, rocks and so on. Various results and mechanical interpretation concerning constitutive laws of this form may be found in [5] and [17], for instance. Here we restrict ourselves to provide three clasical examples of such equations, with our without internal state variables.

The first example is one-dimensional and does not involve internal state variable. It is of the form

$$
\dot{\sigma} = E\dot{\varepsilon} + \mathcal{G}(\sigma, \varepsilon) \tag{3.14}
$$

with

$$
\mathcal{G}(\sigma, \varepsilon) = \begin{cases} -k_1 F_1\big(\sigma - f(\varepsilon)\big) & \text{if } \sigma > f(\varepsilon), \\ 0 & \text{if } g(\varepsilon) \leq \sigma \leq f(\varepsilon), \\ k_2 F_2\big(g(\varepsilon) - \sigma\big) & \text{if } \sigma < g(\varepsilon). \end{cases} \tag{3.15}
$$

Here where $E > 0$ is the Young modulus, $k_1, k_2 > 0$ are viscosity constants, f and g are Lipschitz continuous functions with $g(\varepsilon) < f(\varepsilon)$, and $F_1, F_2 : \mathbb{R}_+ \to \mathbb{R}$ are increasing functions with $F_1(0) = F_2(0) = 0$. Note that the domain of elastic behavior of the material is characterized by the inequalities $g(\varepsilon) \leq \sigma \leq f(\varepsilon)$. Plastic deformations occur only for $\sigma > f(\varepsilon)$ in extension or for $\sigma < g(\varepsilon)$ in compression. Therefore, since the yield limit (in extension and in compression) depends on the deformation, we conclude that the model (3.14), (3.15) represents a model with hardening.

A second example of an elastic-viscoplastic constitutive law without internal state variable is Perzyna's law given by

$$
\dot{\boldsymbol{\varepsilon}} = \mathcal{E}^{-1}\dot{\boldsymbol{\sigma}} + \frac{1}{\delta}(\boldsymbol{\sigma} - \mathcal{P}_{\mathcal{K}}\boldsymbol{\sigma}). \tag{3.16}
$$

Here $\mathcal{E}$ is a fourth order tensor satisfying (3.11), $\mathcal{E}^{-1}$ denotes its inverse, $\delta > 0$ is a viscosity constant, $\mathcal{K}$ is a nonempty, closed, convex set in the space $\mathbb{S}^d$ of symmetric tensors and $\mathcal{P}_{\mathcal{K}}$ represents the projection operator. Notice that in this case the function $\mathcal{G}$ does not depend on $\boldsymbol{\varepsilon}$ and is given by

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = -\frac{1}{\delta}\, \mathcal{E}(\boldsymbol{\sigma} - \mathcal{P}_{\mathcal{K}}\boldsymbol{\sigma}).$$

Since $\boldsymbol{\sigma} = \mathcal{P}_{\mathcal{K}}\boldsymbol{\sigma}$ iff $\boldsymbol{\sigma} \in \mathcal{K}$, from (3.16) we see that viscoplastic deformations occur only for the stress tensors $\boldsymbol{\sigma}$ outside the set $\mathcal{K}$. Thus, the set $\mathcal{K}$ represents the domain of elastic behavior of the material. It is usually defined by

$$\mathcal{K} = \big\{\boldsymbol{\sigma} \in \mathbb{S}^d : \mathcal{F}(\boldsymbol{\sigma}) \le 0\big\} \tag{3.17}$$

where $\mathcal{F} : \mathbb{S}^d \to \mathbb{R}$ is a convex function such that $\mathcal{F}(\mathbf{0}) < 0$. The function $\mathcal{F}$ is called the yield function and the equation $\mathcal{F}(\boldsymbol{\sigma}) = 0$ represents the yield condition.

A concrete example of an elastic-viscoplastic constitutive law of the form (3.1), (3.2) is given by the Perzyna's law with internal state variable,

$$\dot{\boldsymbol{\varepsilon}} = \mathcal{E}^{-1}\dot{\boldsymbol{\sigma}} + \frac{1}{\delta}\,(\boldsymbol{\sigma} - \mathcal{P}_{\mathcal{K}(\boldsymbol{\kappa})}\boldsymbol{\sigma}), \tag{3.18}$$

$$\dot{\boldsymbol{\kappa}}(t) = \frac{2}{3\delta}\,\|\boldsymbol{\sigma} - P_{\mathcal{K}(\boldsymbol{\kappa})}\boldsymbol{\sigma}\|. \tag{3.19}$$

Here $\mathcal{P}_{\mathcal{K}(\boldsymbol{\kappa})}$ represents the projection mapping on the von Mises convex set $\mathcal{K}(\boldsymbol{\kappa})$ defined by equality

$$\mathcal{K}(\boldsymbol{\kappa}) = \big\{\boldsymbol{\sigma} \in \mathbb{S}^d : \big\|\boldsymbol{\sigma}^D\big\| \le \omega(\boldsymbol{\kappa})\sqrt{2}\big\},$$

$\boldsymbol{\sigma}^D$ being the deviator of $\boldsymbol{\sigma}$, and $\omega : \mathbb{R} \to \mathbb{R}$ is a given positive function. Note that, as explained in [13], the variable $\boldsymbol{\kappa}$ given by (3.19) represents the irreversible equivalent strain.

Equation (3.3) is the equilibrium equation and we use it here since we assume that the process is quasistatic. Conditions (3.4) and (3.5) are the displacement boundary condition and traction boundary condition, respectively. We assume that the densities of body forces and surface tractions are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{3.20}$$

Conditions (3.6)–(3.8) were introduced and justified in [34] and, for this reason, we do not present here in detail. We restrict ourselves to mention that (3.6) represents the contact condition in which the normal compliance function p satisfies

$$\left\{\begin{array}{l}
\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\
\text{(b) There exists } L_p > 0 \text{ such that}\\
\qquad \big|p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)\big| \le L_p\,|r_1 - r_2|\\
\qquad\quad \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(c) } \big(p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)\big)(r_1 - r_2) \ge 0\\
\qquad\quad \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\\
\qquad \text{for any } r \in \mathbb{R}.\\
\text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{3.21}$$

This condition was derived by assuming an additive decomposition of the normal stress into two components which satisfy the Signorini condition in the form with a gap function and the normal compliance contact condition with wear, respectively.

Condition (3.7) represents a sliding version of the classical Coulomb law of dry friction. Here η represents the friction coefficient, $\boldsymbol{n}^*$ is the unitary vector defined by

$$\boldsymbol{n}^*(t) = -\frac{\boldsymbol{v}^*(t)}{\|\boldsymbol{v}^*(t)\|}$$

where $\boldsymbol{v}^*$ is the velocity of the foundation, supposed to be a non vanishing time-dependent function in the plane of Γ_3. This condition was derived under the assumption that the velocity of the foundation $\boldsymbol{v}^*(t)$ is large in comparison with the tangential velocity $\dot{\boldsymbol{u}}_\tau(t)$. Here, we assume that the coefficient of friction and velocity of the foundation verify two following conditions:

$$\eta \in L^\infty(\Gamma_3), \qquad \eta(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{3.22}$$

$$\begin{cases} \boldsymbol{v}^* \in C\big(\mathbb{R}_+; \mathbb{R}^3\big) \text{ and there exist } v_1,\ v_2 > 0 \text{ such that} \\ v_1 \leq \big\|\boldsymbol{v}^*(t)\big\| \leq v_2 \quad \forall\, t \in \mathbb{R}_+. \end{cases} \tag{3.23}$$

The differential equation (3.8) represents a version of Archard's law which governs the evolution of the wear function and, again, it was derived under the assumption that the velocity of the foundation $\boldsymbol{v}^*(t)$ is large in comparison with the tangential velocity $\dot{\boldsymbol{u}}_\tau(t)$. Here

$$\alpha(t) = k\big\|\boldsymbol{v}^*(t)\big\|,$$

k being the wear coefficient, assumed to be such that

$$k \in L^\infty(\Gamma_3), \qquad k(\boldsymbol{x}) \geq 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{3.24}$$

Condition (3.9) represents the initial condition for the wear function and shows that at the initial moment the materials involved in the process are new. Next, (3.10) represent the initial conditions for the rest of the unknowns in which $\boldsymbol{u}_0$, $\boldsymbol{\sigma}_0$, $\boldsymbol{\kappa}_0$ denote the initial displacement, the initial stress field and the initial state variable, respectively. We assume in what follows that these initial data have the regularity

$$\boldsymbol{u}_0 \in V, \qquad \boldsymbol{\sigma}_0 \in Q, \qquad \boldsymbol{\kappa}_0 \in L^2(\Omega)^m. \tag{3.25}$$

Finally, we assume that

$$\text{there exists } \tilde{\boldsymbol{\theta}} \in V \text{ such that } \tilde{\theta}_\nu = 1 \text{ a.e. on } \Gamma_3 \tag{3.26}$$

where, we recall, $\tilde{\theta}_\nu = \tilde{\boldsymbol{\theta}} \cdot \boldsymbol{\nu}$. This assumption concerns only the geometry of the problem and was already used in [2], for instance. It is needed in order to derive a mixed variational formulation to Problem $\mathcal{P}$.

4 A Mixed Variational Formulation

We now derive a mixed variational formulation of Problem $\mathcal{P}$. To this end, we define the sets $K \subset V$ and $\Lambda \subset D$, the bilinear form $b : V \times D \to \mathbb{R}$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$K = \{\boldsymbol{v} \in V : v_\nu \le 0 \text{ a.e. on } \Gamma_3\}, \tag{4.1}$$

$$\Lambda = \big\{\boldsymbol{\mu} \in D : \langle \boldsymbol{\mu}, \boldsymbol{v}\rangle_{\Gamma_3} \le 0 \; \forall\, \boldsymbol{v} \in K\big\}, \tag{4.2}$$

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{v}\rangle_{\Gamma_3}, \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D, \tag{4.3}$$

$$\big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{4.4}$$

Next, we assume that $\boldsymbol{\sigma}$, $\boldsymbol{u}$, $\boldsymbol{\kappa}$ and w are regular functions which verify (3.1)–(3.10). Let $t \in \mathbb{R}_+$, $\boldsymbol{v} \in V$ and $\boldsymbol{\mu} \in \Lambda$. We integrate (3.1), (3.2) with initial conditions (3.10) to find that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big), \boldsymbol{\kappa}(s)\big)\, ds + \boldsymbol{\sigma}_0 - \boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.5}$$

$$\boldsymbol{\kappa}(t) = \int_0^t \boldsymbol{G}\big(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big), \boldsymbol{\kappa}(s)\big)\, ds + \boldsymbol{\kappa}_0. \tag{4.6}$$

Moreover, we integrate (3.8) with the initial condition (3.9) to obtain

$$w(t) = \int_0^t \alpha(s) p\big(u_\nu(s) - w(s)\big)\, ds. \tag{4.7}$$

Next, we use Green formula (2.18) and the equilibrium equation (3.3) to see that

$$\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q = \big(\boldsymbol{f}_0(t), \boldsymbol{v}\big)_{L^2(\Omega)^d} + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{4.8}$$

We split the surface integral over Γ_1, Γ_2 and Γ_3. Then we use the equalities $\boldsymbol{v} = \boldsymbol{0}$ on Γ_1, $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on Γ_2, $\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v} = \sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau$ on Γ_3, and definition (4.4) to obtain that

$$\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V + \int_{\Gamma_3} \big(\sigma_\nu(t)v_\nu + \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau\big)\, da \quad \forall\, \boldsymbol{v} \in V. \tag{4.9}$$

Let $\boldsymbol{\lambda}(t) \in D$ be the Lagrange multiplier defined by

$$\big\langle \boldsymbol{\lambda}(t), \boldsymbol{z}\big\rangle_{\Gamma_3} = -\int_{\Gamma_3} \big(\sigma_\nu(t) + p\big(u_\nu(t) - w(t)\big)\big) z_\nu\, da \quad \forall\, \boldsymbol{z} \in W. \tag{4.10}$$

Then, taking into account (4.3) we can write

$$\int_{\Gamma_3} \sigma_\nu(t)v_\nu\, da = -b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) - \int_{\Gamma_3} p\big(u_\nu(t) - w(t)\big)v_\nu\, da \quad \forall\, \boldsymbol{v} \in V \tag{4.11}$$

and, combining this equality with (4.9) and (3.7) we obtain that

$$\begin{aligned}&\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) + \int_{\Gamma_3} p\big(u_\nu(t) - w(t)\big)v_\nu\, da \\ &\quad + \int_{\Gamma_3} \eta p\big(u_\nu(t) - w(t)\big)\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\, da = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V \quad \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{4.12}$$

On the other hand, (4.10), (3.6), (4.1) and (4.2) imply that $\boldsymbol{\lambda}(t) \in \Lambda$. Moreover, using (3.26) and definition (4.3) we deduce that

$$\begin{aligned} b\big(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) &= b\big(\boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) + b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \\ &= \big\langle \boldsymbol{\mu}, \boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}}\big\rangle_{\Gamma_3} - \big\langle \boldsymbol{\lambda}(t), \boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}}\big\rangle_{\Gamma_3} + b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \quad \forall\, \boldsymbol{\mu} \in \Lambda. \end{aligned} \tag{4.13}$$

In addition, the contact condition (3.6), assumption (3.26) and definitions (4.1), (4.2), (4.10) imply that

$$\boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}} \in K, \quad \big\langle \boldsymbol{\mu}, \boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}}\big\rangle_{\Gamma_3} \le 0, \quad \big\langle \boldsymbol{\lambda}(t), \boldsymbol{u}(t) - g\tilde{\boldsymbol{\theta}}\big\rangle_{\Gamma_3} = 0 \quad \forall\, \boldsymbol{\mu} \in \Lambda. \tag{4.14}$$

We combine now (4.13) and (4.14) to deduce that

$$b\big(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \le b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \quad \forall\, \boldsymbol{\mu} \in \Lambda. \tag{4.15}$$

Finally, we gather equalities (4.5)–(4.7), (4.12), and inequality (4.15) to obtain the following mixed variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}^V$ *Find a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$, an internal state variable $\boldsymbol{\kappa} : \mathbb{R}_+ \to L^2(\Omega)^m$, a wear function $w : \mathbb{R}_+ \to L^2(\Gamma_3)$ and a Lagrange multiplier $\boldsymbol{\lambda} : \mathbb{R}_+ \to \Lambda$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \int_0^t \mathcal{G}\big(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big), \boldsymbol{\kappa}(s)\big)\, ds + \boldsymbol{\sigma}_0 - \boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{4.16}$$

$$\boldsymbol{\kappa}(t) = \int_0^t \boldsymbol{G}\big(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(s)\big), \boldsymbol{\kappa}(s)\big)\, ds + \boldsymbol{\kappa}_0, \tag{4.17}$$

$$w(t) = \int_0^t \alpha(s)\, p\big(u_\nu(s) - w(s)\big)\, ds, \tag{4.18}$$

$$\begin{aligned} &\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + \int_{\Gamma_3} p\big(u_\nu(t) - w(t)\big) v_\nu\, da + b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) \\ &\quad + \int_{\Gamma_3} \eta\, p\big(u_\nu(t) - w(t)\big)\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\, da = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{4.19}$$

$$b\big(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \le b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \quad \forall\, \boldsymbol{\mu} \in \Lambda, \tag{4.20}$$

for all $t \in \mathbb{R}_+$.

Note that Problem $\mathcal{P}^V$ represents a system which couples three nonlinear implicit integral equations for the stress field, the internal state variable and the wear function, respectively, with a history-dependent variational equation for displacement field, and a first-order time-dependent variational inequality for the Lagrange multiplier.

5 Existence and Uniqueness

In this section we state and prove the following existence and uniqueness result concerning problem $\mathcal{P}^V$.

Theorem 5.1 *Assume* (3.11)–(3.13), (3.20)–(3.26). *Then, there exists* $e_0 > 0$ *which depends only on* $\mathcal{E}$, Ω, Γ_1 *and* Γ_3 *such that Problem* $\mathcal{P}^V$ *has a unique solution* $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\kappa}, w, \boldsymbol{\lambda})$, *if* $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$. *Moreover, the solution satisfies*

$$(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\kappa}, w, \boldsymbol{\lambda}) \in C\big(\mathbb{R}_+; Q \times V \times L^2(\Omega)^m \times L^2(\Gamma_3) \times \Lambda\big). \tag{5.1}$$

The proof of Theorem 5.1 will be carried out in several steps, based on the abstract results presented in Sect. 2. To present it, we assume in what follows that (3.11)–(3.13), (3.20)–(3.26) hold. The first step of the proof is the following.

Lemma 5.2 *There exists an operator* $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *such that for all functions* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *and* $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$, *equalities* (4.16), (4.17) *hold for all* $t \in \mathbb{R}_+$ *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \mathcal{S}_1\boldsymbol{u}(t), \tag{5.2}$$

$$\boldsymbol{\kappa}(t) = \mathcal{S}_2\boldsymbol{u}(t) \tag{5.3}$$

for all $t \in \mathbb{R}_+$. *Moreover, the operator* $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q \times L^2(\Omega)^m)$ *is a history-dependent operator.*

Proof Lemma 5.2 is a direct consequence of Theorem 2.2 applied with $X = V$, $Y = Q \times L^2(\Omega)^m$,

$$A\boldsymbol{u} = \big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \boldsymbol{\kappa}_0\big),$$

$$G\big(t, \boldsymbol{u}, (\boldsymbol{\sigma}, \boldsymbol{\kappa})\big) = \big(\mathcal{G}\big(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\kappa}\big), \boldsymbol{G}\big(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\kappa}\big)\big)$$

for all $\boldsymbol{u} \in V$, $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in Q \times L^2(\Omega)^m$ and $t \in \mathbb{R}_+$. Indeed, it is easy to see that assumptions (3.11)–(3.13) and (3.25) imply that the operators above are well defined and, moreover, they satisfy conditions (2.2) and (2.3), respectively. □

The next step consists in the following result concerning the wear function.

Lemma 5.3 *There exists an operator* $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ *such that for all functions* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *and* $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$, *equality* (4.18) *holds for all* $t \in \mathbb{R}_+$ *if and only if*

$$w(t) = \mathcal{R}\boldsymbol{u}(t) \tag{5.4}$$

for all $t \in \mathbb{R}_+$. *Moreover, the operator* $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ *is a history-dependent operator.*

Proof Lemma 5.3 is a direct consequence of Theorem 2.2 applied with $X = V$, $Y = L^2(\Gamma_3)$,

$$A\boldsymbol{u} = 0_{L^2(\Gamma_3)}, \qquad G(t, \boldsymbol{u}, w) = \alpha(t)p(u_\nu - w)$$

for all $\boldsymbol{u} \in V$, $w \in L^2(\Gamma_3)$ and $t \in \mathbb{R}_+$. Indeed, it is easy to see that assumptions (3.21)–(3.24) imply that the operators above are well defined and, moreover, they satisfy conditions (2.2) and (2.3), respectively. □

We now complete the statement of Lemmas 5.2 and 5.3 with some estimates concerning the constants involved on the inequalities which provide the history-dependence of the operators $\mathcal{S}$ and $\mathcal{R}$. Let $\mathcal{K}$ and M_n be given by

$$\mathcal{K} = (L_{\mathcal{G}} + L_G)\big(d\|\boldsymbol{E}\|_{\mathbf{Q}_\infty} + 1\big), \tag{5.5}$$

$$M_n = L_p \max\{1, c_0\} \max_{s\in[0,n]} \big\|\alpha(s)\big\|_{L^\infty(\Gamma_3)} \quad \forall n \in \mathbb{N}. \tag{5.6}$$

Then, a simple computation shows that for each $n \in \mathbb{N}$ the inequalities below holds:

$$\big\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t)\big\|_{Q\times L^2(\Omega)^m} \le \mathcal{K}e^{n\mathcal{K}} \int_0^t \big\|\boldsymbol{u}(s) - \boldsymbol{v}(s)\big\|_V\, ds$$

$$\forall \boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+; V)\ \forall t \in [0, n], \tag{5.7}$$

$$\big\|\mathcal{R}\boldsymbol{u}(t) - \mathcal{R}\boldsymbol{v}(t)\big\|_{L^2(\Gamma_3)} \le M_n e^{nM_n} \int_0^t \big\|\boldsymbol{u}(s) - \boldsymbol{v}(s)\big\|_V\, ds$$

$$\forall \boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+; V)\ \forall t \in [0, n]. \tag{5.8}$$

We now state the following equivalence result whose proof represents a direct consequence of Lemmas 5.2 and 5.3.

Lemma 5.4 *Let* $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\kappa}, w, \boldsymbol{\lambda})$ *be functions with regularity* (5.1). *Then* $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\kappa}, w, \boldsymbol{\lambda})$ *is a solution of Problem* $\mathcal{P}^V$ *if and only if*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) + \mathcal{S}_1\boldsymbol{u}(t), \tag{5.9}$$

$$\boldsymbol{\kappa}(t) = \mathcal{S}_2\boldsymbol{u}(t), \tag{5.10}$$

$$w(t) = \mathcal{R}\boldsymbol{u}(t), \tag{5.11}$$

$$\big(\mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + \big(\mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + \int_{\Gamma_3} p\big(u_\nu(t) - \mathcal{R}\boldsymbol{u}(t)\big)v_\nu\, da \tag{5.12}$$

$$+ \int_{\Gamma_3} \eta\, p\big(u_\nu(t) - \mathcal{R}\boldsymbol{u}(t)\big)\boldsymbol{n}^*(t)\cdot \boldsymbol{v}_\tau\, da + b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V \quad \forall \boldsymbol{v} \in V,$$

$$b\big(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \le b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \quad \forall \boldsymbol{\mu} \in \Lambda \tag{5.13}$$

for all $t \in \mathbb{R}_+$.

Note that the interest of Lemma 5.4 arises in the fact that it decouples the unknowns of the Problem $\mathcal{P}^V$. Indeed, a careful examination of the system (5.9)–(5.13) shows that the unknowns $\boldsymbol{\sigma}$, $\boldsymbol{\kappa}$ and w do not appear in the system (5.12)–(5.13), which contains only the unknowns $\boldsymbol{u}$ and $\boldsymbol{\lambda}$. For this reason, the next step in the proof of Theorem 5.1 consists to obtain the unique solvability of the system (5.12)–(5.13).

Lemma 5.5 *There exists* $e_0 > 0$ *which depends only on* $\mathcal{E}$, Ω, Γ_1 *and* Γ_3 *such that if* $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$ *then there exists a unique couple of functions* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *which satisfies* (5.12)–(5.13) *for all* $t \in \mathbb{R}_+$. *Moreover,*

$$(\boldsymbol{u}, \boldsymbol{\lambda}) \in C(\mathbb{R}_+; V \times \Lambda). \tag{5.14}$$

Proof We use the Riesz representation theorem to define the operators $A : V \to V$ and $\tilde{\mathcal{S}} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \big(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{5.15}$$

$$\begin{aligned}\big(\tilde{\mathcal{S}}\boldsymbol{u}(t), \boldsymbol{v}\big)_V &= \big(\mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + \int_{\Gamma_3} p\big(u_\nu(t) - \mathcal{R}\boldsymbol{u}(t)\big)v_\nu\, da \\ &\quad + \int_{\Gamma_3} \eta p\big(u_\nu(t) - \mathcal{R}\boldsymbol{u}(t)\big)\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\, da \quad \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V) \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+.\end{aligned} \tag{5.16}$$

Note that, by assumptions (3.21)–(3.23), the surface integrals in (5.16) are well defined. With these notation it is easy to see that the variational equation (5.12) is equivalent with

$$\big(A\boldsymbol{u}(t), \boldsymbol{v}\big)_V + \big(\tilde{\mathcal{S}}\boldsymbol{u}(t), \boldsymbol{v}\big)_V + b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.17}$$

Therefore, to conclude the proof it is sufficient to show that there exists a unique couple of functions $(\boldsymbol{u}, \boldsymbol{\lambda})$ with regularity (5.14), which satisfies (5.17) and (5.13) for all $t \in \mathbb{R}_+$. The main ingredient in the solution of this system is Theorem 2.4 and, to this end, we check in what follows the assumptions of this theorem.

First, using (3.11) we deduce that the operator A, defined by (5.15), verifies (2.10). Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V)$, $\boldsymbol{v} \in V$, $n \in \mathbb{N}$ and $t \in [0, n]$. According to definition (5.16) of operator $\tilde{\mathcal{S}}$ we have

$$\begin{aligned}&\big(\tilde{\mathcal{S}}\boldsymbol{u}_1(t) - \tilde{\mathcal{S}}\boldsymbol{u}_2(t), \boldsymbol{v}\big)_V \\ &\quad = \big(\mathcal{S}_1\boldsymbol{u}_1(t) - \mathcal{S}_1\boldsymbol{u}_2(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q \\ &\qquad + \int_{\Gamma_3} \big[p\big(u_{1\nu}(t) - \mathcal{R}\boldsymbol{u}_1(t)\big) - p\big(u_{2\nu}(t) - \mathcal{R}\boldsymbol{u}_2(t)\big)\big]v_\nu\, da \\ &\qquad + \int_{\Gamma_3} \eta\,\big[p\big(u_{1\nu}(t) - \mathcal{R}\boldsymbol{u}_1(t)\big) - p\big(u_{2\nu}(t) - \mathcal{R}\boldsymbol{u}_2(t)\big)\big]\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\, da.\end{aligned} \tag{5.18}$$

Using assumptions (3.21)–(3.23), inequality (2.17) and estimates (5.7) and (5.8) we obtain

$$\begin{aligned}&\big|\big(\tilde{\mathcal{S}}\boldsymbol{u}_1(t) - \tilde{\mathcal{S}}\boldsymbol{u}_2(t), \boldsymbol{v}\big)_V\big| \\ &\quad \le \big[c_0^2 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big)\big]\big\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\big\|_V \|\boldsymbol{v}\|_V \\ &\qquad + \big[\mathcal{K}e^{n\mathcal{K}} + c_0 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big)M_n e^{nM_n}\big]\int_0^t \big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\big\|_V\, ds\, \|\boldsymbol{v}\|_V.\end{aligned} \tag{5.19}$$

Thus,

$$\begin{aligned}&\big\|\tilde{\mathcal{S}}\boldsymbol{u}_1(t) - \tilde{\mathcal{S}}\boldsymbol{u}_2(t)\big\|_V \\ &\quad \le \big[c_0^2 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big)\big]\big\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\big\|_V \\ &\qquad + \big[\mathcal{K}e^{n\mathcal{K}} + c_0 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big)M_n e^{nM_n}\big]\int_0^t \big\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\big\|_V\, ds.\end{aligned} \tag{5.20}$$

The previous inequality implies that the operator $\tilde{\mathcal{S}}$ satisfies condition (2.11) with

$$\tilde{d}_n = c_0^2 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big) \tag{5.21}$$

and

$$\tilde{s}_n = \mathcal{K}e^{n\mathcal{K}} + c_0 L_p\big(1 + \|\eta\|_{L^\infty(\Gamma_3)}\big) M_n e^{nM_n}.$$

Next, as it was shown in [21, 22], definition (4.3), implies that the bilinear form $b(\cdot,\cdot)$ satisfies condition (2.12), i.e. there exist constants $M_b > 0$ and $\alpha > 0$ such that

$$\big|b(\boldsymbol{v},\boldsymbol{\mu})\big| \le M_b \|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_D \quad \forall\, \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D \tag{5.22}$$

and

$$\inf_{\boldsymbol{\mu}\in D,\, \boldsymbol{\mu}\neq \boldsymbol{0}_D}\ \sup_{\boldsymbol{v}\in V,\, \boldsymbol{v}\neq \boldsymbol{0}_V} \frac{b(\boldsymbol{v},\boldsymbol{\mu})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\mu}\|_D} \ge \alpha. \tag{5.23}$$

Finally, definition (4.4) and assumptions (3.20), (3.26) yield

$$\boldsymbol{f} \in C(\mathbb{R}_+; V) \quad \text{and} \quad g\tilde{\boldsymbol{\theta}} \in V. \tag{5.24}$$

The previous results allow us to apply Theorem 2.4 with $X = V$, $Y = D$ and $\boldsymbol{h} = g\tilde{\boldsymbol{\theta}}$. According to this theorem there exists $d_0 > 0$ which depends only on $\boldsymbol{E}$, Ω, Γ_1 and Γ_3 such that, if $\tilde{d}_n < d_0$, for all positive integers n, then there exists a unique couple of functions $(\boldsymbol{u},\boldsymbol{\lambda})$ with regularity (5.14), which satisfies (5.17) and (5.13) for all $t \in \mathbb{R}_+$. Denote

$$e_0 = d_0 c_0^{-2} \tag{5.25}$$

which, clearly, depends only on $\boldsymbol{E}$, Ω, Γ_1 and Γ_3. Then, it follows from (5.21) and (5.25) that $\tilde{d}_n < d_0$ iff $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$ which concludes the proof. □

We now have all ingredients to prove our main existence and uniqueness result.

Proof of Theorem 5.1 *Existence.* Assume that $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$, where e_0 is defined by (5.25). Then, using Lemma 5.5 we deduce that there exists a unique couple of functions $(\boldsymbol{u},\boldsymbol{\lambda})$ such that (5.12)–(5.13) hold, for all $t \in \mathbb{R}_+$. Moreover, the solution has the regularity (5.14). Next, we introduce the functions $\boldsymbol{\sigma}$, $\boldsymbol{\kappa}$ and w defined by (5.9)–(5.11). Taking into account assumption (3.11) and the regularity of operators $\mathcal{S}$ and $\mathcal{R}$ we conclude that the triple $(\boldsymbol{\sigma},\boldsymbol{\kappa},w)$ has the regularity $(\boldsymbol{\sigma},\boldsymbol{\kappa},w) \in C(\mathbb{R}_+; Q \times L^2(\Omega)^m \times L^2(\Gamma_3))$. It follows from here that (5.1) holds. Lemma 5.4 implies now the existence part of the theorem.

Uniqueness. The uniqueness of the solution is now a consequence of the unique solvability of system (5.12)–(5.13), guaranteed by Lemma 5.5, combined with Lemma 5.4. □

We conclude from above that, under the assumptions of Theorem 5.1, the contact problem $\mathcal{P}$ has a unique weak solution. Note that inequality $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$, which guarantees the unique weak solvability of Problem $\mathcal{P}^V$, is verified if either the Lipschitz constant L_p or $1+\|\eta\|_{L^\infty(\Gamma_3)}$ is small enough. Therefore, this condition represents a smallness condition on the normal compliance function and/or the coefficient of friction.

6 Particular Cases

The aim of this section is twofold. The first one is to provide examples of contact problems which represent particular cases of Problem $\mathcal{P}$ and whose unique weak solvability could be obtained by using Theorem 5.1. The second one is to compare the mixed variational formulation used in this paper with a different aproach, used in [34].

Elastic contact with wear We start by considering Problem $\mathcal{P}$ in the particular case when the material is elastic, i.e. when $\mathcal{G} \equiv \mathbf{0}$, $G \equiv \mathbf{0}$ and $\boldsymbol{\sigma}_0 = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)$. The classical formulation of this problem is the following.

Problem $\mathcal{P}_e$ *Find a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$, a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and a wear function $w : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) \quad \text{in } \Omega, \tag{6.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \tag{6.2}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \tag{6.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{6.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \qquad \sigma_\nu(t) + p\big(u_\nu(t) - w(t)\big) \le 0, \\ \big(u_\nu(t) - g\big)\big(\sigma_\nu(t) + p\big(u_\nu(t) - w(t)\big)\big) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{6.5}$$

$$-\boldsymbol{\sigma}_\tau(t) = \eta\, p\big(u_\nu(t) - w(t)\big)\,\boldsymbol{n}^*(t) \quad \text{on } \Gamma_3, \tag{6.6}$$

$$\dot{w}(t) = \alpha(t)\, p\big(u_\nu(t) - w(t)\big) \quad \text{on } \Gamma_3, \tag{6.7}$$

$$w(0) = 0 \quad \text{in } \Gamma_3, \tag{6.8}$$

The mixed variational formulation of Problem $\mathcal{P}_e$ follows from Sect. 4 and can be formulated as follows.

Problem $\mathcal{P}_e^V$. *Find a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$, a wear function $w : \mathbb{R}_+ \to L^2(\Gamma_3)$ and a Lagrange multiplier $\boldsymbol{\lambda} : \mathbb{R}_+ \to \Lambda$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \tag{6.9}$$

$$w(t) = \int_0^t \alpha(s)\, p\big(u_\nu(s) - w(s)\big)\, ds, \tag{6.10}$$

$$\begin{aligned} &\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_Q + \int_{\Gamma_3} p\big(u_\nu(t) - w(t)\big) v_\nu\, da + b\big(\boldsymbol{v}, \boldsymbol{\lambda}(t)\big) \\ &\quad + \int_{\Gamma_3} \eta\, p\big(u_\nu(t) - w(t)\big)\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\, da = \big(\boldsymbol{f}(t), \boldsymbol{v}\big)_V \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{6.11}$$

$$b\big(\boldsymbol{u}(t), \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \le b\big(g\tilde{\boldsymbol{\theta}}, \boldsymbol{\mu} - \boldsymbol{\lambda}(t)\big) \quad \forall\, \boldsymbol{\mu} \in \Lambda, \tag{6.12}$$

for all $t \in \mathbb{R}_+$.

The unique solvability of this problem follows from Theorem 5.1, under the assumptions (3.11), (3.20)–(3.24), (3.26), combined with a smallness assumption of the form $L_p(1 + \|\eta\|_{L^\infty(\Gamma_3)}) < e_0$ for the coefficient of friction.

Note that the elastic contact problem $\mathcal{P}_e$ was studied in [34], under the same assumptions. There, using the set of admissible displacement fields given by

$$U = \{\boldsymbol{v} \in V : v_\nu \le g \text{ a.e. on } \Gamma_3\},$$

the following three-fields variational formulation of the problem was derived.

Problem $\tilde{\mathcal{P}}_e^V$ *Find a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a wear function $w : \mathbb{R}_+ \to L^2(\Gamma_3)$ such that*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big), \tag{6.13}$$

$$\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big)_Q + \int_{\Gamma_3} p\big(u_\nu(t) - w(t)\big)\big(v_\nu - u_\nu(t)\big)\, da$$
$$+ \int_{\Gamma_3} \eta p\big(u_\nu(t) - w(t)\big)\boldsymbol{n}^*(t) \cdot \big(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)\big)\, da \ge \big(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\big)_V \quad \forall\, \boldsymbol{v} \in U, \tag{6.14}$$

$$w(t) = \int_0^t \alpha(s) p\big(u_\nu(s) - w(s)\big)\, ds, \tag{6.15}$$

for all $t \in \mathbb{R}_+$.

Then, the existence of a unique solution of the problem $\tilde{\mathcal{P}}_e^V$ was derived in several steps, which could be resumed as follows.

(i) In the first step it is proved that, for a given wear function $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$, there exists a unique displacement filed $\boldsymbol{u}_w \in C(\mathbb{R}_+; U)$ such that

$$\big(\mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}_w(t)\big), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}\big(\boldsymbol{u}_w(t)\big)\big)_Q + \int_{\Gamma_3} p\big(u_{w\nu}(t) - w(t)\big)\big(v_\nu - u_{w\nu}(t)\big)\, da$$
$$+ \int_{\Gamma_3} \eta\, p\big(u_{w\nu}(t) - w(t)\big)\boldsymbol{n}^*(t) \cdot \big(\boldsymbol{v}_\tau - \boldsymbol{u}_{w\tau}(t)\big)\, da \ge \big(\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_w(t)\big)_V \quad \forall\, \boldsymbol{v} \in U, \tag{6.16}$$

for all $t \in \mathbb{R}_+$.

(ii) Then, it was shown that the operator $\Lambda : C(\mathbb{R}_+; L^2(\Gamma_3)) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ defined by

$$\Lambda w(t) = \int_0^t \alpha(s) p\big(u_{w\nu}(s) - w(s)\big)\, ds \tag{6.17}$$

has a unique fixed point $w^* \in C(\mathbb{R}_+; L^2(\Gamma_3))$.

(iii) Finally, defining $\boldsymbol{u}^*$ and $\boldsymbol{\sigma}^*$ by equalities $\boldsymbol{u}^* = \boldsymbol{u}_{w^*}$, $\boldsymbol{\sigma}^* = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^*)$, it was proved that the triple $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, w^*)$ is the uniqe weak solution to Problem $\tilde{\mathcal{P}}_e^V$.

A brief comparasion between problems Problem $\mathcal{P}_e^V$ and Problem $\tilde{\mathcal{P}}_e^V$ leads to the following comments.

(a) Problem $\mathcal{P}_e^V$ represents a four-fields variational formulations of the mechanical contact problem $\mathcal{P}_e$ while, in contrast, Problem $\tilde{\mathcal{P}}_e^V$ represent a three-fields variational formulations of the same problem.

(b) Problem $\tilde{\mathcal{P}}_e^V$ involves a variational inequality with contraints, (6.14). In contrast, Problem $\mathcal{P}_e^V$ involves a variational equation without contraints, (6.11). Removing the constraints in (6.14) was possible by introducing a new variable, the Lagrange multiplier $\boldsymbol{\lambda}$. Considering mixed formulations based on the Lagrange multiplier has important advantages in the numerical solution of the contact problems, as explained in [14, 16, 18].
(c) Under the same assumption on the data, both Problem $\mathcal{P}_e^V$ and Problem $\tilde{\mathcal{P}}_e^V$ have a unique weak solution, with the same regularity. Moreover, their solvability is guaranteed by a smallness assumption on the coefficient of friction. Nevertheless, the question if this assumption represents an intrinsic feature of the contact Problem $\mathcal{P}_e$ or it describes a limitation of the mathematical methods used in solving the problems $\mathcal{P}_e^V$ and $\tilde{\mathcal{P}}_e^V$ is left open.
(d) The solution of Problem $\tilde{\mathcal{P}}_e^V$ is based on arguments of time-dependent variational inequalities of the first kind, combined with the fixed point argument provided by Theorem 2.1. In contrast, the solution of Problem $\mathcal{P}_e^V$ is based on the more elaborate result provided by Theorem 2.4, which already integrates a fixed point argument.
(e) The equivalence between Problems $\mathcal{P}_e^V$ and Problem $\tilde{\mathcal{P}}_e^V$ represents an open question. As far as this equivalence is not proved, we conclude that a contact problem could have different variational formulations and, therefore, the concept of weak solution for such a problem is not an intrinsic one.

Viscoplastic contact with normal compliance and unilateral constraint We now consider Problem $\mathcal{P}$ in the particular case when the material is viscoplastic, without internal state variable, the contact is frictionless and the wear of the contact surfaces in neglecting. Therefore, we take $G \equiv \mathbf{0}$, $\eta \equiv 0$ and $\alpha \equiv 0$ to obtain the following contact model.

Problem $\mathcal{P}_{vp}$ *Find a stress field $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ and a displacement field $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}\big(\dot{\boldsymbol{u}}(t)\big) + \mathcal{G}\big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big)\big) \quad \text{in } \Omega, \tag{6.18}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \tag{6.19}$$

$$\boldsymbol{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \tag{6.20}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{6.21}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \qquad \sigma_\nu(t) + p\big(u_\nu(t)\big) \le 0, \\ \big(u_\nu(t) - g\big)\big(\sigma_\nu(t) + p\big(u_\nu(t)\big)\big) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{6.22}$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \quad \text{on } \Gamma_3, \tag{6.23}$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{6.24}$$

Problem $\mathcal{P}_{vp}$ was considered in [2]. There, besides the unique solvability of the problem, the continuous dependence of the weak solution with respect to both the normal compliance function and the penetration bound was proved. Numerical simulations which provide a numerical evidence of this continuous dependence result were also perfomed.

Signorini frictionless problem with gap We finally consider Problem $\mathcal{P}_{vp}$ in the particular case when the material is elastic and the normal compliance vanishes. Therefore, taking $\mathcal{G} \equiv \mathbf{0}$, $p \equiv 0$ and $\boldsymbol{\sigma} = \mathcal{E}(\boldsymbol{\varepsilon}\boldsymbol{u}_0)$ in (6.18)–(6.24) we obtain the following contact model.

Problem $\mathcal{P}_S$ *Find a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *and a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}\big(\boldsymbol{u}(t)\big) \quad \text{in } \Omega, \tag{6.25}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{6.26}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{6.27}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{6.28}$$

$$\left.\begin{array}{l} u_\nu(t) \le g, \qquad \sigma_\nu(t) \le 0, \\ \big(u_\nu(t) - g\big)\sigma_\nu(t) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{6.29}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on } \Gamma_3, \tag{6.30}$$

Note that Problem $\mathcal{P}_S$ represents the time-dependent version of the famous Signorini frictionless contact problem, see for instance [29] and the references therein. The mixed variational method presented in this paper could be applied to in the study of Problems $\mathcal{P}_{vp}$ and $\mathcal{P}_S$ in order to provide the unique solvability of these problems. It also provides the background for their numerical simulations.

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