# A regularization method for a viscoelastic contact problem

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**Abstract**

We consider a mathematical model which describes the quasistatic contact between a viscoelastic body and a deformable obstacle, the so-called foundation. The material's behaviour is modelled with a viscoelastic constitutive law with long memory. The contact is frictionless and is defined using a multivalued normal compliance condition. We present a regularization method in the study of a class of variational inequalities involving history-dependent operators. Finally, we apply the abstract results to analyse the contact problem.

## Keywords

variational inequality; regularization; weak solution; normal compliance

## I. Introduction

In this paper we introduce two novelties that we describe in what follows. First, we state and prove an abstract regularization result for a class of history-dependent variational inequalities in Hilbert spaces. More exactly, we continue the analysis provided in [1, Section 2.2.3], where regularization arguments are used to prove the unique solvability of the following variational inequality

$$u \in X, \quad (Au, v-u)_X + j(v) - j(u) \geq (f, v-u)_X, \qquad \text{for all } v \in X.$$

Here $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ denotes a real Hilbert space, $A : X \to X$ is a Lipschitz continuous and strongly monotone operator, $j : X \to \mathbb{R}$ is a convex, lower-semicontinuous functional and $f \in X$. With respect to [1] we study the unique solvability of the following history-dependent variational inequality

$$\begin{aligned}&(Au(t), v-u(t))_X + (\mathcal{S}u(t), v-u(t))_X \\ &\quad + j(v) - j(u(t)) \geq (f(t), v-u(t))_X \qquad \text{for all } v \in X\end{aligned} \tag{1.1}$$

using a regularization method involving Gâteaux differentiable functionals. Here, both the data and the solution $u$ depend on the time variable $t \in [0,T]$, where $T > 0$. Moreover, concerning the operator $\mathcal{S}$, the current value $\mathcal{S}u(t)$ at moment $t$ depends on the history of the values of $u$ at moments $0 \le s \le t$. Its presence implies the use of Gronwall's inequality or Lebesgue's convergence theorem.

We consider a contact problem which describes the frictionless contact between a viscoelastic body and a deformable foundation. We model the material's behaviour with a constitutive law with long memory of the

form

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \tag{1.2}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor, $\mathcal{A}$ is the elasticity operator and $\mathcal{B}$ represents the relaxation tensor. Moreover, we assume that the contact process is quasistatic and we study it in the interval of time $[0,T]$. Finally, we use a multivalued normal compliance condition to describe the contact process in normal direction.

This mathematical model was considered in [2]. There, the variational formulation was derived and unique weak solvability was proved. Moreover, the weak solution was approximated using a penalty method. Also, in [3] the dependence of the solution with respect to the data was studied and a fully discrete scheme was introduced. In addition, an optimal order error estimate was derived and numerical simulations were provided for a two-dimensional test problem.

The second novelty of the paper arises from the fact that we analyse the weak solvability of contact problem using regularization arguments. With respect to [2,3] we introduce a regularized contact problem where a single-valued normal compliance condition, including a regularization parameter, is considered. We prove that the weak solution of the regularized problem converges to the weak solution of the original contact problem, as the regularization parameter goes to zero. To this end, we apply the abstract regularization results obtained in the study of (1.1).

Next, we recall that general results on analysis of various classes of variational inequalities, including existence, uniqueness and regularity, can be found in [4-9]. The numerical analysis of variational inequalities, including solution algorithms and error estimates, was treated in [10, 11] and also in [12, 13]. Additional results on optimal control of variational inequalities can be found in [14]. An early attempt to study contact problems within the framework of variational inequalities was made in [15]. The variational analysis of contact models is given in [1, 9, 13, 16, 17]. There, the mathematical analysis of contact problems is provided, including existence and uniqueness results of the weak

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solution. Numerical analysis of contact models, including the study of fully discrete scheme, error estimates and numerical simulations, can be found in [12, 13, 17, 18]. An excellent reference in the study of frictional contact associated with heat generation, material damage, wear and adhesion of contacting surfaces is [19].

The rest of the paper is structured as follows. In Section 2 we describe the classical formulation of the model and list the assumptions on the data. In Section 3 we introduce the regularized problem and derive the weak formulations. Then, we state our main existence, uniqueness and convergence result, Theorem 3.1. In Section 4 we provide the abstract problem and recall its unique solvability obtained in [20]. Then we consider the associated regularized problem and state our main abstract result, Theorem 4.3. The proof of this theorem is given in Section 5. Finally, in Section 6 we illustrate the use of abstract arguments in the proof of Theorem 3.1.

## 2. A viscoelastic contact problem

In this section we introduce the classical formulation of the contact problem and list the assumptions on the data. First of all, we present the notation we shall use and preliminaries related to the contact model. More exactly, we denote by $\mathbb{S}^d(d=1,2,3)$ the space of second-order symmetric tensors on $\mathbb{R}^d$ and we define the following inner products and norms

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\quad \|\boldsymbol{v}\|=(\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}}\quad \text{for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij},\quad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}}\quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d$$

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\Gamma$ and let $\Gamma_1,\Gamma_2$ and $\Gamma_3$ be three measurable parts of $\Gamma$ such that meas $(\Gamma_1)>0$. We use the notation $\boldsymbol{x}=(x_i)$ for a typical point in $\Omega\cup\Gamma$ and we denote by $\boldsymbol{v}=(v_i)$ the outward unit normal at $\Gamma$. The indices $i,j,k,l$ run between 1 and $d$ and the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j}=\partial u_i/\partial x_j$. Moreover, we consider the Hilbert spaces

$$V=\left\{\boldsymbol{v}=(v_i)\in H^1(\Omega)^d:\boldsymbol{v}=\boldsymbol{0}\text{ on }\Gamma_1\right\}$$
$$Q=\left\{\boldsymbol{\tau}=(\tau_{ij})\in L^2(\Omega)^{d\times d}:\tau_{ij}=\tau_{ji}\right\}$$

endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})dx,\quad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega\boldsymbol{\sigma}\cdot\boldsymbol{\tau}dx$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\varepsilon(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v}))\,, \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \qquad \text{for all } \boldsymbol{v} \in H^1(\Omega)^d.$$

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on $\Gamma$, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{v}, \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{v}$. We recall that there exists a positive constant $c_0$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0 \|\boldsymbol{v}\|_V \qquad \text{for all } \boldsymbol{v} \in V. \tag{2.1}$$

For a function $\boldsymbol{\sigma} \in Q$ we use the notation $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ for the normal and tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{v}) \cdot \boldsymbol{v}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{v} - \sigma_\nu \boldsymbol{v}$.

We denote by $\mathbf{Q}_\infty$ the space of fourth-order tensor fields given by

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d\}$$

and it is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1 \le i,j,k,l \le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$$

Moreover,

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q, \qquad \text{for all } \mathcal{E} \in \mathbf{Q}_\infty, \boldsymbol{\tau} \in Q. \tag{2.2}$$

We present the physical setting of the problem. A viscoelastic body occupies the bounded domain $\Omega$ described above. The body is subjected to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on $\Gamma_1$ and surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2$. On $\Gamma_3$ the body is in frictionless contact with a deformable foundation. The contact process is quasistatic and we study it in the interval of time $[0, T]$.

The classical formulation of the contact problem is the following.
Problem $\mathcal{Q}$. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \text{ in } \Omega \tag{2.3}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \mathbf{0} \text{ in } \Omega \tag{2.4}$$

$$\boldsymbol{u}(t) = \mathbf{0} \text{ on } \Gamma_1 \tag{2.5}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{v} = \boldsymbol{f}_2(t) \text{ on } \Gamma_2 \tag{2.6}$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \text{ on } \Gamma_3 \tag{2.7}$$

and there exists $\xi : \Omega \times [0, T] \to \mathbb{R}$ which satisfies

$$\left.\begin{array}{l}\sigma_\nu(t)+p\left(u_\nu(t)\right)+\xi(t)=0 \\ 0 \leq \xi(t) \leq F \\ \xi(t)=0 \quad \text{if } u_\nu(t)<0 \\ \xi(t)=F \quad \text{if } u_\nu(t)>0\end{array}\right\} \quad \text{on } \Gamma_3 \tag{2.8}$$

for all $t \in [0, T]$.

Next, we give a short description of Problem $\mathcal{Q}$ and list the assumptions on the data. Here and below we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Equation (2.3) represents
the viscoelastic constitutive law with long memory introduced in Section 1. The operators $\mathcal{A}$ and $\mathcal{B}$ verify the following conditions

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(a) $\mathcal{A}: \Omega \times \mathbb{S}^{d} \rightarrow \mathbb{S}^{d}$.
(b) There exists $L_{\mathcal{A}}>0$ such that
    $\left\|\mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_{1}\right)-\mathcal{A}
        for all $\boldsymbol{\varepsilon}_{1}, \boldsymbol{\varepsilon}_{2} \in \mathbb{S}^{
(c) There exists $m_{\mathcal{A}}>0$ such that
    $\left(\mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_{1}\right)-\mathcal{A}\
        for all $\boldsymbol{\varepsilon}_{1}, \boldsymbol{\varepsilon}_{2} \in \mathbb{S}^{
(d) The mapping $\boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}
(e) The mapping $\boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \mathbf{0})$ belongs to
        $\mathcal{B} \in C\left([0, T] ; \mathbf{Q}_{\infty}\right)$.
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Various examples and mechanical interpretation concerning constitutive laws in solid mechanics can be found in [1, 16, 19]. Equation (2.4) represents the equation of equilibrium in which Div denotes the divergence operator for tensor-valued functions

$$\operatorname{Div} \boldsymbol{\sigma} = \sigma_{ij,j} \quad \text{for all } \boldsymbol{\sigma} \in Q.$$

Conditions (2.5) and (2.6) are the displacement boundary condition and traction boundary condition, respectively. We assume that the densities of body forces and surface tractions have regularity

$$\boldsymbol{f}_0 \in C\left([0,T]; L^2(\Omega)^d\right), \quad \boldsymbol{f}_2 \in C\left([0,T]; L^2\left(\Gamma_2\right)^d\right). \tag{2.11}$$

Equation (2.7) represents the frictionless condition. Frictionless contact problems were considered, for example, in [1, 19, 21].

Next, (2.8) is the contact condition in which $\sigma_\nu$ denotes the normal stress and $u_\nu$ is the normal displacement. Moreover, the functions $p$ and $F$ satisfy
(a) $p : \mathbb{R} \to \mathbb{R}_+$.
(b) There exists $L_p > 0$ such that

$$\left|p\left(r_1\right)-p\left(r_2\right)\right| \leq L_p\left|r_1-r_2\right| \quad \text{for all } r_1, r_2 \in \mathbb{R} \tag{2.12}$$

(c) $\left(p\left(r_1\right)-p\left(r_2\right)\right)\left(r_1-r_2\right) \geq 0 \quad$ for all $r_1, r_2 \in \mathbb{R}$.
(d) $p(r) = 0$ iff $r \leq 0$.

$$F \in L^{\infty}(\Gamma_3), \quad F(\boldsymbol{x}) \geq 0 \text{ almost every } \boldsymbol{x} \in \Gamma_3. \tag{2.13}$$

This condition can be derived in the following way. Let $t \in [0, T]$. We consider that the normal stress has an additive decomposition of the form

$$\sigma_v(t) = \sigma_v^D(t) + \sigma_v^M(t) \quad \text{ on } \Gamma_3, \tag{2.14}$$

in which $\sigma_\nu^D(t)$ describes the deformability of foundation and $\sigma_\nu^M(t)$ describes the rigidity. We assume that $\sigma_\nu^D(t)$ satisfies a normal compliance contact condition

$$-\sigma_v^D(t) = p\left(u_v(t)\right) \quad \text{ on } \Gamma_3. \tag{2.15}$$

We recall that the normal compliance contact condition was first used in [22] and the term normal compliance was first introduced in [23,24].

Finally, $\sigma_v^M(t)$ satisfies

$$\left\{ \begin{array}{lll} \left|\sigma_v^M(t)\right| \leq F, & \sigma_v^M(t) = 0 & \text{if } u_v(t) < 0, \\ -\sigma_v^M(t) = F & & \text{if } u_v(t) > 0 \end{array} \right. \text{ on } \Gamma_3. \tag{2.16}$$

We combine (2.14)-(2.16) and write $-\sigma_v^M(t) = \xi(t)$ to obtain (2.8). A version of this condition including unilateral constraints was used in [2,3].

Figure I. Representation of the contact condition (2.8)

We present additional details of the contact condition (2.8) which is depicted in Figure 1. We assume that at a given moment $t$ there is a separation between the body and the foundation, i.e. $u_v(t) < 0$. Then, (2.12) part (d) and (2.8) show that $\sigma_\nu(t) = 0$, i.e. the reaction of foundation vanishes. Assume now that at the moment $t$ there is penetration, i.e. $u_\nu(t) > 0$. Then, (2.8) yields

$$-\sigma_v(t) = p\left(u_v(t)\right) + F. \tag{2.17}$$

This equality implies that, at the moment $t$, the reaction of the foundation depends on the penetration and represents a normal compliance-type condition. Moreover, (2.8) shows that if at the moment $t$ we have penetration, then $-\sigma_\nu(t) \geq F$. Indeed, if $u_\nu(t) > 0$, then (2.17) holds and this implies that $-\sigma_\nu(t) \geq F$. We conclude that if $-\sigma_\nu(t) < F$, then there is no penetration and $F$ represents a yield limit of the normal pressure, under which the penetration is not possible. This kind of behaviour characterizes a rigid-elastic foundation.

In conclusion, condition (2.8) shows that when there is a separation between the body's surface and the foundation then the normal stress vanishes; the penetration occurs only if the normal stress reaches the critical value $F$; when there is penetration, the contact follows a multivalued normal compliance contact condition. It can be interpreted physically as follows. The foundation is assumed to be made of a rigid-elastic material which allows penetration, but only if the normal stress arrives to the yield limit $F$.

## 3. Variational formulation and regularization of Problem $\mathcal{Q}$

In this section we derive the variational formulation of Problem $\mathcal{Q}$. Moreover, we consider a regularized problem and state an existence, uniqueness and convergence result. To this end, let $t \in [0,T], \rho > 0$ and $\boldsymbol{v} \in V$ be given. We assume in what follows that ( $\boldsymbol{u}, \boldsymbol{\sigma}$ ) are sufficiently regular functions that satisfy (2.3)-(2.8). We recall the following Green's formula

$$\int_{\Omega} \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) dx + \int_{\Omega} \operatorname{Div} \boldsymbol{\sigma}(t) \cdot \boldsymbol{v} dx = \int_{\Gamma} \boldsymbol{\sigma}(t) \boldsymbol{v} \cdot \boldsymbol{v} da \quad \text{for all } \boldsymbol{v} \in V \tag{3.1}$$

Next, we take $\boldsymbol{v} := \boldsymbol{v} - \boldsymbol{u}(t)$ in (3.1) and use (2.4) to see that

$$\int_{\Omega} \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) dx = \int_{\Omega} \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) dx + \int_{\Gamma} \boldsymbol{\sigma}(t) \boldsymbol{v} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) da.$$

We split the surface integral over $\Gamma_1, \Gamma_2$ and $\Gamma_3$ and taking into account (2.7) we obtain

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) dx &= \int_{\Omega} \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) dx \\ + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) da &+ \int_{\Gamma_3} \sigma_v(t) \left(v_v - u_v(t)\right) da \end{aligned} \tag{3.2}$$

We use (2.8) and assumption (2.13) to deduce that

$$F\left(v_v^+ - u_v^+(t)\right) \geq \xi(t) \left(v_v - u_v(t)\right) \quad \text{on } \Gamma_3, \tag{3.3}$$

where $r^+ = \max\{r, 0\}$ for all $r \in \mathbb{R}$. We again use (2.8) and (3.3) to find that

$$\begin{aligned} &\int_{\Gamma_3} \sigma_\nu(t) \left(v_\nu - u_\nu(t)\right) da \\ &\quad \geq - \int_{\Gamma_3} p\left(u_\nu(t)\right) \left(v_\nu - u_\nu(t)\right) da - \int_{\Gamma_3} F\left(v_\nu^+ - u_\nu^+(t)\right) da \end{aligned} \tag{3.4}$$

We apply Riesz representation theorem to define the function $\boldsymbol{f} : [0,T] \to V$ by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \text{for all } \boldsymbol{v} \in V. \tag{3.5}$$

It follows from (2.11) that this function has regularity

$$\boldsymbol{f} \in C([0,T];V). \tag{3.6}$$

Finally, we combine equality (3.2), inequality (3.4), constitutive law (2.3) and (3.5) to obtain the following variational formulation of Problem $\mathcal{Q}$.

Problem $\mathcal{Q}^V$. Find a displacement field $\boldsymbol{u} : [0,T] \to V$ such that, for all $t \in [0,T]$, the following inequality holds:

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \left(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\right)_Q \tag{3.7}$$
$$+ \int_{\Gamma_3} p\left(u_v(t)\right)\left(v_v - u_v(t)\right) da + \int_{\Gamma_3} F\left(v_v^+ - u_v^+(t)\right) da \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in V$$

The previous variational formulation leads to a history-dependent variational inequality involving a nondifferentiable term. To avoid this difficulty, we use a regularity procedure. Moreover, condition (2.8) can be written in the following equivalent form

$$\left.\begin{array}{l} \sigma_\nu(t) + p\left(u_\nu(t)\right) + \xi(t) = 0 \\ 0 \leq \xi(t) \leq F \\ \xi(t) = F\frac{u_\nu^+(t)}{|u_\nu(t)|} = \frac{F}{2}\left(1 + \frac{u_\nu(t)}{|u_\nu(t)|}\right) \text{ if } u_\nu(t) \neq 0 \end{array}\right\} \text{ on } \Gamma_3. \tag{3.8}$$

Replacing the non-differentiable absolute value $|u_v(t)|$ we obtain the following regularized contact problem.

Problem $\mathcal{Q}_\rho$. Find a displacement field $\boldsymbol{u}_\rho : \Omega \times [0,T] \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}_\rho : \Omega \times [0,T] \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\rho(t)\right) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\rho(s)\right) ds \text{ in } \Omega \tag{3.9}$$

$$\operatorname{Div} \boldsymbol{\sigma}_\rho(t) + \boldsymbol{f}_0(t) = \mathbf{0} \text{ in } \Omega \tag{3.10}$$

$$\boldsymbol{u}_\rho(t) = \mathbf{0} \text{ on } \Gamma_1 \tag{3.11}$$

$$\boldsymbol{\sigma}_\rho(t)\boldsymbol{v} = \boldsymbol{f}_2(t) \text{ on } \Gamma_2 \tag{3.12}$$

$$\boldsymbol{\sigma}_{\rho\tau}(t) = \mathbf{0} \text{ on } \Gamma_3 \tag{3.13}$$

$$\sigma_{\rho v}(t) + p\left(u_{\rho v}(t)\right) + \frac{F}{2}\left(1 + \frac{u_{\rho v}(t)}{\sqrt{u_{\rho v}(t)^2 + \rho^2}}\right) = 0 \text{ on } \Gamma_3 \tag{3.14}$$

for all $t \in [0,T]$.

Note that here and below $u_{\rho\nu}$ is the normal component of the displacement field $\boldsymbol{u}_\rho$ and $\sigma_{\rho\nu}, \boldsymbol{\sigma}_{\rho\tau}$ represent the normal and tangential components of the stress tensor $\boldsymbol{\sigma}_\rho$, respectively. The equations and boundary conditions in problem (3.9)-(3.14) have a similar interpretation as those in problem (2.3)-(2.8). The difference arises in the fact that here we replace the multivalued normal compliance contact condition (2.8) with the single-valued normal compliance condition (3.14). In this condition $\rho$ represents a regularization parameter. We recall that regularizations of Coulomb friction law were considered in [16,25].

Using similar arguments as those used in the case of Problem $\mathcal{Q}$, we obtain the following variational formulation of Problem $\mathcal{Q}_\rho$.
Problem $\mathcal{Q}_\rho^V$. Find a displacement field $\boldsymbol{u}_\rho : [0,T] \to V$ such that, for all $t \in [0,T]$, the following equality holds:

$$\left(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\rho(t)\right), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_Q + \left(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}\left(\boldsymbol{u}_\rho(s)\right) ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_Q \tag{3.15}$$

$$+ \int_{\Gamma_3} p\left(u_{\rho v}(t)\right) v_v da + \int_{\Gamma_3} \frac{F}{2}\left(1 + \frac{u_{\rho v}(t)}{\sqrt{u_{\rho v}^2(t) + \rho^2}}\right) v_v da = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V$$

We have the following existence, uniqueness and convergence result.
Theorem 3.1 Assume that (2.9)-(2.13) hold. Then:
(1) Problem $\mathcal{Q}^V$ has a unique solution $\boldsymbol{u} \in C([0,T];V)$;
(2) for each $\rho > 0$ Problem $\mathcal{Q}_\rho^V$ has a unique solution $\boldsymbol{u}_\rho \in C([0,T];V)$;
(3) the solution of Problem $\mathcal{Q}_\rho^V$ converges to the solution of Problem $\mathcal{Q}^V$, that is

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0 \tag{3.16}$$

for all $t \in [0,T]$.
The convergence result (3.16) is important from a mechanical point of view. More exactly, it shows that the weak solution of viscoelastic contact problem with multivalued normal compliance contact condition may be approached as closely as one wishes by the solution of a viscoelastic contact problem with single-valued normal compliance condition, with a sufficiently small regularization parameter. Note that convergence (3.16) above is understood in the following sense: for each $t \in [0,T]$ and for every sequence $\{\rho_n\} \subset \mathbb{R}_+$converging to zero as $n \to \infty$ we have $\boldsymbol{u}_{\rho_n}(t) \to \boldsymbol{u}(t)$ as $n \to \infty$. The proof of Theorem 3.1 is given in Section 6. To this end, we use the abstract result introduced in Section 4 and proved in Section 5.

# 4. Abstract problem

In this section we state our main abstract result, Theorem 4.3. It represents an extension of Theorem 2.12 of [1] to a class of history-dependent variational inequalities. We consider a real Hilbert space $X$ with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$ and we use notation $C([0,T];X)$ for the space of continuous functions defined on $[0,T]$ with values in $X$. Moreover, let $A : X \to X, \mathcal{S} : C([0,T];X) \to C([0,T];X)$ be two operators, let the functional $j : X \to \mathbb{R}$ and the function $f : [0,T] \to X$. We assume in what follows that $A$ is a strongly monotone and Lipschitz continuous operator.

$$
\begin{cases}
\text{(a) There exists } m_A > 0 \text{ such that} \\
\quad (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \\
\quad \text{for all } u_1, u_2 \in X. \\
\text{(b) There exists } M_A > 0 \text{ such that} \\
\|Au_1 - Au_2\|_X \leq M_A \|u_1 - u_2\|_X \quad \text{for all } u_1, u_2 \in X.
\end{cases}
\tag{4.1}
$$

Moreover, we assume that operator $\mathcal{S}$ satisfies the following condition.

$$
\begin{array}{l}
\text{There exists } L_S > 0 \text{ such that} \\
\quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \leq L_S \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \\
\quad \text{for all } u_1, u_2 \in C([0,T]; X), \text{ for all } t \in [0,T].
\end{array}
\tag{4.2}
$$

Finally, we suppose that

$$
j : X \to \mathbb{R} \text{ is a convex lower semicontinuous function} \tag{4.3}
$$

and

$$
f \in C([0,T]; X). \tag{4.4}
$$

We consider the following problem.

Problem $\mathcal{P}$. Find a function $u : [0,T] \to X$ such that, for all $t \in [0,T]$, the following inequality holds:

$$
\begin{array}{l}
(Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \\
\quad + j(v) - j(u(t)) \geq (f(t), v - u(t))_X \quad \text{for all } v \in X
\end{array}
\tag{4.5}
$$

Following the terminology introduced in [1, 20] we refer to operator $\mathcal{S}$, which satisfies (4.2), as a historydependent operator. In addition, we refer to (4.5) as a history-dependent variational-inequality.

The unique solvability of Problem $\mathcal{P}$ is provided by the following existence and uniqueness result.

Theorem 4.1 Let $X$ be a Hilbert space and assume that (4.1)-(4.4) hold. Then, Problem $\mathcal{P}$ has a unique solution $u \in C([0,T]; X)$.

Theorem 4.1 was proved in [20] by using fixed-point arguments. It represents a crucial tool in studying the weak solvability of a large number of contact problems. We send the reader to [1] for more details.

In the particular case $j \equiv 0$ we have the following consequence of Theorem 4.1.

Corollary 4.2 Let $X$ be a Hilbert space and assume that (4.1)-(4.2) and (4.4) hold. Then, there exists a unique function $u \in C([0,T]; X)$ which satisfies the following equality

$$
(Au(t), v)_X + (\mathcal{S}u(t), v)_X = (f(t), v)_X \quad \text{for all } v \in X.
$$

Let $\rho > 0$ be a parameter. In order to formulate the regularized problem associated to Problem $\mathcal{P}$ we consider the following family of functionals $(j_\rho)$ which satisfies

$$\left\{\begin{array}{l}\text{(a) } j_\rho : X \to \mathbb{R} \text{ is convex Gâteaux differentiable, for each } \rho > 0. \\ \text{(b) } \nabla j_\rho : X \to X \text{ is a Lipschitz continuous operator, for each } \rho > 0.\end{array}\right. \tag{4.6}$$

$$\left\{\begin{array}{l}\text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{such that} \\ \text{(a) } |j_\rho(v) - j(v)| \le G(\rho) \quad \text{for all } v \in X, \text{ for each } \rho > 0. \\ \text{(b) } \lim_{\rho \to 0} G(\rho) = 0.\end{array}\right. \tag{4.7}$$

The assumption made in (4.7) requires the functional $j$ to be approached by a sequence of more regular functionals ( $j_\rho$ ), since they are Gateaux differentiable. Next, we consider the following problem.
Problem $\mathcal{P}_\rho$. Find $u_\rho \in C([0,T];X)$ such that, for all $t \in [0,T]$, equality below holds

$$(Au_\rho(t), v)_X + (\mathcal{S}u_\rho(t), v)_X + (\nabla j_\rho(u_\rho(t)), v)_X = (f(t), v)_X \quad \text{for all } v \in X \tag{4.8}$$

We have the following existence, uniqueness and convergence result.
Theorem 4.3 Let $X$ be a Hilbert space and assume that (4.1)-(4.4), (4.6) and (4.7) hold. Then:
(1) for each $\rho > 0$ Problem $\mathcal{P}_\rho$ has a unique solution;
(2) the solution of Problem $\mathcal{P}_\rho$ converges to the solution of Problem $\mathcal{P}$, that is

$$\|u_\rho(t) - u(t)\|_X \to 0 \quad \text{as } \rho \to 0 \tag{4.9}$$

for all $t \in [0,T]$.

The main feature of Theorem 4.3 consists of showing that the solution of an 'irregular' history-dependent inequality may be approached as limit of the solutions of 'regular' history-dependent variational equalities.

# 5. Proof of Theorem 4.3

In this section we give in several steps the proof of Theorem 4.3. First of all, we provide the unique solvability of (4.8).

Lemma 5.1 For each $\rho > 0$ there exists a unique function $u_\rho \in C([0,T];X)$ which satisfies (4.8) for all $t \in [0,T]$.

Proof. Let $\rho > 0$. Taking into account assumption (4.6) we have that $\nabla j_\rho$ is a monotone operator on $X$. Moreover, using (4.1) and (4.6) we deduce that the operator $A_\rho : X \to X$ defined by

$$A_\rho v := Av + \nabla j_\rho(v) \quad \text{for all } v \in X \tag{5.1}$$

is a strongly monotone and Lipschitz continuous operator on $X$. Lemma 5.1 is now a consequence of Corollary 4.2.

Next, we consider the following intermediate problem.

Problem $\widetilde{\mathcal{P}}_\rho$. Find $\widetilde{u}_\rho : [0,T] \to X$ such that, for all $t \in [0,T]$, the following equality holds:

$$(A\widetilde{u}_\rho(t), v)_X + (\mathcal{S}u(t), v)_X + (\nabla j_\rho(\widetilde{u}_\rho(t)), v)_X = (f(t), v)_X \quad \text{for all } v \in X. \tag{5.2}$$

Note that the difference between (4.8) and (5.2) arises in the fact that in (5.2) the operator $S$ is applied to a known function $u$, which is the solution of (4.5). Thus, equality (5.2) is a time-dependent variational equality. In contrast, (4.8) is a history-dependent variational equality, since the operator $\mathcal{S}$ is applied to the unknown function $u_\rho$.

We have the following existence and uniqueness result.

Lemma 5.2 For each $\rho > 0$ there exists a unique function $\widetilde{u}_\rho \in C([0,T];X)$ which satisfies (5.2) for all $t \in [0,T]$.

Proof. In addition to the operator $A_\rho$, given in (5.1), we define the function $\widetilde{f} : [0,T] \to X$ by equality

$$(\tilde{f}(t), v)_X := (f(t), v)_X + (\mathcal{S}u(t), v)_X \quad \text{for all } v \in X, t \in [0,T].$$

We use assumptions (4.2), (4.4) and Corollary 4.2 to conclude the proof.

The next step is provided by the following weak convergence result.

Lemma 5.3 For each $t \in [0,T]$ the sequence $\{\widetilde{u}_\rho(t)\}$ converges weakly to $u(t)$, i.e.

$$\widetilde{u}_\rho(t) \rightharpoonup u(t) \quad \text{in } X \text{ as } \rho \to 0. \tag{5.3}$$

Proof. Let $\rho > 0$ and $t \in [0,T]$. We use (5.2) with $v := v - \widetilde{u}_\rho(t)$ to obtain

$$\begin{aligned}&(A\widetilde{u}_\rho(t), v - \widetilde{u}_\rho(t))_X + (\mathcal{S}u(t), v - \widetilde{u}_\rho(t))_X \\ &\quad + (\nabla j_\rho(\widetilde{u}_\rho(t)), v - \widetilde{u}_\rho(t))_X = (f(t), v - \widetilde{u}_\rho(t))_X \quad \text{for all } v \in X.\end{aligned} \tag{5.4}$$

Moreover, taking into account assumption (4.6) we deduce that

$$\begin{aligned}&(A\widetilde{u}_\rho(t), v - \widetilde{u}_\rho(t))_X + (\mathcal{S}u(t), v - \widetilde{u}_\rho(t))_X \\ &\quad + j_\rho(v) - j_\rho(\widetilde{u}_\rho(t)) \ge (f(t), v - \widetilde{u}_\rho(t))_X \quad \text{for all } v \in X.\end{aligned} \tag{5.5}$$

We now take $v := 0_X$ in (5.5) and using assumption (4.7) we have

$$\begin{aligned}(A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t))_X \le& -(\mathcal{S}u(t), \widetilde{u}_\rho(t))_X \\ &+ j(0_X) - j(\widetilde{u}_\rho(t)) + (f(t), \widetilde{u}_\rho(t))_X + 2G(\rho).\end{aligned} \tag{5.6}$$

Assumption (4.3) implies that the functional $j$ is bounded below by an affine function, i.e there exists $w \in X$ and $\alpha \in \mathbb{R}$, which do not depend on $t$, such that

$$j(v) \geq (\omega, v)_X + \alpha \quad \text{for all } v \in X. \tag{5.7}$$

Therefore, we deduce that

$$\begin{aligned}(A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t))_X &\leq j(0_X) - (\omega, \widetilde{u}_\rho(t))_X - \alpha \\ &- (\mathcal{S}u(t), \widetilde{u}_\rho(t))_X + (f(t), \widetilde{u}_\rho(t))_X + 2G(\rho).\end{aligned} \tag{5.8}$$

Using assumption (4.1) and Cauchy-Schwarz inequality it follows that

$$\begin{aligned}m_A \|\widetilde{u}_\rho(t)\|_X^2 \leq &(\|A0_X\|_X + \|\omega\|_X + \|\mathcal{S}u(t)\|_X + \|f(t)\|_X) \|\widetilde{u}_\rho(t)\|_X \\ &+ |\alpha| + |j(0_X)| + 2G(\rho).\end{aligned} \tag{5.9}$$

We use (5.9) and inequality $[x, a, b \geq 0 \text{ and } x^2 \leq ax + b \Longrightarrow x^2 \leq a^2 + 2b]$ to deduce that the sequence $\{\widetilde{u}_\rho(t)\}$ is bounded, i.e. there exists $c > 0$, which does not depend on $\rho$, such that

$$\|\widetilde{u}_\rho(t)\|_X \leq c. \tag{5.10}$$

Therefore, it follows that there exists a subsequence of the sequence $\{\widetilde{u}_\rho(t)\}$, still denoted by $\{\widetilde{u}_\rho(t)\}$ and an element $\widetilde{u}(t) \in X$ such that

$$\widetilde{u}_\rho(t) \rightharpoonup \widetilde{u}(t) \quad \text{in } X \text{ as } \rho \to 0. \tag{5.11}$$

In the second part of the proof we investigate the properties of the element $\widetilde{u}(t) \in X$. To this end, we take $v := \widetilde{u}(t)$ in (5.5) and using assumption (4.7) we have

$$\begin{aligned}(A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - \widetilde{u}(t))_X &\leq (\mathcal{S}u(t), \widetilde{u}(t) - \widetilde{u}_\rho(t))_X \\ &+ j(\widetilde{u}(t)) - j(\widetilde{u}_\rho(t)) + 2G(\rho) + (f(t), \widetilde{u}_\rho(t) - \widetilde{u}(t))_X .\end{aligned} \tag{5.12}$$

Next, we pass to the upper limit as $\rho \to 0$ and taking into account (5.11), (4.3) and (4.7) we deduce that

$$\limsup_{\rho \to 0} (A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - \widetilde{u}(t))_X \leq 0 \tag{5.13}$$

Assumption (4.1) and convergence (5.11) yield

$$\liminf_{\rho \to 0} (A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v)_X \geq (A\widetilde{u}(t), \widetilde{u}(t) - v)_X \quad \text{for all } v \in X \tag{5.14}$$

Next, from inequality (5.5) and assumption (4.7) we find that

$$(A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v)_X + (f(t), v - \widetilde{u}_\rho(t))_X + j\left(\widetilde{u}_\rho(t)\right) \le (\mathcal{S}u(t), v - \widetilde{u}_\rho(t))_X + j(v) + 2G(\rho) \quad \text{for all } v \in X \tag{5.15}$$

We pass to the lower limit as $\rho \to 0$ in (5.15) and use assumption (4.7), convergence (5.11) and the lower semicontinuity of $j$. As a result we have

$$\liminf_{\rho \to 0} (A\widetilde{u}_\rho(t), \widetilde{u}_\rho(t) - v)_X \le (\mathcal{S}u(t), v - \widetilde{u}(t))_X + j(v) - j(\widetilde{u}(t)) + (f(t), \widetilde{u}(t) - v)_X \quad \text{for all } v \in X \tag{5.16}$$

We now combine inequalities (5.14) and (5.16) to see that

$$(A\widetilde{u}(t), v - \widetilde{u}(t))_X + (\mathcal{S}u(t), v - \widetilde{u}(t))_X + j(v) - j(\widetilde{u}(t)) \ge (f(t), v - \widetilde{u}(t))_X \quad \text{for all } v \in X \tag{5.17}$$

Next, we take $v := \widetilde{u}(t)$ in (4.5) and $v := u(t)$ in (5.17). Then, adding the resulting inequalities and using assumption (4.1) we obtain that

$$u(t) = \widetilde{u}(t) \tag{5.18}$$

Lemma 5.3 is now a consequence of standard weak convergence arguments. We proceed with the following strong convergence result.
Lemma 5.4 For each $t \in [0, T]$ the sequence $\{\widetilde{u}_\rho(t)\}$ converges strongly in $X$ to $u(t)$, that is

$$\tilde{u}_\rho(t) \to u(t) \quad \text{in } X \text{ as } \rho \to 0. \tag{5.19}$$

Proof. The proof is obtained taking $v := \widetilde{u}(t)$ in (5.14) and using (5.13), (5.18), (4.1) and (5.3).
The last step is provided by the following strong convergence result.
Lemma 5.5 For each $t \in [0, T]$ the sequence $\{u_\rho(t)\}$ converges strongly in $X$ to $u(t)$, that is

$$u_\rho(t) \to u(t) \quad \text{in Xas } \rho \to 0 \tag{5.20}$$

Proof. We use (4.8) to obtain

$$(Au_\rho(t), v - u_\rho(t))_X + (\mathcal{S}u_\rho(t), v - u_\rho(t))_X + \left(\nabla j_\rho\left(u_\rho(t)\right), v - u_\rho(t)\right)_X = (f(t), v - u_\rho(t))_X \quad \text{for all } v \in X \tag{5.21}$$

Next, we take $v := \widetilde{u}_\rho(t)$ in (5.21) and $v := u_\rho(t)$ in (5.4). Then, adding the resulting equalities and using (4.1), (4.6) and Cauchy-Schwarz inequality we deduce that

$$\|u_\rho(t) - \widetilde{u}_\rho(t)\|_X^2 \leq \frac{1}{m_A} \|\mathcal{S}u_\rho(t) - \mathcal{S}u(t)\|_X \|\widetilde{u}_\rho(t) - u_\rho(t)\|_X \tag{5.22}$$

Next, we use (4.2), triangle inequality and a Gronwall's argument to obtain

$$\|u_\rho(t) - u(t)\|_X \leq \|\widetilde{u}_\rho(t) - u(t)\|_X + \frac{L_S}{m_A} e^{\frac{TL_S}{m_A}} \int_0^t \|\widetilde{u}_\rho(s) - u(s)\|_X \, ds \tag{5.23}$$

We now use (5.10), (5.23), Lemma 5.4 and Lebesgue's convergence theorem to obtain (5.20), which concludes the proof.

We are now in position to present the proof of Theorem 4.3.

Proof. (1) The unique solvability of Problem $\mathcal{P}_\rho$ is a consequence of Lemma 5.1.
(2) The convergence (4.9) is a consequence of Lemma 5.5.

Therefore, we conclude that the proof of Theorem 4.3 is complete.

## 6. Proof of Theorem 3.1

In this section we give the proof of Theorem 3.1. To this end, we use Theorem 4.3 with $X = V$. First of all we define operators $A : V \to V, \mathcal{S} : C([0,T]; V) \to C([0,T]; V)$, and the functional $j : V \to \mathbb{R}$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} p\,(u_v)\, v_v da \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \tag{6.1}$$

$$(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V = \left( \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \right)_Q \tag{6.2}$$

for all $\boldsymbol{u} \in C([0,T]; V), \boldsymbol{v} \in V$

$$j(\boldsymbol{v})_- = \int_{\Gamma_3} F v_v^+ da \quad \text{for all } \boldsymbol{v} \in V \tag{6.3}$$

Then, it is easy to see that Problem $\mathcal{Q}^V$ is equivalent to the problem of finding a function $\boldsymbol{u} : [0,T] \to V$ such that, for all $t \in [0,T]$, the following inequality holds

$$\begin{aligned} &(A\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ &\quad + j(\boldsymbol{v}) - j(\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in V. \end{aligned} \tag{6.4}$$

Moreover, for each $\rho > 0$, we define operator $P_\rho : V \to V$ by

$$(P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} \frac{F}{2} \left( 1 + \frac{u_v}{\sqrt{u_v^2 + \rho^2}} \right) v_v da \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \tag{6.5}$$

Therefore, Problem $\mathcal{Q}_\rho^V$ is equivalent to the problem of finding a function $\boldsymbol{u}_\rho : [0,T] \to V$ such that, for all $t \in [0,T]$, the following equality holds

$$(A\boldsymbol{u}_\rho(t), \boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}_\rho(t), \boldsymbol{v})_V + (P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{6.6}$$

Using assumptions (2.9), (2.12) and inequality (2.1) we deduce that the operator $A$, defined in (6.1), verifies (4.1) with $M_A = L_{\mathcal{A}} + c_0^2 L_p$ and $m_A = m_{\mathcal{A}}$.

Next, a simple calculation based on inequality (2.2) and assumption (2.10) shows that

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_V \leq d \max_{r\in[0,T]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \, ds \tag{6.7}$$
$$\text{for all } \boldsymbol{u}_1, \boldsymbol{u}_2 \in C([0,T];V), t \in [0,T]$$

The previous inequality implies that $\mathcal{S}$ satisfies (4.2) with $L_S = d\max_{r\in[0,T]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty}$.

As in [2] we use assumption (2.13) and inequality (2.1) to see that the functional $j$ defined by (6.3), is a seminorm on $V$ and verifies

$$j(\boldsymbol{v}) \leq c_0 \, (\text{meas}\,(\Gamma_3))^{1/2} \, \|F\|_{L^\infty(\Gamma_3)} \|\boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{v} \in V. \tag{6.8}$$

We deduce that $j$ satisfies (4.3). Taking into account the previous results and (3.6) we conclude that the hypotheses of Theorem 4.1 are fulfilled. Therefore, the variational inequality (6.4) has a unique solution $\boldsymbol{u} \in C([0,T];V)$.

Next, we show the unique solvability of variational equality (6.6). To this end, let $\rho > 0$ and $\boldsymbol{u}, \boldsymbol{v} \in V$. Using (2.1) we deduce that the operator $P_\rho$, defined in (6.5), verifies

$$(P_\rho \boldsymbol{u} - P_\rho \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq 0 \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \tag{6.9}$$

and

$$\|P_\rho \boldsymbol{u} - P_\rho \boldsymbol{v}\|_V \leq \frac{c_0^2}{\rho} \|F\|_{L^\infty(\Gamma_3)} \|\boldsymbol{u} - \boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V. \tag{6.10}$$

Therefore, the operator

$$\boldsymbol{v} \to A\boldsymbol{v} + P_\rho \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in V$$

is strongly monotone and Lipschitz continuous. We apply Corollary 4.2 to conclude that the variational equality (6.6) has a unique solution $\boldsymbol{u}_\rho \in C([0,T];V)$.

We are now in position to present the proof of Theorem 3.1.

(1) The unique solvability of Problem $\mathcal{Q}^V$ follows from the unique solvability of (6.4).

(2) The unique solvability of Problem $\mathcal{Q}_\rho^V$ follows from the unique solvability of

(6.6).
(3) Let $\rho > 0$. We define the functional $j_\rho : V \to \mathbb{R}$ by

$$j_\rho(\boldsymbol{v}) = \int_{\Gamma_3} \frac{F}{2}\left(\sqrt{v_v^2+\rho^2} - \rho + v_v\right) da \quad \text{for all } \boldsymbol{v} \in V \tag{6.11}$$

We deduce that $j_\rho$ is Gâteaux differentiable and

$$(\nabla j_\rho(\boldsymbol{u}), \boldsymbol{v})_V = \int_{\Gamma_3} \frac{F}{2}\left(1 + \frac{u_v}{\sqrt{u_v^2+\rho^2}}\right) v_v da \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \tag{6.12}$$

Taking into account (6.5) and (6.12) we see that (6.6) is equivalent with

$$(A\boldsymbol{u}_\rho(t), \boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}_\rho(t), \boldsymbol{v})_V + (\nabla j_\rho(\boldsymbol{u}_\rho(t)), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{6.13}$$

Moreover, (6.5), (6.12) and (6.9) imply the convexity of $j_\rho$. Therefore, $j_\rho$ and $\nabla j_\rho$ satisfy (4.6). Finally, using (6.3), (6.11) and assumption (2.13) we deduce that the functionals $j$ and $j_\rho$ verify (4.7) with

$$G(\rho) = \frac{\rho}{2}\int_{\Gamma_3} F da$$

Convergence (3.16) is now a direct consequence of Theorem 4.3.
A numerical analysis and simulations of convergence result (3.16) will be provided in our next paper. Moreover, the extension of (4.9) to convergence results on the space $C([0,T];X)$ remains an open problem which will be investigated in the future.

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