Analysis of a contact problem with wear and unilateral constraint

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ABSTRACT

This paper represents a continuation of our previous work, where a mathematical model which describes the equilibrium of an elastic body in frictional contact with a moving foundation was considered. An existence and uniqueness result was proved, together with a convergence result. The proofs were carried out by using arguments of elliptic variational inequalities. In this current paper, we complete our model by taking into account the wear of the foundation. This makes the problem evolutionary and leads to a new and nonstandard mathematical model, which couples a time-dependent variational inequality with an integral equation. We provide the unique weak solvability of the model by using a fixed point argument, among others. Then, we penalize the unilateral contact condition and prove that the penalized problem has a unique solution which converges to the solution of the original problem, as the penalization parameter converges to zero.

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1. Introduction

Contact processes between deformable bodies or between a deformable body and a foundation abound in industry and everyday life. Their modeling is rather complex and, usually, leads to strongly nonlinear boundary value problems. Basic reference in the field includes [1–5] and, more recently.[6–9] There, the mathematical analysis of various models of contact is provided, including existence and uniqueness results of the solution. The references [2,3,7] deal also with the numerical analysis of various models of contact, including the study of fully discrete schemes, error estimates and numerical simulations.

Contact processes are accompanied by a number of phenomena among which the main one is the friction. Nevertheless, more is involved in contact than just friction. Indeed, during a contact process elastic or plastic deformations of the surface asperities may happen. Also, some or all of the following may take place: squeezing of oil or other fluids, breaking of the asperities' tips and production of debris, motion of the debris, formation or welding of junctions, creeping, fracture, etc. Moreover, frictional contact is associated with heat generation, material damage, wear and adhesion of contacting surfaces.

As the contact process evolves, the contacting surfaces evolve too, via their wear. Wear in sliding systems is often very slow but it is persisting, continuous and cumulative. There may be increase in the conformity of the surfaces and their smoothness, or increase in the surface roughness, fogging of the surface, generation of scratches and grooves, initiation of cracks and generation of debris which may change the contact characteristics. Asperities under large contact stresses may deform plastically or break. In the first case, the surface morphology changes and, therefore, both the contact stress and the friction traction are affected. These may be incorporated into a history or memory-dependent friction coefficient. In the second case, when asperities break, the surfaces wear out, debris are produced, and again the surface structure changes over time. This must be taken into account if the long time behavior of the system is to be realistically predicted.

To model the wear of the contacting surfaces the wear function $w = w(\boldsymbol{x}, t)$ is introduced, measuring the depth, in the normal direction, of the removed material. Therefore, it measures the change in the surface geometry, and represents the cumulative amount of material removed, per unit surface area, in the neighborhood of the point $\boldsymbol{x}$ up to time t. Since the amounts of material removed are small, as an approximation, one may treat it as a change in the gap. It is usually assumed that the rate of wear of the surface is proportional to the contact pressure and to the relative slip rate, that is to the dissipated frictional power. This leads to the rate form of Archard's law of surface wear,

$$\dot{w} = k\,|\sigma_\nu|\,\|\boldsymbol{v}\|, \tag{1.1}$$

where k is the wear coefficient, a very small positive constant in practice. Also, σ_ν represents the normal stress on the contact surface and $\|\boldsymbol{v}\|$ denotes the relative slip rate. The initial condition is $w(\boldsymbol{x}, 0) = w_0(\boldsymbol{x})$, and $w_0(\boldsymbol{x}) = 0$ when the surface is new or the initial shape is used as the reference configuration. The wear implies the evolution of contacting surfaces and these changes affect the contact process. Thus, due to its crucial role, there exists a large engineering and mathematical literature devoted to this topic. We resume to mention here the references [8,10–22], among others.

A mathematical model which describes the equilibrium of an elastic body in frictional contact with a moving foundation was recently considered in [23]. There, the contact was modeled with a normal compliance condition with unilateral constraints, associated to a sliding version of Coulomb's law of dry friction. The unique weak solvability of the model was proved, by using arguments of elliptic quasivariational inequalities. The current paper represents a continuation of [23]. Here, we complete the model studied in [23] by taking into account the wear of the foundation. We model the wear process with a version of Archard's law (1.1), as is customary in the mathematical literature. This leads to a new and interesting mathematical model which, in contrast to the model in [23], is evolutionary. Providing the variational analysis of this new model represents the main aim of this paper.

The rest of the manuscript is structured as follows. In Section 2, we present the notation and some preliminary material. In Section 3, we introduce the model of sliding frictional contact with wear, list the assumptions on the data and derive its variational formulation. The unique weak solvability of the contact problem is presented in Section 4. There, we state and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on arguments on time-dependent variational inequalities and fixed point. Finally, in Section 5 we present our second result, Theorem 5.1. It states the convergence of the solution of a penalized frictional contact problem with wear to the solution of the contact model considered in Section 3, as the penalization parameter converges to zero.

2. Notations and preliminaries

In this section, we present the notation we shall use and some preliminary material. Everywhere in this paper we use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$. For $d \in \mathbb{N}$, we denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$. Moreover, the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i\,, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\,, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with Lipschitz continuous boundary Γ and let Γ_1, Γ_2, and Γ_3 be three measurable parts of Γ such that $meas\,(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Also, we use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ. In particular, we recall that the inner products on the Hilbert spaces $L^2(\Omega)^d$ and $L^2(\Gamma)^d$ are given by

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}x, \qquad (\boldsymbol{u}, \boldsymbol{v})_{L^2(\Gamma)^d} = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v}\, \mathrm{d}a,$$

and the associated norms will be denoted by $\|\cdot\|_{L^2(\Omega)^d}$ and $\|\cdot\|_{L^2(\Gamma)^d}$, respectively. Moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},$$
$$Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, \mathrm{d}x,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ is the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Recall that the completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary Γ. We denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential component of $\boldsymbol{v}$ on Γ, respectively, defined by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. By the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\, \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{2.1}$$

For a regular function $\boldsymbol{\sigma} : \Omega \cup \Gamma \to \mathbb{S}^d$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ, respectively, and we recall that $\sigma_\nu = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v}\, \mathrm{d}x = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}\, \mathrm{d}a \quad \forall\, \boldsymbol{v} \in V. \tag{2.2}$$

We end this section with two abstract results which will be used in the rest of the paper. The first one represents an existence, uniqueness, and convergence result for elliptic variational inequalities. To introduce it, we consider a real Hilbert space X endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We assume that:

$$K \text{ is a nonempty, closed, convex subset of } X. \tag{2.3}$$

$$A : X \to X \text{ is a strongly monotone Lipschtz continuous operator.} \tag{2.4}$$

$$\begin{cases} (a) \ \ G : X \to X \text{ is a monotone Lipschtz continuous operator.} \\ (b) \ \ (Gu, v - u)_X \le 0 \qquad \forall\, u \in X,\ v \in K. \\ (c) \ \ Gu = 0_X \ \text{ iff } \ u \in K. \end{cases} \tag{2.5}$$

$$f \in X. \tag{2.6}$$

With these given data we consider the problem of finding an element u such that

$$u \in K, \quad (Au, v-u)_X \geq (f, v-u)_X, \quad \forall\, v \in K \tag{2.7}$$

and, for each $\rho > 0$, we consider the problem of finding an element u_ρ such that

$$u_\rho \in X, \quad Au_\rho + \frac{1}{\rho} Gu_\rho = f. \tag{2.8}$$

The following result, proved in [9], will be used in Sections 4 and 5 of this paper.

Theorem 2.1: *Let X be a Hilbert space and assume that (2.3)–(2.5) hold. Then:*

(1) *The variational inequality (2.7) has a unique solution.*
(2) *For each $\rho > 0$ there exists a unique element u_ρ which solves the nonlinear Equation (2.8).*
(3) *The solution u_ρ of (2.8) converges strongly to the solution u of (2.7), i.e.*

$$u_\rho \to u \quad \text{in } X \quad \text{as} \quad \rho \to 0. \tag{2.9}$$

The second abstract result we need is a fixed point result. To introduce it we consider a Banach space X. We use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X and, for a subset $K \subset X$, we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. We also use the notation $C^1(\mathbb{R}_+; X)$ for the space of continuous differentiable functions defined on $\mathbb{R}_+$ with values in X.

The following fixed-point result will be used in Section 5 of the paper.

Theorem 2.2: *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be a nonlinear operator with the following property: for each $n \in \mathbb{N}$ there exists $c_n > 0$ such that*

$$\|\Lambda u(t) - \Lambda v(t)\|_X \leq c_n \int_0^t \|u(s) - v(s)\|_X \mathrm{d}s, \tag{2.10}$$

for all $u, v \in C(\mathbb{R}_+; X)$ and for all $t \in [0, n]$. Then the operator Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$.

Theorem 2.2 represents a simplified version of Corollary 2.5 in [24]. We underline that in (2.10) and below, the notation $\Lambda\eta(t)$ represents the value of the function $\Lambda\eta$ at the point t, i.e. $\Lambda\eta(t) = (\Lambda\eta)(t)$.

3. The model

In this section we introduce the contact problem, list the assumptions on the data and derive its variational formulation.

The physical setting is as follows. An elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2, and Γ_3 such that *meas* $(\Gamma_1) > 0$ and, in addition, Γ_3 is plane. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. It is fixed on Γ_1 and surfaces tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictional contact with a moving obstacle, the so-called foundation. We denote by $\boldsymbol{v}^*$ the velocity of the foundation, which is supposed to be a non-vanishing time-dependent function in the plane of Γ_3. The friction implies the wear of the foundation that we model with a surface variable, the wear function. Its evolution is governed by a simplified version of Archard's law that we shall describe below. Moreover, we assume that the foundation is deformable and, therefore, its penetration is allowed. We model the contact with a normal compliance condition with unilateral constraint, which takes into account the wear of the foundation. We associate this condition to a sliding version of Coulomb's law of dry friction. We adopt the framework of the small strain theory and we assume

that the contact process is quasistatic and it is studied in the interval of time $\mathbb{R}_+ = [0, \infty)$. Then, the classical formulation of the contact problem under consideration is the following.
Problem $\mathcal{P}$. *Find a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$, *a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$, *and a wear function* $w : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \text{in } \Omega, \tag{3.1}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{3.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t) - w(t)) \le 0, \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t) - w(t))\Big) = 0 \end{array}\right\} \quad \text{on } \Gamma_3, \tag{3.5}$$

$$-\boldsymbol{\sigma}_\tau(t) = \mu\, p(u_\nu(t) - w(t))\, \boldsymbol{n}^*(t) \quad \text{on } \Gamma_3, \tag{3.6}$$

$$\dot{w}(t) = \alpha(t)\, p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_3, \tag{3.7}$$

for all $t \in \mathbb{R}_+$ *and, in addition,*

$$w(0) = 0 \qquad \text{on} \quad \Gamma_3. \tag{3.8}$$

Here and below, for simplicity, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$. Moreover, the functions $\boldsymbol{n}^*$ and α are given by

$$\boldsymbol{n}^*(t) = -\frac{\boldsymbol{v}^*(t)}{\|\boldsymbol{v}^*(t)\|}, \qquad \alpha(t) = k\,\|\boldsymbol{v}^*(t)\| \qquad \forall\, t \in \mathbb{R}_+, \tag{3.9}$$

where k represents the wear coefficient.

We now provide a brief explanation for the equations and conditions in Problem $\mathcal{P}$. First, Equation (3.1) represents the elastic constitutive law of the material in which $\mathcal{F}$ denotes a given nonlinear operator. Equation (3.2) is the equilibrium equation in which Div represents the divergence operator for tensor-valued functions. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively.

Next, condition (3.5) represents the contact condition in which $g > 0$ and p is a positive Lipschitz continuous increasing function which vanishes for a negative argument. This condition can be derived in the following way. First, we assume that the obstacle is made of a hard material covered by a layer of soft material of thickness g. Thus, at each moment t, the normal stress has an additive decomposition of the form

$$\sigma_\nu(t) = \sigma_\nu^R(t) + \sigma_\nu^S(t) \quad \text{on } \Gamma_3, \tag{3.10}$$

in which the function $\sigma_\nu^R(t)$ describes the reaction to penetration of the hard material and $\sigma_\nu^S(t)$ describes the reaction of the soft material. The hard material does not wear and is perfectly rigid. Therefore, the penetration is limited by the bound g and σ_ν^R satisfies the Signorini condition in the form with a gap function, i.e.

$$u_\nu(t) \le g, \qquad \sigma_\nu^R(t) \le 0, \qquad \sigma_\nu^R(t)(u_\nu(t) - g) = 0 \qquad \text{on } \Gamma_3. \tag{3.11}$$

The soft material is elastic and could wear. Therefore, we assume that $\sigma_\nu^S(t)$ satisfies a normal compliance contact condition with wear, that is

$$-\sigma_\nu^S(t) = p(u_\nu(t) - w(t)) \qquad \text{on } \Gamma_3. \tag{3.12}$$

This condition shows that at each moment t, the reaction of the soft layer depends on the current value of the penetration, represented by $u_\nu(t) - w(t)$. Indeed, we assume that a wear process of the soft layer of the foundation takes place and the debris are immediately removed from the system. Thus, the penetration becomes $u_\nu(t) - w(t)$, instead of $u_\nu(t)$ as in the case without wear. Condition (3.12) describes the fact that the surface geometry of foundation is affected by wear, see [8] for details. We now combine (3.10) and (3.12) to see that

$$\sigma_\nu^R(t) = \sigma_\nu(t) + p(u_\nu(t) - w(t)) \quad \text{on } \Gamma_3. \tag{3.13}$$

Then we substitute equality (3.13) in (3.11) to obtain the contact condition (3.5).

We now describe the frictional contact condition (3.6). First, we recall the classical Coulomb law of dry friction,

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \le \mu\,|\sigma_\nu(t)|,\\ &-\boldsymbol{\sigma}_\tau(t) = \mu\,|\sigma_\nu(t)|\,\frac{\dot{\boldsymbol{u}}_\tau(t)-\boldsymbol{v}^*(t)}{\|\dot{\boldsymbol{u}}_\tau(t)-\boldsymbol{v}^*(t)\|} \ \text{ if } \ \dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t) \neq \boldsymbol{0} \end{aligned}\right\} \quad \text{on } \Gamma_3. \tag{3.14}$$

Here μ represents the friction coefficient, $\dot{\boldsymbol{u}}_\tau(t)$ is the tangential velocity, and $\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t)$ represents the relative tangential velocity or the relative slip rate. We assume that at each moment t the velocity of the foundation, $\boldsymbol{v}^*(t)$, is large in comparison with the tangential velocity $\dot{\boldsymbol{u}}_\tau(t)$ and, for this reason, we approximate the relative slip rate by $\boldsymbol{v}^*(t)$. Therefore, using the approximations $\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t) \approx -\boldsymbol{v}^*(t) \neq \boldsymbol{0}$, $\|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t)\| \approx \|\boldsymbol{v}^*(t)\|$, the friction law (3.14) implies that

$$\boldsymbol{\sigma}_\tau(t) = \mu\,|\sigma_\nu(t)|\,\frac{\boldsymbol{v}^*(t)}{\|\boldsymbol{v}^*(t)\|} \qquad \text{on } \Gamma_3.$$

Therefore, using the definition (3.9) of the vector $\boldsymbol{n}^*(t)$ yields

$$-\boldsymbol{\sigma}_\tau(t) = \mu\,|\sigma_\nu(t)|\,\boldsymbol{n}^*(t) \qquad \text{on } \Gamma_3. \tag{3.15}$$

Next, we note that as far as the contact of the elastic body is in the status of normal compliance (i.e. $u_\nu(t) < g$), condition (3.5) shows that

$$-\sigma_\nu(t) = p(u_\nu(t) - w(t)) \qquad \text{on } \Gamma_3 \tag{3.16}$$

and, therefore, substituting this equality in (3.15) we deduce that (3.6) holds. We extend this condition to the case when the contact is unilateral, i.e. when $u_\nu(t) = g$. In this way, we fully justify the friction law (3.6).

Next, to obtain the differential Equation (3.7) we start from the Archard's law, (1.1), i.e.

$$\dot{w}(t) = k\,|\sigma_\nu(t)|\,\|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t)\| \qquad \text{on } \Gamma_3. \tag{3.17}$$

Then, using again the approximation $\|\dot{\boldsymbol{u}}_\tau(t) - \boldsymbol{v}^*(t)\| \approx \|\boldsymbol{v}^*(t)\|$, Equation (3.17) leads to

$$\dot{w}(t) = k\,|\sigma_\nu(t)|\,\|\boldsymbol{v}^*(t)\| \qquad \text{on } \Gamma_3.$$

We now use the definition (3.9) of the function α to obtain

$$\dot{w}(t) = \alpha(t)\,|\sigma_\nu(t)| \qquad \text{on } \Gamma_3. \tag{3.18}$$

Next, we note that as far as the contact of the elastic body is in the status of normal compliance (3.16) holds and, therefore, substituting this equality in (3.18) we deduce (3.7). We extend this equality in the case of the unilateral contact, i.e. in the case when $u_\nu(t) = g$. In this way we fully justify the differential Equation (3.7) which governs the evolution of the wear function.

Finally (3.8) represents the initial condition for the wear function, which shows that at the initial moment the foundation is new.

We note that considering an arbitrary contact surface Γ_3 and a thickness $g = g(\boldsymbol{x})$ depending on the spatial variable does not cause additional mathematical difficulties in the analysis of Problem $\mathcal{P}$. Nevertheless, we decided to assume that Γ_3 is plane and g is a constant since these assumptions arise in a large number of the industrial process and lead to a simple geometry which helps the reader to better understand the wear phenomenon.

We now turn to the variational formulation of Problem $\mathcal{P}$ and, to this end, we list the assumptions on the data. First, we assume that the elasticity operator $\mathcal{F}$ and the normal compliance function satisfy the following condition.

$$\begin{cases} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \ \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q. \end{cases} \tag{3.19}$$

$$\begin{cases} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \le L_p\, |r_1 - r_2| \\ \qquad\quad \forall\, r_1,\ r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0 \\ \qquad \forall\, r_1,\ r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3, \forall\, r \in \mathbb{R}. \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{3.20}$$

The densities of body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{3.21}$$

Finally, the friction coefficient, the wear coefficient, and the foundation velocity verify

$$\mu \in L^\infty(\Gamma_3), \quad \mu(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{3.22}$$

$$k \in L^\infty(\Gamma_3), \quad k(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{3.23}$$

$$\boldsymbol{v}^* \in C(\mathbb{R}_+; \mathbb{R}^d) \text{ and there exists } v > 0 \text{ such that } \|\boldsymbol{v}^*(t)\| \ge v, \ \forall\, t \in \mathbb{R}_+. \tag{3.24}$$

Note that assumption (3.24) is compatible with the physical setting described above since, at each time moment, the velocity of the foundation is assumed to be large enough. In addition, (3.9), (3.23), and (3.24) imply that

$$\boldsymbol{n}^* \in C(\mathbb{R}_+; \mathbb{R}^d), \quad \alpha \in C(\mathbb{R}_+; L^\infty(\Gamma_3)) \tag{3.25}$$

and, moreover,

$$\alpha(t) \ge 0 \quad \text{a:e: on } \Gamma_3, \quad \text{for all } t \in \mathbb{R}_+. \tag{3.26}$$

Next, we introduce the set of admissible displacements fields defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}. \tag{3.27}$$

In addition, we use the Riesz representation theorem to define the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equality

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d}, \tag{3.28}$$

for all $\boldsymbol{v} \in V$ and $t \in \mathbb{R}_+$. It follows from assumption (3.21) that $\boldsymbol{f}$ has the regularity

$$\boldsymbol{f} \in C(\mathbb{R}_+; V). \tag{3.29}$$

Assume in what follows that $(\boldsymbol{\sigma}, \boldsymbol{u}, w)$ are sufficiently regular functions which satisfy (3.1)–(3.8) and let $\boldsymbol{v} \in U$ and $t > 0$ be given. We use Green formula (2.2) and the equilibrium Equation (3.2) to obtain

$$\int_\Omega \boldsymbol{\sigma}(t)(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,\mathrm{d}x - \int_\Omega \boldsymbol{f}_0(t)(\boldsymbol{v} - \boldsymbol{u}(t))\,\mathrm{d}x = \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\,\mathrm{d}a \quad \forall\, \boldsymbol{v} \in U.$$

Next, we split the boundary integral over Γ_1, Γ_2, and Γ_3. Since $\boldsymbol{v} - \boldsymbol{u}(t) = \boldsymbol{0}$ on Γ_1, $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on Γ_2, taking into account (3.28) we deduce that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q = (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\,\mathrm{d}a. \tag{3.30}$$

Note that

$$\boldsymbol{\sigma}(t)\,\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_3 \tag{3.31}$$

and, using contact condition (3.5) and the definition (3.27) of the set U, we have

$$\begin{aligned} \sigma_\nu(t)(v_\nu - u_\nu(t)) &= (\sigma_\nu(t) + p(u_\nu(t) - w(t)))(v_\nu - g) \\ &\quad + (\sigma_\nu(t) + p(u_\nu(t) - w(t)))(g - u_\nu(t)) - p(u_\nu(t) - w(t))(v_\nu - u_\nu(t)) \\ &\geq -p(u_\nu(t) - w(t))(v_\nu - u_\nu(t)) \quad \text{on} \quad \Gamma_3. \end{aligned} \tag{3.32}$$

Therefore, taking into account identity (3.31), inequality (3.32), and the friction law (3.6) we obtain that

$$\begin{aligned} \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\,\mathrm{d}a &\geq -\int_{\Gamma_3} p(u_\nu(t) - w(t))(v_\nu - u_\nu(t))\,\mathrm{d}a \\ &\quad - \int_{\Gamma_3} \mu\, p(u_\nu(t) - w(t))\boldsymbol{n}^*(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\,\mathrm{d}a. \end{aligned} \tag{3.33}$$

We now combine (3.30) and (3.33) to obtain that

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + \int_{\Gamma_3} p(u_\nu(t) - w(t))(v_\nu - u_\nu(t))\,\mathrm{d}a \\ &\quad + \int_{\Gamma_3} \mu\, p(u_\nu(t) - w(t))\boldsymbol{n}^*(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\,\mathrm{d}a \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U. \end{aligned} \tag{3.34}$$

In addition, we note that the boundary condition (3.3), the first inequality in (3.5) and (3.27) imply that $\boldsymbol{u}(t) \in U$. Finally, we integrate the differential Equation (3.7) with the initial condition (3.8) to obtain that

$$w(t) = \int_0^t \alpha(s)\, p(u_\nu(s) - w(s))\,\mathrm{d}s. \tag{3.35}$$

We now gather the constitutive law (3.1), the variational inequality (3.34), and the integral Equation (3.35) to obtain the following variational formulation of the contact problem $\mathcal{P}$.

Problem $\mathcal{P}^V$. *Find a stress field $\boldsymbol{\sigma}:\mathbb{R}_+\to Q$, a displacement field $\boldsymbol{u}:\mathbb{R}_+\to U$, and a wear function $w:\mathbb{R}_+\to L^2(\Gamma_3)$ such that*

$$\boldsymbol{\sigma}(t)=\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{3.36}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q+\int_{\Gamma_3} p(u_\nu(t)-w(t))(v_\nu-u_\nu(t))\,\mathrm{d}a$$
$$+\int_{\Gamma_3}\mu\, p(u_\nu(t)-w(t))\boldsymbol{n}^*(t)\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\,\mathrm{d}a\geq(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V\quad\forall\,\boldsymbol{v}\in U, \tag{3.37}$$

$$w(t)=\int_0^t\alpha(s)\,p(u_\nu(s)-w(s))\,\mathrm{d}s, \tag{3.38}$$

for all $t\in\mathbb{R}_+$.

The unique solvability of Problem $\mathcal{P}^V$ will be proved in the next section. A triple $(\boldsymbol{\sigma},\boldsymbol{u},w)$ which satisfy (3.36)–(3.38) is called *weak solution* of Problem $\mathcal{P}$.

We end this section with some additional comments on our contact model. Assume that (3.1)–(3.8) has a classical solution. Then, since α and p are positive functions, it follows from (3.7) what $\dot{w}(t)\geq 0$ for all t, i.e. the wear is increasing, in each point of the contact surface. Moreover, if at a moment t_0 we have $w(t_0)=g$, then, using Equation (3.7) and the properties (3.20) of the function p, it can be easily proved that $w(t_0)=g$ for all $t\geq t_0$. This behavior shows that the wear of the foundation is limited by the constraint $w(t)\leq g$, which fits with the assumption that rigid layer of the foundation does not wear.

4. An existence and uniqueness result

In this section, we state and prove the following existence and uniqueness result.

Theorem 4.1: *Assume that (3.19)–(3.24) hold. Then there exists a constant μ_0 which depends only on Ω, Γ_1, Γ_3, $\mathcal{F}$, and p such that, if*

$$\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0, \tag{4.1}$$

then Problem $\mathcal{P}^V$ has a unique solution. Moreover, the solution has the regularity

$$\boldsymbol{\sigma}\in C(\mathbb{R}_+;Q),\quad \boldsymbol{u}\in C(\mathbb{R}_+;U),\quad w\in C^1(\mathbb{R}_+;L^2(\Gamma_3)), \tag{4.2}$$

and, in addition,

$$w(t)\geq 0\quad\text{a.e. on }\Gamma_3,\quad\text{for all}\quad t\in\mathbb{R}_+. \tag{4.3}$$

We conclude from above that Problem $\mathcal{P}$ has a unique weak solution, provided that assumptions of Theorem 4.1 are satisfied. In addition, note that condition (4.1) represents a smallness condition on the coefficient of friction which is frequently needed in the study of static or quasistatic frictional contact problems with elastic materials. The question if this condition describes an intrinsic feature of the frictional contact process or it represents a limitation of our mathematical tools represents an open question which, clearly, has to be investigated in the future.

The proof of Theorem 4.1 will be carried out in several steps. We assume in the rest of this section that (3.19)–(3.24) hold. In the first step, we consider a given wear function $w\in C(\mathbb{R}_+;L^2(\Gamma_3))$ and we construct the following intermediate variational problem.

Problem $\mathcal{P}_w^V$**.** *Find a displacement field* $\boldsymbol{u}_w : \mathbb{R}_+ \to U$ *such that*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_w(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_w(t)))_Q + \int_{\Gamma_3} p(u_{w\nu}(t) - w(t))(v_\nu - u_{w\nu}(t))\,\mathrm{d}a$$
$$+ \int_{\Gamma_3} \mu\, p(u_{w\nu}(t) - w(t))\boldsymbol{n}^*(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_{w\tau}(t))\,\mathrm{d}a \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_w(t))_V \quad \forall\, \boldsymbol{v} \in U, \tag{4.4}$$

for all $t \in \mathbb{R}_+$.

In the study of Problem $\mathcal{P}_w^V$, we have the following existence and uniqueness result.

Lemma 4.2: *There exists a constant* μ_0 *which depends only on* Ω, Γ_1, Γ_3, $\mathcal{F}$, *and* p *such that, if* (4.1) *holds, then there exists a unique solution to Problem* $\mathcal{P}_w^V$ *which satisfies* $\boldsymbol{u}_w \in C(\mathbb{R}_+; U)$.

Proof: Let $t \in \mathbb{R}_+$ and consider the operator $A_{wt} : V \to V$ defined by

$$(A_{wt}\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} p(u_\nu - w(t))v_\nu\,\mathrm{d}a$$
$$+ \int_{\Gamma_3} \mu\, p(u_\nu - w(t))\boldsymbol{n}^*(t) \cdot \boldsymbol{v}_\tau\,\mathrm{d}a \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.5}$$

We use assumptions (3.19), (3.20), (3.22), and inequality (2.1) to see that the operator A_{wt} is Lipschitz continuous, i.e. it verifies the inequality

$$\|A_{wt}\boldsymbol{u}_1 - A_{wt}\boldsymbol{u}_2\|_V \leq \big(L_{\mathcal{F}} + c_0^2 L_p(1 + \|\mu\|_{L^\infty(\Gamma_3)})\big)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V, \tag{4.6}$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$. Next, we introduce the constant μ_0 defined by

$$\mu_0 = \frac{m_{\mathcal{F}}}{c_0^2 L_p}, \tag{4.7}$$

and note that it depends only on Ω, Γ_1, Γ_3, $\mathcal{F}$, and p. Assume that (4.1) holds. Then, we obtain

$$c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} < m_{\mathcal{F}}. \tag{4.8}$$

We use again assumptions (3.19), (3.20), and (3.20) and inequalities (2.1) and (4.8) to deduce that the operator A_{wt} is strongly monotone, i.e. it satisfies the inequality

$$(A_{wt}\boldsymbol{u}_1 - A_{wt}\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V \geq (m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)})\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2, \tag{4.9}$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in V$.

Using these ingredients, by Theorem 2.1(1), we deduce that there exists a unique element $\boldsymbol{u}_{wt} \in U$ such that

$$(A_{wt}\boldsymbol{u}_{wt}, \boldsymbol{v} - \boldsymbol{u}_{wt})_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_{wt})_V \quad \forall\, \boldsymbol{v} \in U. \tag{4.10}$$

Denote $\boldsymbol{u}_{wt} = \boldsymbol{u}_w(t)$. Then, it follows from (4.10) and (4.5) that the element $\boldsymbol{u}_w(t) \in U$ is the unique element which solves the variational inequality (4.4).

We now prove the continuity of the function $t \mapsto \boldsymbol{u}_w(t) : \mathbb{R}_+ \to V$. To this end, let t_1, $t_2 \in \mathbb{R}_+$ and denote $\boldsymbol{u}_i = \boldsymbol{u}_w(t_i)$, $w_i = w(t_i)$, $\boldsymbol{f}_i = \boldsymbol{f}(t_i)$, $\boldsymbol{n}_i^* = \boldsymbol{n}^*(t_i)$, for $i = 1, 2$. We use standard arguments in (4.4) to find that

$$
\begin{aligned}
(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_Q &\le (\boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V \\
&+ \int_{\Gamma_3} [p(u_{1\nu} - w_1) - p(u_{2\nu} - w_2)][(u_{2\nu} - w_2) - (u_{1\nu} - w_1)]\,\mathrm{d}a \\
&+ \int_{\Gamma_3} [p(u_{1\nu} - w_1) - p(u_{2\nu} - w_2)](w_2 - w_1)\,\mathrm{d}a \\
&+ \int_{\Gamma_3} \mu\,[p(u_{1\nu} - w_1)\boldsymbol{n}_1^* - p(u_{2\nu} - w_2)\boldsymbol{n}_1^*]\cdot(\boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau})\,\mathrm{d}a \\
&+ \int_{\Gamma_3} \mu\,[p(u_{2\nu} - w_2)\boldsymbol{n}_1^* - p(u_{2\nu} - w_2)\boldsymbol{n}_2^*]\cdot(\boldsymbol{u}_{2\tau} - \boldsymbol{u}_{1\tau})\,\mathrm{d}a.
\end{aligned}
$$

Therefore, (3.19), (3.20), (3.22), and (2.1) yield

$$
\begin{aligned}
&(m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)})\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2 \\
&\le \Big(c_0 L_p (1 + \|\mu\|_{L^\infty(\Gamma_3)})\|w_1 - w_2\|_{L^2(\Gamma_3)} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_V \\
&\quad + c_0\, p(g)\, \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{n}_1^* - \boldsymbol{n}_2^*\|\Big)\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V + L_p \|w_1 - w_2\|_{L^2(\Gamma_3)}^2.
\end{aligned}
$$

We now use (4.8) and the elementary inequality

$$
x, y, z \ge 0 \text{ and } x^2 \le yx + z \implies x^2 \le y^2 + 2z
$$

to deduce that

$$
\begin{aligned}
&\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2 \\
&\le a\Big(\|w_1 - w_2\|_{L^2(\Gamma_3)} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\| + \|\boldsymbol{n}_1^* - \boldsymbol{n}_2^*\|\Big)^2 + b\,\|w_1 - w_2\|_{L^2(\Gamma_3)}^2,
\end{aligned}
\tag{4.11}
$$

where a and b denote two positive constants which do not depend on t_1 and t_2. This inequality combined with (3.25), (3.29), and the regularity $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$ show that $\boldsymbol{u}_w \in C(\mathbb{R}_+; V)$. Thus, we conclude the existence part in Lemma 4.2. The uniqueness part follows from the unique solvability of (4.10) for each $t \in \mathbb{R}_+$. □

We assume in what follows that (4.1), (4.7) hold and we consider the operator $\Lambda : C(\mathbb{R}_+; L^2(\Gamma_3)) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ defined by

$$
\Lambda w(t) = \int_0^t \alpha(s)\, p(u_{w\nu}(s) - w(s))\,\mathrm{d}s,
\tag{4.12}
$$

for all $w \in C(\mathbb{R}_+; L^2(\Gamma_3))$, where $\boldsymbol{u}_w$ is the unique solution of Problem $\mathcal{P}_w^V$. We have the following fixed point result, which represents the second step in the proof of Theorem 4.1.

Lemma 4.3: *The operator Λ has a unique fixed point $w^* \in C(\mathbb{R}_+; L^2(\Gamma_3))$.*

Proof: Let $w_1, w_2 \in C(\mathbb{R}_+; L^2(\Gamma_3))$. For simplicity we denote by $\boldsymbol{u}_i$, $i = 1, 2$ the solutions of problems $\mathcal{P}_{w_i}^V$, i.e. $\boldsymbol{u}_i = \boldsymbol{u}_{w_i}$. Let $n \in \mathbb{N}$ and let $t \in [0, n]$. Taking into account (4.12), (3.9), and (3.20) we deduce that

$$
\begin{aligned}
&\|\Lambda w_1(t) - \Lambda w_2(t)\|_{L^2(\Gamma_3)} \\
&\le \nu_n^* \Big(c_0 \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\,\mathrm{d}s + \int_0^t \|w_1(s) - w_2(s)\|_{L^2(\Gamma_3)}\,\mathrm{d}s\Big),
\end{aligned}
\tag{4.13}
$$

where

$$v_n^* = L_p \|k\|_{L^\infty(\Gamma_3)} \max_{r\in[0,n]} \|\boldsymbol{v}^*(r)\|.$$

On the other hand, using arguments similar to those used in the proof of (4.11) yield

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \le \sqrt{a+b}\, \|w_1(t) - w_2(t)\|_{L^2(\Gamma_3)}. \tag{4.14}$$

We now combine the inequalities (4.13) and (4.14) to deduce that

$$\|\Lambda w_1(t) - \Lambda w_2(t)\|_{L^2(\Gamma_3)} \le v_n^* \big(c_0\sqrt{a+b} + 1\big) \int_0^t \|w_1(s) - w_2(s)\|_{L^2(\Gamma_3)}. \tag{4.15}$$

Lemma 4.3 is now a direct consequence of Theorem 2.2. □

We now have all ingredients needed to provide the proof of our main existence and uniqueness result.

Proof of Theorem 4.1: *Existence.* Let $w^* \in C(\mathbb{R}_+; L^2(\Gamma_3))$ be the unique fixed point of the operator Λ and let $\boldsymbol{u}^*, \boldsymbol{\sigma}^*$ defined by

$$\boldsymbol{u}^*(t) = \boldsymbol{u}_{w^*}(t), \tag{4.16}$$

$$\boldsymbol{\sigma}^*(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^*(t)), \tag{4.17}$$

for all $t \in \mathbb{R}_+$. We recall that $w^* = \Lambda w^*$ and using equalities (4.12) and (4.16) we deduce that

$$w^*(t) = \int_0^t \alpha(s)\, p(u^*_{w\nu}(s) - w^*(s))\, \mathrm{d}s, \tag{4.18}$$

for all $t \in \mathbb{R}_+$. We show that the triple $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, w^*)$ satisfies (3.36)–(3.38). First, we note that (3.36) is a direct consequence of (4.17). Then, we write the inequality (4.4) for $w = w^*$ and use the notation (4.16), (4.17) to see that (3.37) holds. Finally, (3.38) follows from (4.18). We conclude from above that the triple $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*, w^*)$ represents a solution of Problem $\mathcal{P}^V$, as claimed. The regularity expressed in (4.2) is a direct consequence of the Lemma 4.2 combined with assumption (3.19) and formula (4.18). Finally, condition (4.3) follows from (4.18), since α and p are positive functions, as it results from (3.26) and (3.20)(a).

Uniqueness. The uniqueness of the solution follows from the unique solvability of Problem $\mathcal{P}_w^V$, provided in Lemma 4.2, combined with the uniqueness of the fixed point of operator Λ defined by (4.12). □

5. A convergence result

In this section we provide a convergence result in the study of Problem $\mathcal{P}$, based on the penalization of the unilateral constraint and arguments similar to those used in [25]. To this end, we assume in what follows that (3.19)–(3.24) and (4.1) hold where, recall, μ_0 is given by (4.7). Then, it follows from Theorem 4.1 that Problem $\mathcal{P}^V$ has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}, w)$, with regularity (4.2). Next, for each $\rho > 0$ we consider the following contact problem.

Problem $\mathcal{P}_\rho$. *Find a stress field* $\boldsymbol{\sigma}_\rho : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$*, a displacement field* $\boldsymbol{u}_\rho : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$*, and a wear function* $w_\rho : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) \quad \text{in } \Omega, \tag{5.1}$$

$$\operatorname{Div}\boldsymbol{\sigma}_\rho(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in } \Omega, \tag{5.2}$$

$$\boldsymbol{u}_\rho(t) = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}_\rho(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on } \Gamma_2, \tag{5.4}$$

$$\sigma_{\rho\nu}(t) + p(u_{\rho\nu}(t) - w_\rho(t)) + \frac{1}{\rho}p(u_{\rho\nu}(t) - g) = 0 \quad \text{on } \Gamma_3, \tag{5.5}$$

$$-\boldsymbol{\sigma}_{\rho\tau}(t) = \mu\, p(u_{\rho\nu}(t) - w_\rho(t))\, \boldsymbol{n}^*(t) \quad \text{on } \Gamma_3, \tag{5.6}$$

$$\dot{w}_\rho(t) = \alpha(t)\, p(u_{\rho\nu}(t) - w_\rho(t)) \quad \text{on } \Gamma_3, \tag{5.7}$$

for all $t \in \mathbb{R}_+$ *and, in addition,*

$$w_\rho(0) = 0 \qquad \text{on} \quad \Gamma_3. \tag{5.8}$$

Note that here and below $u_{\rho\nu}$ is the normal component of the displacement field $\boldsymbol{u}_\rho$ and $\sigma_{\rho\nu}$, $\boldsymbol{\sigma}_{\rho\tau}$ represent the normal and tangential components of the stress tensor $\boldsymbol{\sigma}_\rho$, respectively. Moreover, recall that the functions $\boldsymbol{n}^*$ and α are defined by (3.9). The equations and boundary conditions in problem (5.1)–(5.8) have a similar interpretation as those in problem (3.1)–(3.8). The difference arises in the fact that here we replace the contact condition (3.5) with condition (5.5), i.e. we remove the unilateral constraint. In (5.5) ρ represents a penalization parameter which may be interpreted as a deformability coefficient of the foundation, and then $\frac{1}{\rho}$ is the surface stiffness coefficient.

In order to provide the variational formulation of Problem $\mathcal{P}_\rho$ we define the operator $G : V \to V$ by equality

$$(G\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu - g)v_\nu \mathrm{d}a \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V. \tag{5.9}$$

Then, using the assumptions (3.20) on the normal compliance function, we deduce that the operator G has the following properties:

$$\begin{cases} (a)\;\; G : V \to V \text{ is a monotone Lipschitz continuous operator.} \\ (b)\;\; (G\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_V \leq 0 \qquad \forall\, \boldsymbol{u} \in V,\, \boldsymbol{v} \in U. \\ (c)\;\; G\boldsymbol{u} = \boldsymbol{0}_V \;\; \text{iff } \; \boldsymbol{u} \in U. \end{cases} \tag{5.10}$$

The proof of these properties is straightforward and, therefore, we skip it.

Next, using notation (5.9) and arguments similar to those used in Section 3 we obtain the following variational formulation of Problem $\mathcal{P}_\rho$.

Problem $\mathcal{P}_\rho^V$**.** *Find a stress field* $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$*, a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$*, and a wear function* $w_\rho : \mathbb{R}_+ \to L^2(\Gamma_3)$ *such that*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \tag{5.11}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_\rho(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} p(u_{\rho\nu}(t) - w_\rho(t))v_\nu \,\mathrm{d}a \\ &\quad + \int_{\Gamma_3} \mu\, p(u_{\rho\nu}(t) - w_\rho(t))\boldsymbol{n}^*(t)\cdot \boldsymbol{v}_\tau \,\mathrm{d}a + \frac{1}{\rho}(G\boldsymbol{u}_\rho(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V, \end{aligned} \tag{5.12}$$

$$w_\rho(t) = \int_0^t \alpha(s)\, p(u_{\rho\nu}(s) - w_\rho(s))\,\mathrm{d}s, \tag{5.13}$$

for all $t \in \mathbb{R}_+$.

The following theorem states the unique solvability of Problem $\mathcal{P}_\rho^V$ and describes the behavior of its solution as $\rho \to 0$.

Theorem 5.1: *Assume that (3.19)–(3.24) and (4.1) hold, with μ_0 given by (4.7). Then:*

(1) *For each $\rho > 0$ there exists a unique solution of Problem $\mathcal{P}_\rho^V$. Moreover, the solution has the regularity*

$$\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q), \quad \boldsymbol{u}_\rho \in C(\mathbb{R}_+; V), \quad w_\rho \in C^1(\mathbb{R}_+; L^2(\Gamma_3)). \tag{5.14}$$

(2) *For each $n \in \mathbb{N}$ there exists $\omega_n > 0$ such that*

$$\|\boldsymbol{u}_\rho(t)\|_V \le \omega_n \qquad \forall\, t \in [0,n],\ \forall\, \rho > 0. \tag{5.15}$$

(3) *The solution $(\boldsymbol{\sigma}_\rho, \boldsymbol{u}_\rho, w_\rho)$ of the Problem $\mathcal{P}_\rho^V$ converges to the solution $(\boldsymbol{\sigma}, \boldsymbol{u}, w)$ of the Problem $\mathcal{P}^V$, that is*

$$\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q + \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \|w_\rho(t) - w(t)\|_{L^2(\Gamma_3)} \to 0 \quad as \quad \rho \to 0, \tag{5.16}$$

for all $t \in \mathbb{R}_+$.

The proof of Theorem 5.1 will be carried out in several steps. We assume in what follows that (3.19)–(3.24) and (4.1) hold. Recall that everywhere below w represents the third component of the solution of Problem $\mathcal{P}^V$, provided by Theorem 4.1. Therefore, w is fixed and, moreover, it satisfies condition (4.3). In the first step, we consider the following intermediate variational problem.

Problem $\widetilde{\mathcal{P}}_{w\rho}^V$. *Find a displacement field $\widetilde{\boldsymbol{u}}_\rho : \mathbb{R}_+ \to V$ such that*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + \int_{\Gamma_3} p(\widetilde{u}_{\rho\nu}(t) - w(t))v_\nu \,\mathrm{d}a$$
$$+ \int_{\Gamma_3} \mu\, p(\widetilde{u}_{\rho\nu}(t) - w(t))\boldsymbol{n}^*(t)\cdot \boldsymbol{v}_\tau \,\mathrm{d}a + \frac{1}{\rho}(G\widetilde{\boldsymbol{u}}_\rho(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,$$

for all $t \in \mathbb{R}_+$.

Note that Problem $\widetilde{\mathcal{P}}_{w\rho}^V$ is similar to Problem $\mathcal{P}_w^V$ considered in Section 4. Its solution depends on ρ and w but, for simplicity, we do not indicate explicitly its dependence with respect to w.

We have the following existence, uniqueness, and convergence result.

Lemma 5.2:

(1) *For each $\rho > 0$ Problem $\widetilde{\mathcal{P}}_{w\rho}^V$ has a unique solution which satisfies $\widetilde{\boldsymbol{u}}_\rho \in C(\mathbb{R}_+; V)$.*
(2) *For each $n \in \mathbb{N}$ there exists $\omega_n > 0$ which does not depend on w such that*

$$\|\widetilde{\boldsymbol{u}}_\rho(t)\|_V \le \omega_n \quad \forall\, t \in [0,n],\ \forall\, \rho > 0. \tag{5.17}$$

(3) *The solution $\widetilde{\boldsymbol{u}}_\rho$ of Problem $\widetilde{\mathcal{P}}_{w\rho}^V$ converges to the second component of the solution of the Problem $\mathcal{P}^V$, that is*

$$\|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \quad as \quad \rho \to 0, \tag{5.18}$$

for all $t \in \mathbb{R}_+$.

Proof: (1) Let $\rho > 0$ be fixed and let $t \in \mathbb{R}_+$. Using the definition (4.5) of the operator A_{wt}, we deduce that the variational Equation (5.17) is equivalent to equation

$$\widetilde{\boldsymbol{u}}_\rho(t) \in V, \quad A_{wt}\widetilde{\boldsymbol{u}}_\rho(t) + \frac{1}{\rho} G\widetilde{\boldsymbol{u}}_\rho(t) = \boldsymbol{f}(t). \tag{5.19}$$

Recall that inequalities (4.6), (4.8) and (4.9) show that A_{wt} is a strongly monotone Lipschitz continuous operator on V. Therefore, taking into account (5.10), we are in a position to apply Theorem 2.1(2) with $X = V$ and $K = U$. In this way, we deduce the existence of a unique element $\widetilde{\boldsymbol{u}}_\rho(t)$ which solves (5.19). The continuity of the function $t \mapsto \widetilde{\boldsymbol{u}}_\rho(t) : \mathbb{R}_+ \to V$ follows from estimates similar to those used in the proof of Lemma 4.2.

(2) Let $n \in \mathbb{N}$ and let $t \in [0, n]$ be fixed. Also, let $\rho > 0$. We use Equation (5.19) to deduce that

$$(A_{wt}\widetilde{\boldsymbol{u}}_\rho(t), \widetilde{\boldsymbol{u}}_\rho(t))_V + \frac{1}{\rho}(G\widetilde{\boldsymbol{u}}_\rho(t), \widetilde{\boldsymbol{u}}_\rho(t))_V = (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\rho(t))_V.$$

Next, we use the properties (5.10)(a) and (c) of the operator G to see that

$$(G\widetilde{\boldsymbol{u}}_\rho(t), \widetilde{\boldsymbol{u}}_\rho(t))_V \geq 0$$

and, therefore,

$$(A_{wt}\widetilde{\boldsymbol{u}}_\rho(t), \widetilde{\boldsymbol{u}}_\rho(t))_V \leq (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\rho(t))_V. \tag{5.20}$$

We now use inequality (4.9) with $\boldsymbol{u}_1 = \widetilde{\boldsymbol{u}}_\rho(t)$ and $\boldsymbol{u}_2 = \boldsymbol{0}_V$ to see that

$$(A_{wt}\widetilde{\boldsymbol{u}}_\rho(t) - A_{wt}\boldsymbol{0}_V, \widetilde{\boldsymbol{u}}_\rho(t))_V \geq (m_{\mathcal{F}} - c_0^2 L_p \|\mu\|_{L^\infty(\Gamma_3)})\|\widetilde{\boldsymbol{u}}_\rho(t)\|_V^2. \tag{5.21}$$

Therefore, using (5.20), (5.21), (4.1), and (4.7) yields

$$\|\widetilde{\boldsymbol{u}}_\rho(t)\|_V \leq c\,(\|\boldsymbol{f}(t)\|_V + \|A_{wt}\boldsymbol{0}_V\|_V) \tag{5.22}$$

where $c > 0$ is a constant which is independent on n, t, ρ, and w. Also, since w satisfies (4.3), by the definition (4.5) of the operator A_{wt} and the property (3.20)(e) of the function p it follows that

$$\|A_{wt}\boldsymbol{0}_V\|_V \leq \|\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{0}_V)\|_Q. \tag{5.23}$$

Inequality (5.17) follows now from inequalities (5.22), (5.23), and the regularity (3.29).

(3) Let $\rho > 0$ be fixed and let $t \in \mathbb{R}_+$. We substitute (3.36) in (3.37) then we use the definition (4.5) of the operator A_{wt} to see that

$$\boldsymbol{u}(t) \in U, \quad (A_{wt}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V, \qquad \forall\, \boldsymbol{v} \in U. \tag{5.24}$$

The convergence (5.18) is now a direct consequence of (5.19), (5.24), and Theorem 2.1(3). □

We are now in a position to provide the main result in this section.

Proof of Theorem 5.1: (1) The unique solvability of Problem $\mathcal{P}_\rho^V$ follows from arguments similar to those used in the proof of Theorem 4.1 and, therefore, we skip it. Nevertheless, we note that the main ingredient of the proof is an existence and uniqueness result similar to that in Lemma 5.2, combined with the fixed point argument in Theorem 2.2.

(2) We note that (5.11) and (5.12) imply that the function $\boldsymbol{u}_\rho$ satisfies an inequality of the from (5.17) in which w is replaced by w_ρ. Therefore, using the same arguments as those in the proof of (5.17), we deduce that inequality (5.15) holds.

(3) Let $t \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ be such that $t \in [0, n]$. Also, let $\rho > 0$. Using the triangle inequality we obtain that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}_\rho(t)\|_V + \|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}(t)\|_V. \tag{5.25}$$

On the other hand, arguments similar to those used in the proof of (4.14) yield

$$\|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}_\rho(t)\|_V \leq \sqrt{a+b}\,\|w_\rho(t) - w(t)\|_{L^2(\Gamma_3)} \tag{5.26}$$

where a and b represent positive constants which do not depend on n, t, and ρ. Moreover, using the integral Equations (3.38), (5.13), the hypothesis (3.20) on the normal compliance function p, the regularity (3.25) on α and the Gronwall's argument we obtain that

$$\|w_\rho(t) - w(t)\|_{L^2(\Gamma_3)} \le c_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{5.27}$$

Here and below c_n represents a positive constant which depends on n but does not depend on t and ρ, and whose value may change from line to line.

Combining now inequalities (5.26) and (5.27) yields

$$\|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}_\rho(t)\|_V \le c_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{5.28}$$

Therefore, from (5.25) and (5.28), we have

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le c_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds + \|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}(t)\|_V. \tag{5.29}$$

We use again a Gronwall's argument to deduce that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}(t)\|_V + c_n \int_0^t e^{c_n(t-s)} \|\widetilde{\boldsymbol{u}}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{5.30}$$

Note that $e^{c_n(t-s)} \le e^{c_n t} \le e^{n c_n}$ for all $s \in [0, t]$ and, therefore, (5.30) implies that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\rho(t) - \boldsymbol{u}(t)\|_V + c_n e^{n c_n} \int_0^t \|\widetilde{\boldsymbol{u}}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds. \tag{5.31}$$

On the other hand, (5.17) and (5.18) allow us to use Lebesgue's convergence theorem to find that

$$\int_0^t \|\widetilde{\boldsymbol{u}}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds \to 0 \qquad \text{as} \quad \rho \to 0. \tag{5.32}$$

Therefore, using (5.31), (5.18), and (5.32) we conclude that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \qquad \text{as} \quad \rho \to 0. \tag{5.33}$$

Next, we use (3.36), (5.11), and the properties (3.19) of operator $\mathcal{F}$ to obtain that

$$\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q \le L_{\mathcal{F}} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V. \tag{5.34}$$

We now combine inequalities (5.34) and (5.27) to see that

$$\begin{aligned} &\|\boldsymbol{\sigma}_\rho(t) - \boldsymbol{\sigma}(t)\|_Q + \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \|w_\rho(t) - w(t)\|_{L^2(\Gamma_3)} \\ &\quad \le c_n \Big(\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V + \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \, ds \Big). \end{aligned} \tag{5.35}$$

Finally, (5.15) and (5.33) allow us to use Lebesgue's convergence theorem. Thus, we pass to the limit in (5.35) as $\rho \to 0$ to obtain (5.16). □

In addition to the mathematical interest in the convergence result (5.16) it is important from the mechanical point of view, since it shows that the weak solution of the elastic contact problem with deformability coefficient.

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