Lagrange Interpolation on Arbitrary, Equidistant and Chebyshev Nodes

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Definition

Let $\mathcal{F}$ be a function space and let $\Phi \subseteq \mathcal{F}$,
Let $f \in \mathcal{F}$
Find $\phi \in \Phi$, so
$\|f - \phi\|$ is minimal, ϕ is an approximation
Where $\|\cdot\|$ is a semi-norm on $\mathcal{F}$.
For Lagrange interpolation on $\{x_0 \ldots x_n\} \subset [a, b]$ define

$$\|f\| = \sum_{k=0}^{n} |f(x_k)|$$

$$\|f - \phi\| = 0 \Leftrightarrow f(x_k) = \phi(x_k)$$

Existence and uniqueness

Let $f : [a, b] \to \mathbb{R}$ and
$\{x_0 ... x_n\} \subseteq [a, b]$ with $\forall i \neq j \Rightarrow x_i \neq x_j$, then
$\exists! P_n \in \Pi_n$ with $\forall k \in 0..n \; P_n(x_k) = f(x_k)$

$$P_n = \sum_{k=0}^{n} a_k x^k$$

Then

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \dots & \dots & \dots & \dots \\ 1 & x_2 & \dots & x_2^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ ..... \\ f(x_n) \end{bmatrix}$$

$$\det(W) = \prod_{0 \leq i < j \leq n} (x_j - x_i) \neq 0$$

Fundamental polynomials

Lagrange fundamental polynomials:
$l_i \in \Pi_n$ so $l_i(x_j) = \delta_{ij}$

$$l_i(x) = \prod_{k=0, k\neq i}^{n} \frac{x - x_k}{x_i - x_k}$$

Let

$$\sum_{k=0}^{n} \alpha_k l_k = 0 \Rightarrow \sum_{k=0}^{n} \alpha_k l_k(x_i) = 0,\ \forall i = 0..n \Rightarrow a_i = 0,\ \forall i = 0..n$$

$\{l_i | i = 0..n\}$ linearly independent $\Rightarrow$ $\{l_i | i = 0..n\}$ base in Π_n

$$P_n(x) = \sum_{k=0}^{n} f(x_k) l_k(x)$$

Important nodes

Equispaced nodes: $x_0 = a,\ h = \frac{b-a}{n},\ x_k = x_0 + kh$

Fekete nodes: Maximize $\prod_{i \neq j} |x_i - x_j|^{\frac{2}{n(n+1)}}$

Fekete-Leja nodes: Approximation to Fekete nodes.
Greedy algorithm, start with x_0.
Add x_n to maximize $\prod_{j=0}^{n-1} |x_j - x_n|$

Fejer nodes: Let $D \subseteq \mathbb{C} = \{e^{it} | t \in [0, 2\pi]\}$
$\phi : [a, b] \in D,\ \{z_k\} \subseteq D,\ x_k = \phi^{-1}(z_k)$

Chebyshev nodes: Particular Fejer nodes
z_k equispaced in the Riemmanian metric on D

Chebyshev nodes

Chebyshev points:

$$z_k \in D \; equispaced \Rightarrow x_k = \Re(z_k) = \frac{1}{2}(z_k + z_k^{-1})$$

$$x_k = \cos\left(\frac{k\pi}{n}\right)$$

Chebyshev polynomials:

$$T_n(x) = \arccos\left(n\cos(x)\right)$$

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Chebyshev nodes are maxima and minima for Chebyshev polynomials.

Properties:

$$|T_n(x)| \leq 1, \;\; T_n(x) = \sum_{k=0}^{n} a_k x^k \Rightarrow a_n = 2^{n-1}$$

Orthogonality:

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = 0, \;\; \forall n \neq m$$

Chebyshev series:

$$\forall f \in S \subseteq \mathcal{L}^2[-1,1], \;\; f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

where

$$a_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx, \;\; a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{T_k(x) f(x)}{\sqrt{1-x^2}} dx, \;\; \forall k \in \mathbb{N}^*$$

Interpolation on Chebyshev nodes

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$$

Chebyshev truncation:

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

Let x_k be Chebyshev nodes and $P_n \in \Pi_n$ so $P_n(x_k) = f(x_k)$

$$P_n(x) = \sum_{k=0}^{n} c_k T_k(x)$$

$$T_i(x_k) = T_{2jn-i}(x_k) = T_{2jn+i}(x_k)$$

$$c_0 = \sum_{j=0}^{\infty} a_{2jn}, \; c_n = \sum_{j=0}^{\infty} a_{(2j+1)n}, \; c_k = a_k + \sum_{j=0}^{\infty} (a_{2jn-k} + a_{2jn+k})$$

Barycentric form for Chebyshev nodes

Barycentric form:

$$l_k(x) = \frac{\prod_{j=0,j\neq k}^{n}(x - x_j)}{\prod_{j=0,j\neq k}^{n}(x_k - x_j)}$$

$$l(x) = \prod_{i=0,i}^{n}(x - x_j)$$

$$l_k(x) = \frac{l(x)}{l'(x_k)(x - x_k)}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f(x_k)l(x)}{l'(x_k)(x - x_k)}$$

Second barycentric form:

$$\lambda_k = \frac{1}{l'(x_k)}$$

$$\sum_{k=0}^{n} l_k(x_j) = 1, \ \forall j \Rightarrow \sum_{k=0}^{n} l_k(x) = 1$$

$$l_i(x) = \frac{l_i(x)}{\sum_{k=0}^{n} l_k(x)} = \frac{\frac{l(x)\lambda_i}{x-x_i}}{\sum_{k=0}^{n} \frac{l(x)\lambda_k}{x-x_k}}$$

$$P_n(x) = \frac{\sum_{k=0}^{n} \frac{f(x_k)\lambda_k}{x-x_k}}{\sum_{k=0}^{n} \frac{\lambda_k}{x-x_k}}$$

Chebyshev barycentric form

Recall:

$$T_{n-1}(x_k) = T_{2n-(n-1)}(x_k)$$

$$T_{n+1} - T_{n-1}(x_k) = 0 = l(x_k)$$

$$l(x) = 2^{-n} T_{n+1} - T_{n-1}(x)$$

$$T'_{n+1}(x_k) - T'_{n-1}(x_k) = 4n(-1)^k, \ \ k \in \{0, n\}$$

$$T'_{n+1}(x_k) - T'_{n-1}(x_k) = 2n(-1)^k, \ \ k \notin \{0, n\}$$

$$\lambda_k = \frac{2^{n-1}(-1)^k}{n} \ \ k \notin \{0, n\}$$

$$P_n(x) = \frac{\frac{f(-1)}{2(x+1)} + \frac{f(1)(-1)^n}{2(x-1)} + \sum_{k=1}^{n-1} \frac{f(x_k)(-1)^k}{x-x_k}}{\frac{1}{2(x+1)} + \frac{(-1)^n}{2(x-1)} + \sum_{k=1}^{n-1} \frac{(-1)^k}{x-x_k}}$$

Divided differences

Define: $f[x_i] = f(x_i)$ and $f[x_i \dots x_{i+k}] = \frac{f[x_{i+1} \dots x_{i+k}] - f[x_i \dots x_{i+k-1}]}{x_{i+k} - x_i}$
Newton form of the polynomial:

$$P_n(x) = \sum_{k=0}^{n} \alpha_k \prod_{i=0}^{k-1} (x - x_i)$$

With:

$$\alpha_k = f[x_0 \dots x_k]$$

$$f[x_0 \dots x_n] =$$

$$= \int_0^1 \int_0^{t_1} \dots \int_0^{t_{n-1}} f^n(x_0 + t_1(x_1 - x_0) + \dots t_n(x_n - x_{n-1}))dt_n \dots dt_1$$

$$\exists \xi \in [x_0, x_n] \Rightarrow f[x_0 \dots x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Equispaced nodes

Equispaced nodes: $x_{k+1} - x_k = h$, $x = x_0 + sh \Rightarrow x - x_k = (s-k)h$

$$f[x_0 \dots x_n] = \frac{\sum_{k=0}^{n} (-1)^k C_n^k f(x_k)}{n! h^n}$$

Forward formula:

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} \left(f[x_0 \dots x_k] \left(\prod_{i=0}^{k-1} (s-i) \right) h^{k+1} \right)$$

Backward formula:

$$P_n(x) = f[x_n] + \sum_{k=1}^{n} \left(f[x_n \dots x_{n-k}] \left(\prod_{i=0}^{k-1} (s-n+i) \right) h^{k+1} \right)$$

Central differences form: Let $x_{-n}, \ldots x_{-1}, x_0, x_1 \ldots x_n$
$x \in [x_{-1}, x_1]$

Stirling's formula:

$$P_{2n+1}(x) = f[x_0] + \sum_{k=1}^{n} \left(f[x_{-k} \ldots x_0 \ldots x_k] \left(\prod_{i=0}^{k-1} (s^2 - i^2) \right) h^{2k+1} \right) +$$

$$\sum_{k=0}^{n-1} \left(\frac{f[x_{-k} \ldots x_0 \ldots x_{k+1}] + f[x_{-k-1} \ldots x_0 \ldots x_k]}{2} s \left(\prod_{i=1}^{k-1} (s^2 - i^2) \right) h^{2k} \right)$$

Neville's method

Let $M = \{x_0, x_1 \dots x_n\} \subseteq [a, b]$
Let $P_M = P_n$
Then

$$P_M(x) = \frac{P_{M \setminus \{x_i\}}(x)(x - x_i) - P_{M \setminus \{x_j\}}(x)(x - x_j)}{x_j - x_i}$$

$f(x_0)$	$\frac{f(x_0)(x-x_0)-f(x_1)(x-x_1)}{x_1-x_0}$	..	$\frac{P_{x_1,\dots x_n}(x_0)(x-x_0)-P_{x_0,\dots x_{n-1}}(x_n)(x-x_n)}{x_n-x_0}$
$f(x_1)$	...	..	
...	...		
$f(x_n)$			

Remainder term

Let $R_n(x) = f(x) - P(x)$
Construct:

$$g(t) = (f(t) - P_n(t)) - R_n(x)\frac{(t-x_0)(t-x_1)\dots(t-x_n)}{(x-x_0)(x-x_1)\dots(x-x_n)}$$

$g(t)$ has $n+2$ zeros, Rolle's theorem implies

$$\exists \xi \in [x_0, x_n],\ g^{(n+1)}(\xi) = 0$$

$$g^{(n+1)}(t) = f^{(n+1)}(t) - R_n(x)\frac{(n+1)!}{(x-x_0)(x-x_1)\dots(x-x_n)}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{k=0}^{n}(x-x_k)$$

Alternative forms of the remainder

$$P_n(x) = f[x_0] + \left(\sum_{k=1}^{n} f[x_0 \dots x_k] \left(\prod_{j=0}^{k-1} (x - x_j) \right) \right) + r \prod_{j=0}^{n} (x - x_j)$$

$$r = f[x, x_0, \dots x_n]$$

$$f(x) - P_n(x) = f[x, x_0, \dots x_n] \prod_{j=0}^{n} (x - x_j)$$

Integral form of the remainder

$$f(x) = f(x_0) + f(x) - f(x_0) = f[x_0] + \int_0^1 f'(x_0 + t_1(x - x_0))dt_1(x - x_0)$$

$$= f[x_0] + \int_0^1 f'(x_0 + t_1(x_1 - x_0)) + f'(x_0 + t_1(x - x_0)) -$$
$$- f'(x_0 + t_1(x_1 - x_0))dt_1(x - x_0) = \dots$$

$$R_n(x) = \int_0^1 \int_0^{t_1} \dots \int_0^{t_{n-1}} f^n(x_0 + t_1(x_1 - x_0) + \dots$$
$$+ t_n(x_n - x_{n-1}))dt_n \dots dt_1(x - x_0)(x - x_1) \dots (x - x_n)$$

Hermite's formula

$$g_k(z) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(z-x_0)(z-x_1)\dots(z-x_{k-1})(z-x_{k+1})\dots(z-x_n)(x-z)}$$

From Cauchy's integral formula:

$$g_k(x_k) = \frac{1}{2\pi i}\oint_{\Gamma_k} \frac{g_k(z)}{z-x_k}dz$$

$$\frac{l(x)}{l'(x_k)(x-x_k)} = \frac{1}{2\pi i}\oint_{\Gamma_k} \frac{l(x)}{l(z)(x-z)}dz$$

$$f(x_k)g_k(x_k) = f(x_k)l_k(x) = \frac{1}{2\pi i}\oint_{\Gamma_k} \frac{f(z)l(x)}{l(z)(x-z)}dz$$

Let Γ' enclose all x_k but not x

$$h(z) = \frac{f(z)l(x)}{l(z)(x-z)}$$

h has singularities enclosed by Γ' at x_k $\forall k = 0 \dots n$

$$\frac{1}{2\pi i}\oint_{\Gamma'} h(z)dz = \sum_{k=0}^{n} Res(h, x_k) = \sum_{k=0}^{n} f(x_k)g_k(x_k)$$

$$P_n(x) = \frac{1}{2\pi i}\oint_{\Gamma'} \frac{f(z)l(x)}{l(z)(x-z)}dz$$

Hermite's remainder

Let Γ enclose all x_k and x

$$\frac{1}{2\pi i}\oint_\Gamma h(z)dz = Res(h,x) + \sum_{k=0}^{n} Res(h,x_k)$$

$$\frac{1}{2\pi i}\oint_\Gamma h(z)dz = -\frac{f(x)l(x)}{l(x)} + P_n(x)$$

$$R_n(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)l(x)}{l(z)(x-z)}dz$$

First convergence result

Let $S = \{z \in \mathbb{C} | dist(z, [-1, 1] \leq 2)\}$
f analytic on $K \supsetneq S$ and $\Gamma \subseteq K$, $\Gamma \cap S = \emptyset$

$$x \in [-1, 1] \Rightarrow |x - x_k| \leq 2$$

$$z \in \Gamma \Rightarrow |z - x_k| > 2$$

$$\exists \gamma < 1, \left| \frac{x - x_k}{z - z_k} \right| \leq \gamma \Rightarrow \left| \frac{l(x)}{l(z)} \right| \leq \gamma^{n+1}$$

$$|R_n(x)| = \left| \frac{1}{2\pi i} \oint_\Gamma \frac{f(z) l(x)}{l(z)(x - z)} dz \right| \leq \frac{\gamma^{n+1}}{2\pi i} \left| \oint_\Gamma \frac{f(z)}{(x - z)} dz \right| \to 0$$

$$\lim_{n \to \infty} P_n(x) = f(x)$$

Weierstrass' theorem

Let $f \in C[-1,1]$ then $\forall \epsilon > 0$ $\exists p \in \Pi$ with $\|f - p\|_\infty < \epsilon$
Consider $\hat{f}$ an extension on $\mathbb{R}$ with compact support for f

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \hat{f}(x)$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \hat{f}(y) e^{-\frac{(x-y)^2}{4t}} dy$$

$$u \in C^\infty \Rightarrow \|T_n(x) - u(x,t)\|_\infty < \frac{\epsilon}{2}, \ \|f(x) - u(x,t)\|_\infty < \frac{\epsilon}{2}$$

$$\|T_n(x) - f(x)\|_\infty < \epsilon$$

Equioscillation

$f \in C[-1,1] \Rightarrow \ \exists! p^* \in \Pi_n$ so $\|f - p^*\|$ is minimal and $\exists x_k$ for $k = 0 \ldots n+1$ with $f(x_k) - p^*(x_k) = \|f - p^*\|$ and $(f(x_k) - p^*(x_k))(f(x_{k+1}) - p^*(x_{k+1})) \leq 0$

Suppose p^* equioscillates on x_k and $\exists q \in \Pi_n$ with $\|f - p^*\| > \|f - q\|$ then $(p^* - q)(x_k)(p^* - q)(x_{k+1}) < 0 \Rightarrow \Leftarrow$

Suppose p^* is the best approximation and it equioscillates at $x_k \ k = 0..i, \ i \leq n$ Construct $\delta(x) = (x - x_0)(x - x_1) \ldots (x - x_k)$

$$\|f - p^*\| < \|f - p^* + \epsilon\delta\|$$

for small $\epsilon \Rightarrow \Leftarrow$

Pessimistic Result

Let $f(x) = g(\theta) = f(cos(\theta))$
Let S_n be the truncation of the Fourier series of $g(\theta)$

$$S_n(\theta) = \sum_{k=0}^{n} e^{ik\theta} \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi) e^{-ik\phi} d\phi$$

(Losinsky-Kharshiladze)

$$\|S_n\| > c \ln(n) \Rightarrow \|P_n\| > \frac{2}{\pi^2} \ln(n-1) - \frac{1}{2}$$

If x_n is an uniform mesh then

$$\|P_n\| > 2^n$$

For every sequence of nodes x_{kn} there is a $f \in C[-1,1]$ with $P_n f$ divergent

Rate of convergence

Let f be a function so $f^{(k)} \in BV[-1,1]$, $V_{-1}^{1} f(x) = v$ and x_i be Chebyshev nodes, then for $n > k$

$$\|f - P_n\|_\infty < \frac{4v}{\pi k (n-k)^k}$$

Let f be analytic in a Bernstein ellipse E_ρ with $|f(z)| < M$

$$\|f - P_n\|_\infty < \frac{4M\rho^{-n}}{\rho - 1}$$

Lebesgue's constant

$$\Lambda = \sup_{\|f\|_\infty = 1} \|P_n(f)\|_\infty$$

The Lebesgue function:

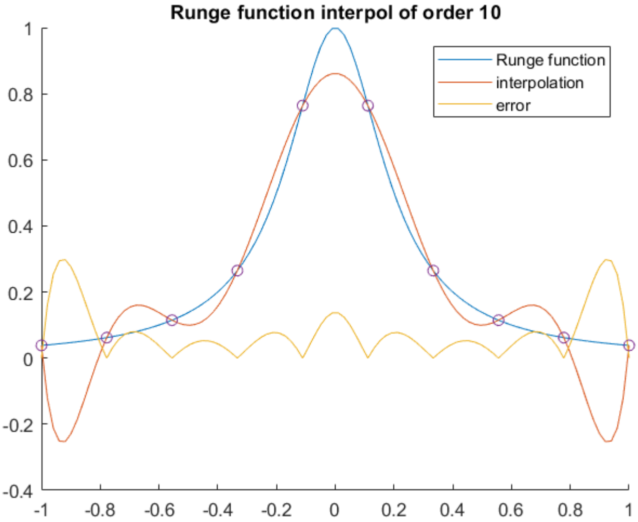
$$\lambda(x) = \sum_{k=0}^{n} |l_k(x)| \Rightarrow \Lambda = \sup_{x \in [-1,1]} \lambda(x)$$

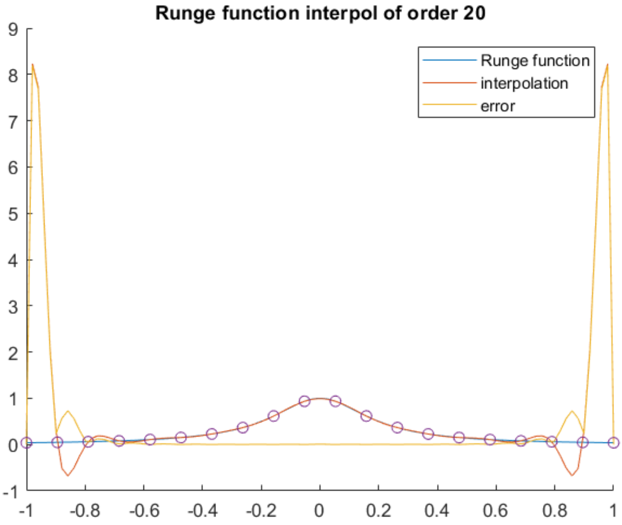
$$\|f - P_n\| < (\Lambda + 1)\|f - p^*\|$$

If P_n is an interpolant on Chebyshev nodes

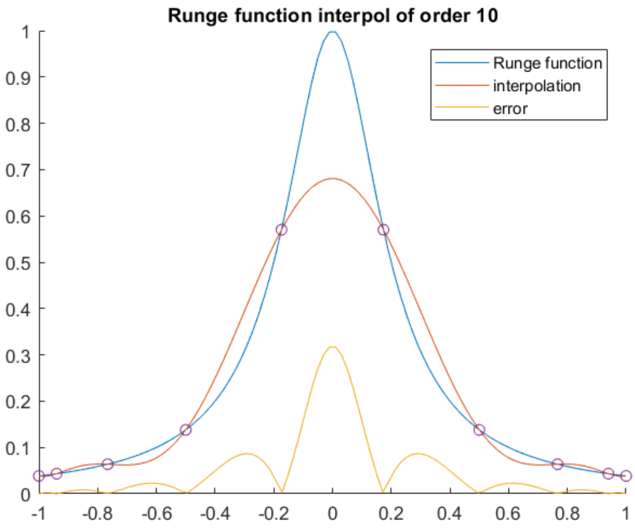
$$\|f - P_n\| < \left(2 + \tfrac{2}{\pi} \ln\left(n+1\right)\right) \|f - p^*\|$$

Runge's phenomenon





Runge function interpol of order 20
Runge function
interpolation
error
9
8
7
6
5
4
3
2
1
0
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Runge function interpol of order 10
Runge function
interpolation
error
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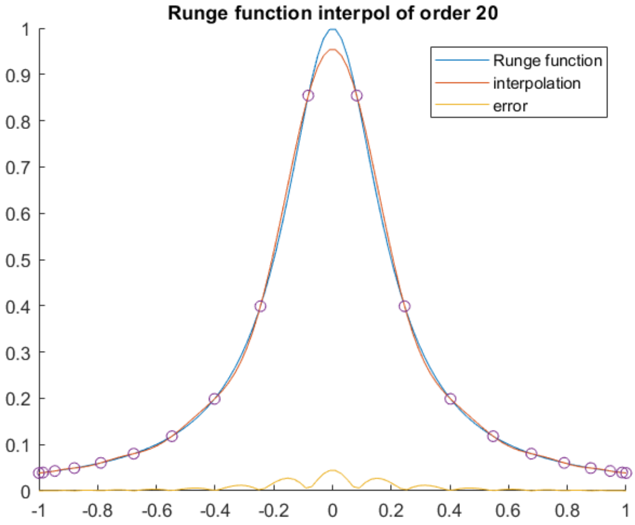
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Runge function interpol of order 20
Runge function
interpolation
error
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0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
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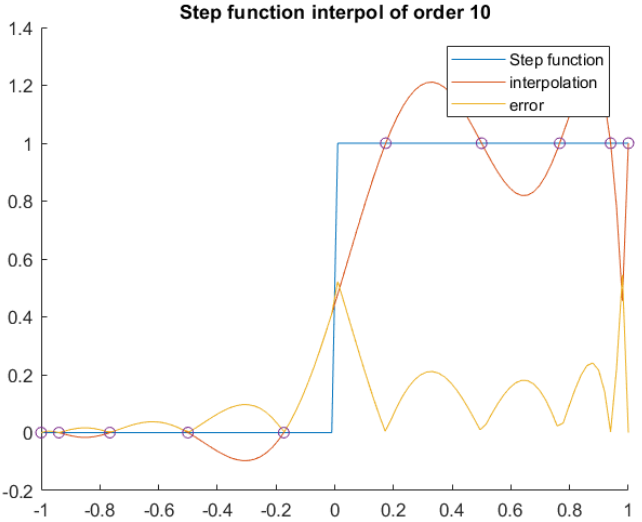
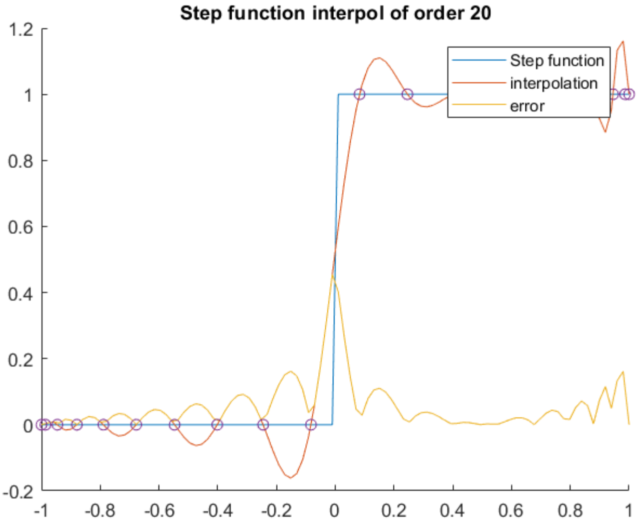
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Gibbs's phenomenon





Step function interpol of order 20
Step function
interpolation
error
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

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