Numerical linear algebra II

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Eigenvalues

Let $A \in M^{nxn}$, λ is an eigenvalue

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Algebraic multiplicity of an eigenvalue = the algebraic multiplicity of the root

$$\det(A^* - \lambda I) = \det(A - \bar{\lambda} I)^*$$

Eigenvalues of A are complex conjugates of eigenvalues of A^*

Eigenvectors

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Eigenvectors from different eigenvalues are linearly-independent
Geometric multiplicity = number of linearly-independent eigenvectors for an eigenvalue

Characteristic Polynomial

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$$p(t)I = (tI - A)B = (tI - A)\left(\sum_{k=0}^{n-1} t^k B_k\right)$$

$$t^n B_{n-1} + \sum_{k=1}^{n-1} t^k (B_{k-1} - AB_k) - AB_0$$

Characteristic Polynomial

$$p(t)I = \sum_{k=0}^{n} t^n c_n I$$

$$B_{n-1} = c_n I,\ B_{k-1} - AB_k = c_k I,\ -AB_0 = c_0$$

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$$p(A) = \sum_{k=0}^{n} c_n A^n = A^n B_{n-1} + \sum_{k=1}^{n-1} (A^k B_{k-1} - A^{k+1} B_k) - AB_0 = 0$$

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Viète:

$$\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$$

$$\det(A) = \lambda_1 \ldots \lambda_n$$

Similarity Transforms

$$A \sim B \Leftrightarrow A = M^{-1}BM$$

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1. reflexive : $A = I^{-1}AI$
2. transitive:
   $A = M^{-1}BM,\ B = N^{-1}CN \Rightarrow A = (NM)^{-1}C(NM)$
3. symmetric: $A = M^{-1}BM \Rightarrow B = (M^{-1})^{-1}AM^{-1}$

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Similarity is a change of basis for linear operators

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Similarity is a change of basis for linear operators

$$\det(\lambda I - M^{-1}AM) = \det(M^{-1}(\lambda I - A)M) = \det(\lambda I - A)$$

Eigenvalue decomposition

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Let V be the matrix of eigenvectors

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If $\forall\ k \leq n\ a_k = g_k$ then V has n linearly independent columns

$$V^{-1}AV = \Lambda$$

Schur's lemma

$$A = M^{-1}TM$$

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If A is a $1x1$ matrix it is upper-triangular
Induction for $A \in M^{(n-1)x(n-1)}$
Let $Av_1 = \lambda v_1$ and $(v_1, b_2 \ldots b_n)$ be a base

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B is upper-triangular in $(v_2 \dots v_n)$

$$v_i = \sum_{k=2}^{n} \gamma_{ki} b_k$$

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$$Av_i = A\sum_{k=2}^{n} \gamma_{ki} b_k = \sum_{k=2}^{n} A\gamma_{ki} b_k = \sum_{k=2}^{n}(\gamma_{ki}\alpha_i v + B\gamma_{ki} b_k)$$

$$Av_i = \sum_{k=2}^{n} \gamma_{ki}\alpha_i v + B\sum_{k=2}^{n} \gamma_{ki} b_k = \sum_{k=2}^{n} \gamma_{ki}\alpha_i v + Bv_i$$

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$$RTR^{-1} = U$$

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Normal matrices

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T is diagonal

$$T = \Lambda \Rightarrow A = Q\Lambda Q^*$$

Jordan Canonical Form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda_i & 1 & \dots & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & \lambda_i & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}$$

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$$A = MJM^{-1} = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_k \end{bmatrix}$$

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One eigenvector for one Jordan block J_k

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$$Ax_2 = \lambda x_2 + x_1 \Rightarrow (A - \lambda I)^2 x_2 = 0$$

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$$J - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\ker(J - \lambda I) = \operatorname{span}(1, 0)$$

$$\dim \ker(J - \lambda I) = 1 \neq 2$$

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If $\operatorname{im}(A = \lambda I) \cap \ker(A - \lambda I) = \{0\}$ then λ has $n - r$ independent eigenvectors.

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If $\operatorname{im}(A = \lambda I) \cap \ker(A - \lambda I) = \{0\}$ then λ has $n - r$ independent eigenvectors.

$$Q = \operatorname{im}(A - \lambda I) \cap \ker(A - \lambda I) \neq \{0\}$$

$\dim Q = s < r$ there are s vectors from Q in $(p_1, \ldots, p_r)$

Let $p_{i_j} \in Q$ and $(A - \lambda I)q_{i_j} = p_{i_j}$, $\{q_{i_j}\}$ is linearly-independent.

Jordan Form

Let $p_{i_j} \in Q$ and $(A - \lambda I)q_{i_j} = p_{i_j}$, $\{q_{i_j}\}$ is linearly-independent.
(p_i) are eigenvectors, (q_{i_j}) are generalized eigenvectors

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Let $(z_i) \in \ker(A - \lambda I) \setminus Q$ be a base
If $x \in \ker C^{k+1}, x \notin \ker C^k, y \in \ker C^k$ then let
$\alpha x + \beta y = 0 \Rightarrow C^k(\alpha x + \beta y) = 0 \Rightarrow \alpha C^k x = 0 \Rightarrow \alpha = 0 \Rightarrow \beta = 0$,
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$x,\ y$ are independent
$q_{i_j},\ p_i,\ z_i$ are independent

$$\dim\ker(A-I) + \dim\operatorname{im}(A-\lambda I) + \dim Q = n$$

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$\dim \operatorname{im}(A - \lambda I)^k$ defines the Jordan blocks.
Let $J_1,\ J_2$ Jordan forms of A
$J_1,\ J_2$ are similar, so $\dim \operatorname{im}(J_1 - \lambda I)^k = \dim \operatorname{im}(J_2 - \lambda I)^k$

Uniqueness of the Jordan Form

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Let $J_1,\ J_2$ Jordan forms of A
$J_1,\ J_2$ are similar, so $\dim \operatorname{im}(J_1-\lambda I)^k = \dim \operatorname{im}(J_2-\lambda I)^k$ $J_1,\ J_2$ have the same blocks for all eigenvalues.

Eigenvalues of normal matrices

Let $A = A^*$

$$Ax = \lambda x \Rightarrow x^* A^* = \bar{\lambda} x^* \Rightarrow x^* A x = \bar{\lambda} x^* x \Rightarrow (\lambda - \bar{\lambda}) x = 0 \Rightarrow \lambda \in \mathbb{R}$$

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$$Qx = \lambda x \Rightarrow (Qx)^* = \bar{\lambda} x^* \Rightarrow x^* Q^* Q x = \lambda \bar{\lambda} x^* x \Rightarrow |\lambda| = x$$

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Let A be positive-definite

$$Ax = \lambda x \Rightarrow x^* A x = \lambda \|x\|_2 \Rightarrow \lambda > 0$$

Singular Value Decomposition

Let $A \in M^{nxm}$ with full-rank $x \neq 0 \Rightarrow Ax \neq 0$
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Av_i is an eigenvector for AA^*

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$x^* A^* Ax > 0$ AA^* and $A^* A$ are positive-definite

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$$u_i = \frac{Av_i}{\sigma_i} \Rightarrow \sigma_i \sigma_j u_i^* u_j = (Av_i)^* Av_j = v_i^* A^* Av_j = v_i \lambda_j v_j = 0$$

Singular Value Decomposition

$$Ax = 0 \Rightarrow A^*Ax = 0 \Leftrightarrow x \in \ker A \Rightarrow x \in \ker A^*A$$

$$Ax = y,\ A^*Ax = 0 \Rightarrow x^*A^*Ax = 0 \Rightarrow \|Ax\| = 0 \Rightarrow Ax = 0$$

$$x \in \ker A^*A \Rightarrow x \in \ker A$$

Singular Value Decomposition

$$Ax = 0 \Rightarrow A^*Ax = 0 \Leftrightarrow x \in \ker A \Rightarrow x \in \ker A^*A$$

$$Ax = y,\ A^*Ax = 0 \Rightarrow x^*A^*Ax = 0 \Rightarrow \|Ax\| = 0 \Rightarrow Ax = 0$$

$$x \in \ker A^*A \Rightarrow x \in \ker A$$

$$\ker A = \ker A^*A \Rightarrow \operatorname{rank} A = \operatorname{rank} AA^* = \operatorname{rank} A^*A$$

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$$A[v_1 \ldots v_r] = [u_1 \ldots u_r] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

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$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\sigma = |\lambda|$$

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$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

Properties of different matrix norms:

$$\|Ax\| \leq \|A\| \|x\|$$

Matrix norms

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Let $\| \cdot \|$ be induced, $\|I\| = \sup_{\|x\|=1} \|x\| = 1$

Frobenius Norm

The Frobenius norm $\|\cdot\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^2\right)^{\frac{1}{2}}$ is not induced

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$$\operatorname{tr} AA^* = \sum_{i=1}^n \sum_{j=1}^m a_{ij}\bar{a}_{ij} = \|A\|_F^2 = \operatorname{tr} A^*A \Rightarrow \|A\|_F = \sqrt{\sum_{k=1}^r \sigma_k^2}$$

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$$\|AQ\|_F = \sqrt{\operatorname{tr} AQQ^*A^*} = \sqrt{\operatorname{tr} AA^*} = \|A\|_F = \|QA\|_F$$

$$\|A^*\|_F = \sqrt{\operatorname{tr} A^*A} = \|A^*\|_F$$

The 2-norm

$$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\| \Rightarrow \|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\sup_{x\neq 0} \frac{x^*A^*Ax}{x^*x}}$$

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$$\frac{\sqrt{\sum_{i=1}^r |\lambda_i y_i|^2}}{\sqrt{\sum_{i=1}^r |y_i|^2}} \leq \sqrt{\lambda_{\max}} \frac{\sqrt{\sum_{i=1}^r |y_i|^2}}{\sqrt{\sum_{i=1}^r |y_i|^2}} = \sqrt{\lambda_{max}} \Rightarrow \|A\|_2 = \sigma_{\max}$$

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If A is normal $\|A\|_2 = |\lambda_{\max}|$

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$$\|Ax\|_F = \sqrt{\sum_{i=1}^{n} \left(\sum_{j=1}^{m} |a_{ij} x_j| \right)^2} \leq \sqrt{\sum_{i=1}^{n} \left(\left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{j=1}^{n} |x_j|^2 \right) \right)} =$$

$$= \sqrt{\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \right) \left(\sum_{j=1}^{n} |x_j|^2 \right)} = \|A\|_F \|x\|_F$$

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Frobenius norm is consistent: $\|AB\|_F \leq \|A\|_F\|B\|_F$

More Matrix Norms

The maximum norm $\|A\|_{\max} = \max_{ij} |a_{ij}|$ is not induced

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ AB = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix},\ \|AB\| = 2 \geq \|A\|\|B\| = 1$$

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$$\alpha_k = \sum_{i=1}^{m} |a_{ik}|, \;\; \alpha_{\max} = \max_k \alpha_k$$

$$\|A\|_1 \leq \sum_{j=1}^{n} |x_j| \alpha_j \leq \alpha_{\max} \sum_{j=1}^{n} |x_j| = \alpha_{\max}$$

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An upper-triangular matrix has the eigenvalues on the diagonal

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Where $V = U + \Lambda$ and U is strictly upper-triangular

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Let $D = \text{diag}\,(\epsilon,\ \ldots,\ \epsilon^n)$, $D^{-1} = \text{diag}\,(\epsilon^{-1},\ \ldots,\ \epsilon^{-n})$

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$$\delta = \frac{\epsilon}{2\|U\|_\infty}$$

$$\|D^{-1}Q^*AQD\|_\infty \leq \rho(A) + \frac{\epsilon}{2} < \rho(A) + \epsilon$$

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$$\|M\|_\epsilon = 0 \Rightarrow \|D^{-1}Q^*MQD\|_\infty = 0 \Rightarrow D^{-1}Q^*MQD = O \Rightarrow M = O$$

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$$\|M+N\|_\epsilon = \|D^{-1}Q^*(M+N)QD\|_\infty = \|D^{-1}Q^*MQD+$$
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$\|\cdot\|_\epsilon$ is a matrix norm

Matrix Norms and Spectral Radius

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Matrix Norms and Spectral Radius

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$\|\cdot\|_\epsilon$ is a consistent matrix norm

$$\rho(A) \leq \|A\|, \ \forall \|\cdot\|$$

$$\forall\ \epsilon > 0\ \exists\ \|\cdot\|_\epsilon,\ \|A\|_\epsilon < \rho(A) + \epsilon$$

Matrix Norms and Spectral Radius

$$\|MN\|_\epsilon = \|D^{-1}Q^*MNQD\|_\infty = \|(D^{-1}Q^*MQD)(D^{-1}QNQD)\|_\infty \leq$$
$$\leq \|D^{-1}Q^*MQD\|_\infty \|D^{-1}Q^*NQD\|_\infty = \|M\|_\epsilon \|N\|_\epsilon$$

$\|\cdot\|_\epsilon$ is a consistent matrix norm

$$\rho(A) \leq \|A\|, \ \forall \|\cdot\|$$

$$\forall \ \epsilon > 0 \ \exists \ \|\cdot\|_\epsilon, \ \|A\|_\epsilon < \rho(A) + \epsilon$$

$$\rho(A) = \inf_{\|\cdot\|} \|A\|$$

Spectral Radius

$$Ax = \lambda x \Rightarrow AAx = \lambda^2 x \Rightarrow A^k x = \lambda^k x$$

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Let A normal and $\|x\|_2 = 1$, $A = U\Lambda U^*$, $y = U^* x \Rightarrow \|y\|_2 = 1$

$$\|Ax\|_2^2 = x^* A^* A x = x^* U \Lambda^* U^* x = y^* \Lambda^* \Lambda y = \sum_{k=1}^{n} |\lambda_k|^2 |y_k|^2$$

$$\sum_{k=1}^{n} |\lambda_k|^2 |y_k|^2 \leq \rho(A)^2 \sum_{k=1}^{n} |y_k|^2 = \rho(A)^2 \|y\|_2^2$$

$$\|Ax\|_2 \leq \rho(A)\|x\|$$

Matrix Convergence

Let $\lim_{k\to\infty} A^k = 0$ and $Av = \lambda v \neq 0$

$$\lim_{k\to\infty} A^k = 0 \Rightarrow \lim_{k\to\infty} A^k v = 0 \Rightarrow \lim_{k\to\infty} \lambda^k v = 0 \Rightarrow \lim_{k\to\infty} \lambda^k = 0$$

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$$J_i = \begin{bmatrix} \lambda_i & 1 & \\ & \ddots & \\ & & \lambda_i \end{bmatrix}, J_i^k = \begin{bmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{g-1}\lambda_i^{k-g+1} \\ & \ddots & \\ & & \lambda_i^k \end{bmatrix}$$

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If $\rho(A) = 1$ and A is not diagonalizable $\lim_{k\to\infty} kP(k)\lambda_i^k = \infty$ so
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Neumann Series

Let $B \in M^n$ with $\|B\| < 1$

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$$(I - B)\sum_{k=0}^{n} B^k = \sum_{k=0}^{n}(B^k - B^{k+1}) = I - B + + \ldots + B^n - B^{n+1} = I - B^{n-1}$$

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Gelfand's formula

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Corollaries:

$$\rho(AB) = \lim_{k\to\infty} \|(AB)^k\|^{\frac{1}{k}} \leq \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} \|B^k\|^{\frac{1}{k}} =$$

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$$\sum_{i=1}^n |a_{ik}| \le \sqrt{n} \sqrt{\sum_{i=1}^n |a_{ik}|^2} = \sqrt{n}\sqrt{A^*A_{kk}} \le \sqrt{n}\sqrt{\lambda_{\max}(A^*A)}$$

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$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_\infty} \le \sup_{x\neq 0} \frac{\|Ax\|_1}{\frac{1}{n}\|x\|_1} = n\|A\|_1$$

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Schatten Norms

Let $A = U\Sigma V^*$

$$\|A\|_{sp} = \left(\sum_{k=1}^{n} \sigma_k^p\right)^{\frac{1}{p}} = (\mathrm{tr}((AA^*)^{\frac{p}{2}}))^{\frac{1}{p}}$$

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$$\forall \lambda,\ \lambda \in G$$

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$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$$

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Eigenvalue Condition Number

$$A = M\Lambda M^{-1} \Rightarrow M^{-1}AM = \Lambda \Rightarrow M^{-1}A = M^{-1}\Lambda$$

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$$M^{-1}(A + \delta A)M = \begin{bmatrix} \lambda + y^*\delta A x & \epsilon & \dots \\ \epsilon & & \dots\dots \end{bmatrix}$$

Eigenvalue Condition Number

$$SM^{-1}(A + \delta A)MS^{-1} = \begin{bmatrix} \lambda + y^*\delta Ax & s\epsilon & \dots \\ s^{-1}\epsilon & & \dots\dots\dots \end{bmatrix}, S = \text{diag}\,(s, 1, \dots)$$

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$$\kappa(\lambda) = \frac{\|x\|\|y\|}{y^*x} \Rightarrow \kappa(A) = \|M\|\|M^{-1}\|$$

Power Method

Let $A \in M^n$, $b_0 \in C^n$, λ the dominant unique eigenvalue

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$$b_k = \frac{\lambda^k c_1 m_1 + MJ^k(c_2 e_2 + \ldots + c_n e_n)}{\|\lambda^k c_1 m_1 + MJ^k(c_2 e_2 + \ldots + c_n e_n)\|}$$

Power Method

$$b_k = \frac{\lambda^k c_1}{|\lambda^k c_1|} \frac{v_1 + \frac{1}{c_1} M (\frac{1}{\lambda} J)^k (c_2 e_2 + \ldots + c_n e_n)}{\| v_1 + \frac{1}{c_1} M (\frac{1}{\lambda} J)^k (c_2 e_2 + \ldots + c_n e_n) \|}$$

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$$J = \begin{bmatrix} \lambda & 0 & \ldots \\ 0 & J_2 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & & J_i \end{bmatrix} \Rightarrow (\frac{1}{\lambda} J)^k = \begin{bmatrix} 1 & 0 & \ldots \\ 0 & (\frac{1}{\lambda} J_2)^k & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & & (\frac{1}{\lambda} J_i)^k \end{bmatrix}$$

Power Method

$$b_k = \frac{\lambda^k c_1}{|\lambda^k c_1|} \frac{v_1 + \frac{1}{c_1} M(\frac{1}{\lambda} J)^k (c_2 e_2 + \ldots + c_n e_n)}{\|v_1 + \frac{1}{c_1} M(\frac{1}{\lambda} J)^k (c_2 e_2 + \ldots + c_n e_n)\|}$$

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$$\sim (\frac{1}{\lambda}J_2)^k \to 0 \sim \left|\frac{\lambda_2}{\lambda}\right| \to 0$$

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Let $A \in M^n$, $A = A^*$

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Let U, $\dim(U) = k$, $\dim \operatorname{span}\left(q_k, \ \ldots, \ q_n\right) = n - k + 1$

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Deflation techniques

Let $Ax = \lambda x$, $Au = \mu u$, $A_k = A_{k,:}$, $A_k x = \lambda x_k$

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Householder transformation

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hdotsfor{4} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

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$$A = Q^* H Q$$

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