Unsteady Boundary Layer Flow and Heat Transfer Over a Stretching Sheet

Cornelia Revnic*, Teodor Grosan† and Ioan Pop[1†]

* "Tiberiu Popoviciu" Institute of Numerical Analysis, P.O.Box. 68-1, 400110, Cluj-Napoca, Romania
† Babes-Bolyai University, Applied Mathematics, R-3400 Cluj, CP 253, Romania

Abstract. Unsteady two-dimensional boundary layer flow and heat transfer over a stretching flat plate in a viscous and incompressible fluid of uniform ambient temperature is investigated in this paper. It is assumed that the plate is isothermal and is stretched in its own plane. Using appropriate similarity variables, the basic partial differential equations are transformed into a set of two ordinary differential equations. These equations are solved numerically for some values of the governing parameters, using Rungge-Kutta method of fourth order. Flow and heat transfer characteristics are determined and represented in some tables and figures. It is found that the structure of the boundary layer depends on the ratio of the velocity of the potential flow near the stagnation point to that of the velocity of the stretching surface. In addition, it is shown that the heat transfer from the plate increases when the Prandtl number increases. Our results are shown to include the steady situation as a special case considered by other authors. Comparison with known results is very good.

Keywords: heat transfer, stretching surface, the external inviscid flow, stagnation-point flow, boundary layer.
PACS: 02.60Lj, 44.20+b, 44.27+g, 47,10Ad, 47.85-g

INTRODUCTION

The unsteady boundary layers are important in several physical problems in aero - nautics, missile dynamics, acoustics etc. The work in this area was initiated by Moore [1], Lighthill [2] and Lin [3]. Critical reviews of unsteady boundary layers were presented by Stuart [4], Riley [5], Telionis [6], [7] and Pop [8]. In recent years certain aspects of the unsteady flows were investigated by Ma and Hui [9] and Ludlow et al. [10] using the classical method of Lie-group. The essence of the Lie-group method is that each of the variables in the initial equation is subjected to an infinitesimal transformation and the demand that the equation is invariant under these transformations leads to the determination of the possible symmetries (see Ludlow et al. [10]). The fundamental governing equations for fluid mechanics are the Navier-Stokes equations. This nonlinear set of partial differential equations have no general solutions, and only a small number of exact solutions have been found (see Wang [11]). Exact solutions are important for the following reasons: (i) the solutions represent fundamental fluid-dynamic flows. Also, owing to the uniform validity of exact solutions, the basic phenomena described by the Navier-Stokes equations can be more closely studied. (ii) the exact solutions serve as standards for checking the accuracies of the many approximate methods, whether they are numerical, asymptotic, or empirical.

Flow of a viscous fluid over a stretching sheet has an important bearing on several technological processes. In particular in the extrusion of a polymer in a melt-spinning process, the extruded from the die is generally drawn and simultaneously stretched into a sheet which is then solidified through quenching or gradual cooling by direct contact with water. Further, glass blowing, continuous casting of metals and spinning of fibres involve the flow due to a stretching surface, see Lakshmisha et al. [12]. In all these cases, a study of the flow field and heat transfer can be of significant importance since the quality of the final product depends to a large extent on the skin friction coefficient and the surface heat transfer rate. Crane [13] presented a simple closed form exponential solution of the steady two-dimensional flow caused solely by a linearly stretching sheet in an otherwise quiescent incompressible fluid. The simplicity of the geometry and the possibility of obtaining further exact solutions through simple generalizations have generated a lot of interest in extending it to more general situations. Such extensions include consideration of more general stretching velocity, application to non-Newtonian fluids, and inclusion of other physical effects such as suction or blowing, magnetic fields, etc. Unsteady two-dimensional boundary layer flow over a stretching surface has been studied by Na and Pop [14], Wang et al. [15], Elbashbeshy and Badiz [16], Sharidan et al. [17] and Ali and Magyari [18], while Lakshmisha et al. [12], Devi et al. [19] and Takhar et al. [20] have considered the unsteady three-dimensional-flow due to the impulsive motion of a stretching surface. The aim of the present analysis is to study the unsteady flow and heat transfer in the stagnation-point flow on a heated stretched surface in a viscous and incompressible fluid when both velocities of the stretching sheet and of the external flow (inviscid flow) are proportional to the distance from the stagnation-point and inversely to time. The geometry is similar to that proposed by Mahapatra and Gupta [21] for the steady two-dimensional stagnation-point flow towards a stretching sheet. The parabolic partial differential equations governing the flow and heat transfer have been reduced to a system of two ordinary differential equations which are solved using an implicit finite-difference scheme in combination with the shooting method.

PROBLEM FORMULATION

We consider the unsteady two-dimensional forced convection flow and heat transfer of a viscous and incompressible fluid near a stagnation point on a surface coinciding with the plain $y = 0$, the flow being confined to $y > 0$. Two equal and opposite forces are applied along the x - axis at the initial time $t = 0$, so that the surface is stretched keeping the origin fixed as shown in Fig.1. It is assumed that the uniform temperature of the plane is T_w , while the temperature of the ambient fluid is T_∞, where $T_w > T_\infty$ (heated plate). It is also assumed that the viscous dissipation effects are neglected. Under these assumptions, the system of boundary layer equations are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

subject to the initial and boundary conditions are of the form:

$$\begin{aligned} t &< 0: \ u=0,\ v=0,\ T=T_\infty \ \ for \ \ any \ \ y>0 \\ t &> 0: \ u=u_w(t,x),\ v=0,\ T=T_w \ \ for \ \ y=0 \\ t &= 0: \ u=u_{ws}(x),\ v=0,\ T=T_w \\ & \quad u\to u_e(t,x),\ T\to T_\infty \ \ as \ \ y\to\infty \end{aligned} \tag{4}$$

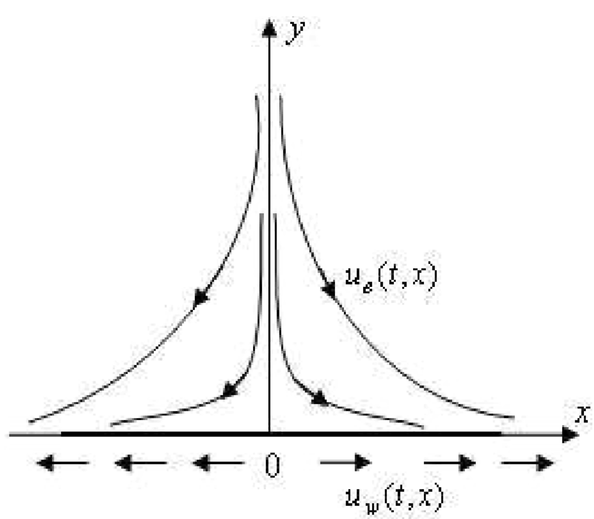

FIGURE 1. Physical model and coordinate system.

where u and v are the velocity components along the $x-$ and $y-$ axis, T is the fluid temperature, ν is the kinematic viscosity, $u_{ws} = cx$ (c is a positive constant) and α is the thermal diffusivity. Following Surma Devi [19] et al., we assumed that $u_w(t,x)$ and $u_e(t,x)$ are given by

$$u_w(t,x) = \frac{cx}{(1-\gamma t)}, \quad u_e(t,x) = \frac{ax}{(1-\gamma t)} \tag{5}$$

where a is a positive constant. The momentum and energy equations can be transformed to the corresponding ordinary differential equations by the following substitutions:

$$\begin{aligned} \psi &= (c\nu/(1-\gamma t))^{1/2} x f(\eta), \\ \theta(\eta) &= (T-T_\infty)/(T_w-T_\infty), \\ \eta &= (c/\nu(1-\gamma t))^{1/2} y \end{aligned} \tag{6}$$

where ψ is the stream function which is defined in the usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$.
Substituting (6) into Eqs. (2) and (3), we obtain the following two ordinary differential equations:

$$f''' + f f'' + \frac{a^2}{c^2} - f'^2 + \frac{\gamma}{c}\left(\frac{a}{c} - \frac{\eta}{2} f'' - f'\right) = 0 \tag{7}$$

$$\frac{1}{Pr}\theta'' + f\,\theta' - \frac{\gamma}{2c}\eta\theta' = 0 \tag{8}$$

subject to the boundary conditions (4) which become

$$f(0) = 0, f'(0) = 1, \theta(0) = 1 \tag{9}$$

$$f'(\infty) = \frac{a}{c}, \theta(\infty) = 0 \tag{10}$$

where Pr is the Prandtl number and primes denote differentiation with respect to η.
The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho u_{ws}^2}, \; Nu_x = \frac{x q_w}{k(T_w - T_\infty)}, \tag{11}$$

where τ_w is the skin friction and q_w is the heat transfer from the plate which are given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}, \; q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{12}$$

with μ and k being the dynamic viscosity and thermal conductivity, respectively. Using (6), we get

$$\begin{aligned} (1-\gamma t)^{3/2} Re_x^{1/2} C_f &= f''(0), \\ (1-\gamma t)^{1/2} Re_x^{-1/2} Nu_x &= -\theta'(0) \end{aligned} \tag{13}$$

Where $Re = (cx)x/\nu$ is the low Reynolds number. It is important to notice that for the steady-state case,Eqs. (7) and (8) reduced to

$$f''' + f f'' - f'^2 + \frac{a^2}{c^2} = 0 \tag{14}$$

$$\frac{1}{Pr}\theta'' + f\,\theta' = 0 \tag{15}$$

with the boundary conditions (9)-(10). Equations (14) and (15) with the boundary conditions (9)-(10) where established by Mahapatra and Gupta [21].

TABLE 1. Values of $f''(0)$ for some values of a/c when the flow is steady. ( ) values reported by Mahapatra and Gupta [21].

a/c	**0.10**	**0.20**	**0.50**	**2.00**
$f''(0)$	-0.9696	-0.9182	-0.6673	2.0175
	(-0.9694)	(-0.9181)	(-0.6673)	(2.0175)

TABLE 2. Values of $\theta'(0)$ for some values of a/c and Pr when the flow is steady. ( ) values reported by Mahapatra and Gupta [21].

a/c / Pr	**0.05**	**0.5**	**1**	**1.5**
0.1	-0.081	-0.381	-0.603	-0.777
	(-0.081)	(-0.383)	(-0.603)	(-0.777)
0.5	-0.137	-0.472	-0.691	-0.863
	(-0.136)	(-0.473)	(-0.692)	(-0.863)
2	-0.248	-0.711	-0.978	-1.171
	(-0.241)	(-0.709)	(-0.974)	(-1.171)

SOLUTION

The systems of ordinary differential equations (7)-(8) and (14)-(15) subject to the boundary condition (9)-(10) have been solved numerically for some values of the parameters a/c, t and Pr using Rungge-Kutta method of fourth order combined with the shooting technique. For the physical considearation we take $\gamma = -1$. Some values of $f''(0)$ and $\theta'(0)$ are given in Tables 1 and 2 for the case of the steady flow.

We can see from these tables that there is a very good agreement between our results and those obtained by Mahapatra and Gupta [21]. Therefor, we are confident that the results obtained using the present method are accurate.

Figures 2 - 5 show the velocities profiles f and f' along with the corresponding streamlines patterns for the case of unsteady flow, Eqs. (7) and (8). The values of the parameters are $a = 0.1$, $c = 1$ and $t = 0, 1, 2, 3$. It is interesting to notice that the solution of Eq. (7) is not unique. Thus, there are two solutions, one Fig. 2 representing an attached flow and the other one Fig. 4 the reversed flow. These is in agreement with the results obtained by Ma and Hui [9] for the unsteady two-dimensional boundary layer flow near a stagnation point of a fixed plat plate.

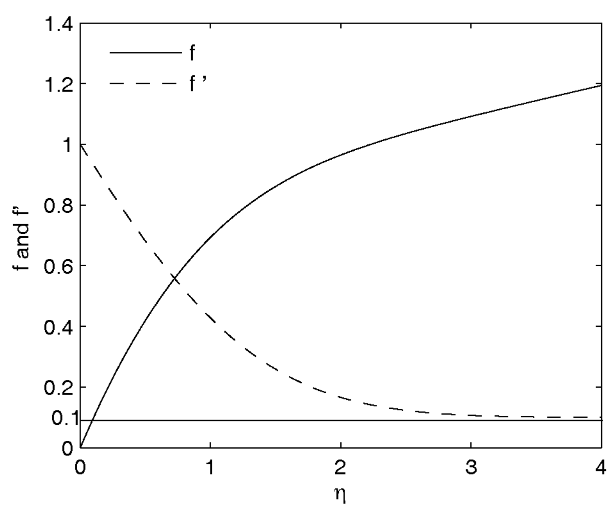

FIGURE 2. The first solution of $f(\eta)$ and $f'(\eta)$ for $a/c = 0.1$.

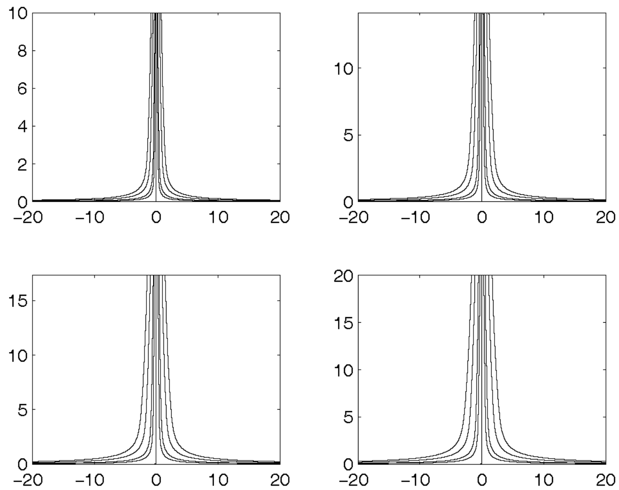

FIGURE 3. The streamlines function for: $t = 0, 1, 2$ and 3 corresponding to the first solution for $a/c = 0.1$.

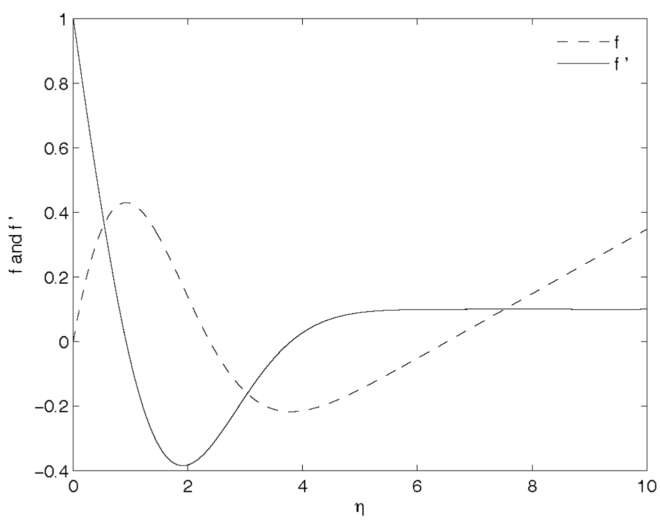

FIGURE 4. The second solution of $f(\eta)$ and $f'(\eta)$ for $a/c = 0.1$.

Fig. 6 illustrates the dimensionless temperature profiles $\theta(\eta)$ for some values of Pr when $a/c = 2$. We notice that temperature profile increase when Pr decreases. Further, Fig. 7 shows the variation of the heat transfer from the wall $-\theta'(0)$ with a/c and different values of Pr. It is evident from Fig. 7 that an increase in Pr result in a decrease in the thermal boundary layer thickness and as a consequence the heat transfer from the wall increases with Pr.

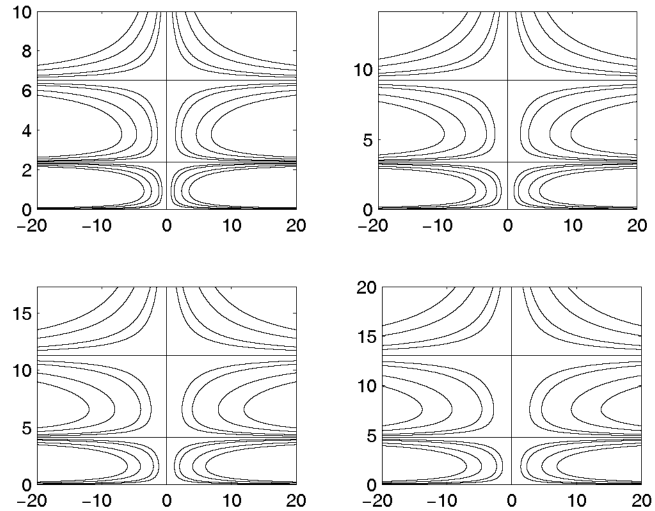

FIGURE 5. The streamlines function for: $t = 0, 1, 2$ and 3 corresponding to the first solution for $a/c = 0.1$.

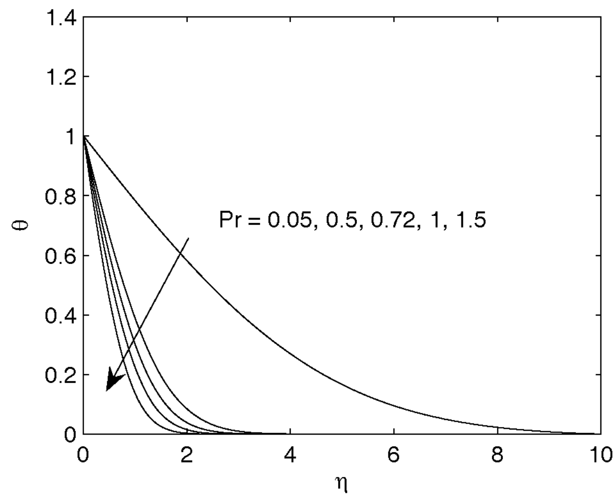

FIGURE 6. Temperature profiles of $\theta(\eta)$ for several values of Pr and $a/c = 2$ in respect with η.

CONCLUSION

The unsteady two-dimensional stagnation-point flow and heat transfer of a viscous and incompressible fluid over an isothermal stretching flat plate in its own plane has been numerically analyzed in detailed. Following Surma Devi et al. [19] similarity variables where used to reduced the governing partial differential equations to ordinary differential equations. Solving numerically these equations, we have been able to determine the velocity and temperature profiles, skin friction and heat transfer from the plate. For the case of steady-state flow, we have compared our present results with those of Mahapatra and Gupta [21]. The agreement between the results is excelent. Effects of a/c and Pr on the flow and heat transfer characteristic have been examened and discussed in detail. It is shown that for small values of a/c the solution of the ordinary differential equation is not unique. One solution represents an attached flow and the other one a reversed flow. It shoud be noticed that we have determined solutions of the problem for more values of the governing parameters but in order to save space, the reported results are limited only to some values of these parameters.

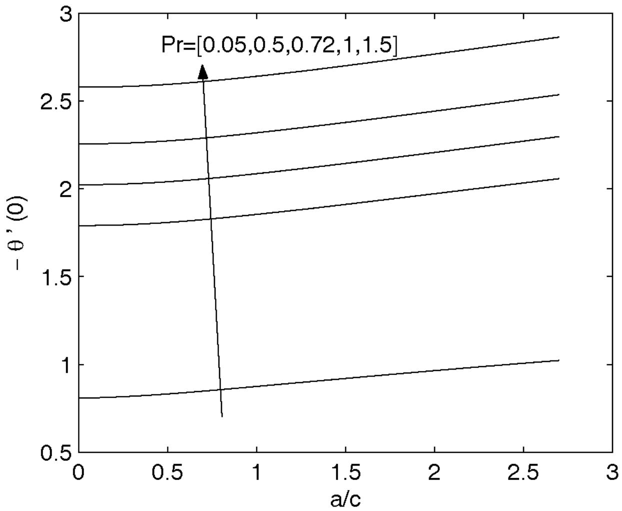

FIGURE 7. Variation of the heat transfer with a/c for several values of Pr.

REFERENCES

1. F. K. Moore, *Unsteady laminar boundary layer flow*, NACA TN, 1951, 3471.
2. M. J. Lightill, Proc. Roy. Soc. **224A**, 1–23 (1954).
3. C. C. Lin, Proc. 9th Int. Cong. Appl. Mech **4**, 155–168 (1956).
4. J. T. Stuart **1**, 1–40 (1971).
5. N. Riley, SIAM Review **17**, 274–297 (1975).
6. D. P. Telionis, J. Fluids Engng. **101**, 29–43 (1979).
7. D.P. Telionis, *Unsteady Viscous Flows,* Springer, Berlin, 1981.
8. I. Pop, *Theory of Unsteady Boundary Layers* (in Romanian), Ed. Stiintifica si Enciclopedica, Bucuresti–Romania, 1983.
9. P.K.H. Ma, and W.H. Hui, J. Fluid Mech **216**, 537–559 (1990).
10. D. K. Ludlow, P. A. Clarkson, and A.–P. Bassom, Q. Jl. Mech. Appl. Math. **53**, 175–206 (2000).
11. C. Y. Wang, Annu. Rev. Fluid Mech **23**, 159–177 (1991).
12. K. N. Lakshmisha, S. Venkateswaran, and G. Nath, ASME J. Heat Transfer **110**, 590–595 (1988).
13. L.J. Crane, J. Appl. Math. Phys, (ZAMP) **21**, 645–647 (1970).
14. I. Pop, and T.Y. Na, **23**, 413–422 (1996).
15. C. Y. Wang, Q. Du, M. Miklavčič, and C.–C. Chang, SIAM J. Appl. Math. **57**, 1–14 (1997).
16. E. M. A. Elbashbeshy, and M.A.A. Bazid, Heat Mass Transfer **41**, 1–4 (2004).
17. S. Sharidan, T. Mahmood, and I. Pop, Int. J. Appl. Mech. Engng., **11**, 647–654 (2006).
18. M. E. Ali, and E. Magyari, Int. J. Heat Mass Transfer **50**, 188–195 (2007).
19. C. D. Surma Devi, H.S. Takhar, and G. Nath, Int. J. Heat Mass Transfer **29**, 1996– 1999 (1986).
20. H. S. Takhar, A.J. Chamkha, and G. Nath, Acta Mechanica **146**, 59–71 (2001).
21. T. Ray Mahapatra, and A. S. Gupta, Heat Mass Transfer **38**, 517–521 (2002).