The equivalence between Mann–Ishikawa iterations and multistep iteration

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Abstract

We show that the convergence of Mann, Ishikawa iterations are equivalent to the convergence of a multistep iteration, for various classes of operators.

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1. Introduction

Let X be a Banach space, B a nonempty, convex subset of X, and T a selfmap of B. The two most popular iteration procedures for obtaining fixed points of T, if they exist, are Mann iteration [5], defined by

$$u_1 \in B, \quad u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad n \geqslant 1 \tag{1.1}$$

and Ishikawa iteration [4], defined by

$$z_1 \in B, \quad z_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n T z_n, \quad n \geqslant 1 \tag{1.2}$$

for certain choices of $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$.

For X a Hilbert space, B a convex compact subset of X, T a Lipschitzian pseudocontractive selfmap of B, Ishikawa [4] was able to show that (1.2) converges strongly to the unique fixed point of T in B, provided that (i$'$) $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ for all $n \geqslant 1$, (ii) $\lim \beta_n = 0$, and (iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Previous attempts to establish the same result for Mann iteration had proved unsuccessful. Finally, in year 2000, in [1] an example was provided of a Lipschitzian pseudocontraction for which the Mann iteration fails to converge to the fixed point.

Although condition (i$'$) was required in order to obtain the result of Ishikawa, it was noted that one could relax condition (i$'$) by replacing it with (i) $0 \leqslant \alpha_n, \beta_n \leqslant 1$ and still obtain strong convergence for many different maps. Moreover, by proving a convergence theorem for this modified Ishikawa method, and then setting $\beta_n = 0$ one obtained as a corollary the corresponding theorem for Mann iteration. The literature abounds with such papers.

A reasonable conjecture is that the Ishikawa iteration methods satisfying (i) and the corresponding Mann iterations are equivalent for all maps for which either method provides convergence to a fixed point.

In an attempt to verify this conjecture the authors, in a series of papers [9–14] have shown the equivalence for several classes of maps.

In year 2000, M.A. Noor introduced in [7] the three-step procedure

$$v_1 \in B, \quad t_n = (1-\gamma_n)v_n + \gamma_n T v_n,$$

$$w_n = (1-\beta_n)v_n + \beta_n T t_n,$$

$$v_{n+1} = (1-\alpha_n)v_n + \alpha_n T w_n, \quad n \geqslant 1. \tag{1.3}$$

The presence of (1.3) raises an interesting question.

Is there a map for which (1.3) converges to a fixed point, but for which (1.2), with (i$'$) fails to converge?

The answer to that question is unknown, but we shall show in this paper that (1.3), (1.2) and (1.1) are equivalent for all classes of functions for which (1.3) has been used in [7,8]. In fact, we prove a more general result, by using a multi-step procedure of arbitrary fixed order $p \geqslant 2$, defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n^1,$$

$$y_n^i = (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1}, \quad i = 1, \ldots, p-2,$$

$$y_n^{p-1} = (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n. \tag{1.4}$$

The sequence $\{\alpha_n\}$ is such that for all $n \in \mathbb{N}$

$$\{\alpha_n\} \subset (0,1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \tag{1.5}$$

and for all $n \in \mathbb{N}$

$$\{\beta_n^i\} \subset [0,1), \;\; 1 \leqslant i \leqslant p-1, \quad \lim_{n\to\infty} \beta_n^1 = 0. \tag{1.6}$$

Taking $p=3$ in (1.4) we obtain iteration (1.3). Taking $p=2$ in (1.4) we obtain (1.2).

The map $J:X\to 2^{X^*}$ given by $Jx:=\{f\in X^*\colon \langle x,f\rangle=\|x\|^2, \|f\|=\|x\|\}, \forall x\in X$, is called *the normalized duality mapping*. The Hahn–Banach theorem assures that $Jx\neq\emptyset, \forall x\in X$.

Definition 1.1. A map $T:B\to B$ is called strongly pseudocontractive if there exist $k\in(0,1)$ and $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle\leqslant(1-k)\|x-y\|^2, \quad \forall x,y\in B. \tag{1.7}$$

A map $S:X\to X$ is called strongly accretive if there exist $k\in(0,1)$ and $j(x-y)\in J(x-y)$ such that

$$\langle Sx-Sy, j(x-y)\rangle\geqslant k\|x-y\|^2, \quad \forall x,y\in D(S). \tag{1.8}$$

In (1.7) when $k=0$, then T is called *pseudocontractive*. In (1.8) when $k=1$, S is called *accretive*.

Lemma 1.2 (Weng [15]). *Let $\{a_n\}$ be a nonnegative sequence which satisfies the following inequality*

$$a_{n+1}\leqslant(1-\lambda_n)a_n+\sigma_n, \tag{1.9}$$

where $\lambda_n\in(0,1)$, $\forall n\in\mathbb{N}, \sum_{n=1}^{\infty}\lambda_n=\infty$, and $\sigma_n=o(\lambda_n)$. Then $\lim_{n\to\infty}a_n=0$.

The following Lemma is from [6].

Lemma 1.3 (Morales and Jung [6]). *If X is a real Banach space, then the following relation is true*:

$$\|x+y\|^2\leqslant\|x\|^2+2\langle y, j(x+y)\rangle, \quad \forall x,y\in X, \forall j(x+y)\in J(x+y). \tag{1.10}$$

2. Main results

Theorem 2.1. *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T:B\to B$ be a continuous and strongly pseudocontractive operator. If $\{\alpha_n\}\subset(0,1)$ satisfies* (1.5) *and $\{\beta_n^i\}\subset[0,1)$, $i=1,\ldots,p-1$, satisfy* (1.6) *and $u_1=x_1\in B$, then the following are equivalent*:
(i) *the Mann iteration* (1.1) *converges to the fixed point of T*,
(ii) *the iteration* (1.4) *converges to the fixed point of T.*

Proof. Corollary 1 of [2] assures the existence of a fixed point. The uniqueness of the fixed point comes from (1.7).

Since B is convex and bounded and T is a selfmap of B, $u_n\in B$ for each n, and hence $\{u_n\}$ is bounded. The condition $T:B\to B$ and the assumption that B is bounded and convex lead us to conclusion $\{\|Tu_n\|\}$ is bounded.

Denote

$$P = \sup_{n\in\mathbb{N}} \{\|x_n\|\}. \tag{2.1}$$

We will prove that $\{\|x_n\|\}$ is bounded. Supposing now that

$$x_n \in B \tag{2.2}$$

we will prove that

$$x_{n+1}, y_n^i (i = 1, \ldots, p-1) \in B. \tag{2.3}$$

The fact that B is a convex set, $T : B \to B$ and relation (1.4) lead to

$$y_n^{p-1} = (1 - \beta_n^{p-1}) x_n + \beta_n^{p-1} T x_n \in B, \tag{2.4}$$

similarly, we obtain

$$y_n^{p-2} = (1 - \beta_n^{p-2}) x_n + \beta_n^{p-2} T y_n^{p-1} \in B. \tag{2.5}$$

Recursively, we have

$$y_n^i = (1 - \beta_n^i) x_n + \beta_n^i T y_n^{i+1} \in B, \quad i = 1, \ldots, p-3. \tag{2.6}$$

Thus $y_n^1 \in B$. Using the assumption $T : B \to B$ we obtain that $T y_n^1 \in B$. Hence

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n^1 \in B \tag{2.7}$$

because already $x_n \in B$. Thus $P < +\infty$. Set

$$M := \max \left\{ P, \sup_{n\in\mathbb{N}} \{\|T x_n\|\}, \sup_{n\in\mathbb{N}} \{\|T y_n^i\| :\ 1 \leqslant i \leqslant p-1\}, \right.$$
$$\left. \sup_{n\in\mathbb{N}} \{\|u_n\|\}, \sup_{n\in\mathbb{N}} \{\|T u_n\|\} \right\} \tag{2.8}$$

to obtain

$$M < +\infty. \tag{2.9}$$

Because X^* is uniformly convex the duality map is a single-valued map [3]. Using (1.1), (1.4), (1.7) and (1.10) with

$$x := (1 - \alpha_n)\ (x_n - u_n),$$

$$y := \alpha_n (T y_n^1 - T u_n),$$

$$x + y = x_{n+1} - u_{n+1} \tag{2.10}$$

we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\|^2 &= \|(1 - \alpha_n)(x_n - u_n) + \alpha_n (T y_n^1 - T u_n)\|^2 \\ &\leqslant (1 - \alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle T y_n^1 - T u_n, J(x_{n+1} - u_{n+1}) \rangle \\ &= (1 - \alpha_n)^2 \|x_n - u_n\|^2 + 2\alpha_n \langle T y_n^1 - T u_n, J(x_{n+1} - u_{n+1}) \end{aligned}$$

$$
\begin{aligned}
&-J(y_n^1-u_n)\rangle+2\alpha_n\langle Ty_n^1-Tu_n,J(y_n^1-u_n)\rangle\\
&\leqslant(1-\alpha_n)^2\|x_n-u_n\|^2+2\alpha_n(1-k)\|y_n^1-u_n\|^2\\
&\quad+2\alpha_n\langle Ty_n^1-Tu_n,J(x_{n+1}-u_{n+1})-J(y_n^1-u_n)\rangle. \qquad (2.11)
\end{aligned}
$$

Set

$$\sigma_n:=2\alpha_n\langle Ty_n^1-Tu_n,J(x_{n+1}-u_{n+1})-J(y_n^1-u_n)\rangle. \qquad (2.12)$$

Proposition 12.3 of [3] assures that, when X^* is uniformly convex, then J is single-valued map and is uniformly continuous on every bounded set of X. Since $\{Ty_n^1-Tu_n\}$ is bounded, to have $\lim_{n\to\infty}\sigma_n=0$ is sufficient to prove that

$$J(x_{n+1}-u_{n+1})-J(y_n^1-u_n)\to 0,(n\to\infty). \qquad (2.13)$$

$$
\begin{aligned}
&\|(x_{n+1}-u_{n+1})-(y_n^1-u_n)\|\\
&\quad=\|(x_{n+1}-y_n^1)-(u_{n+1}-u_n)\|\\
&\quad=\|-\alpha_nx_n+\alpha_nTy_n^1+\beta_n^1x_n-\beta_n^1Ty_n^2+\alpha_nu_n-\alpha_nTu_n\|\\
&\quad\leqslant\alpha_n(\|x_n\|+\|Ty_n^1\|+\|u_n\|+\|Tu_n\|)+\beta_n^1(\|x_n\|+\|Ty_n^2\|)\\
&\quad\leqslant(\alpha_n+\beta_n^1)4M\to 0,\quad \text{as } n\to\infty. \qquad (2.14)
\end{aligned}
$$

The uniform continuity of $J(\cdot)$ guarantees that (2.13) is satisfied.

Relations (1.1), (1.4) and (1.10) with

$$
\begin{aligned}
&x:=(1-\beta_n^1)(x_n-u_n),\\
&y:=\beta_n^1(Ty_n^2-u_n),\\
&x+y=y_n^1-u_n, \qquad (2.15)
\end{aligned}
$$

lead to

$$
\begin{aligned}
\|y_n^1-u_n\|^2&=\|(1-\beta_n^1)(x_n-u_n)+\beta_n^1(Ty_n^2-u_n)\|^2\\
&\leqslant(1-\beta_n^1)^2\|x_n-u_n\|^2+2\beta_n^1\langle Ty_n^2-u_n,J(y_n^1-u_n)\rangle\\
&\leqslant\|x_n-u_n\|^2+2\beta_n^1\|Ty_n^2-u_n\|\|y_n^1-u_n\|\\
&\leqslant\|x_n-u_n\|^2+2\beta_n^1(\|Ty_n^2\|+\|u_n\|)(\|y_n^1\|+\|u_n\|)\\
&\leqslant\|x_n-u_n\|^2+\beta_n^18M^2. \qquad (2.16)
\end{aligned}
$$

We already know that $\|Ty_n^2\|\leqslant M$ and $\|y_n^1\|\leqslant M,\forall n\in\mathbb{N}$. Observe that we do not need further evaluations for $y_n^3,\dots,y_n^{p-1},x_n$. This is the crucial point in this proof: starting the computations in (1.4), from x_{n+1} we do not need to evaluate more than two steps. The other steps are included in (2.4), (2.5), (2.6), and (2.7), to prove $\|Ty_n^2\|\leqslant M,\forall n\in\mathbb{N}$.

Substituting (2.16) and (2.12) in (2.11), we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\|^2 &\leqslant (1-\alpha_n)^2\|x_n-u_n\|^2+2\alpha_n(1-k)\|x_n-u_n\|^2\\ &\quad +\sigma_n+\alpha_n\beta_n^1 16M^2(1-k)\\ &=(1-2k\alpha_n+\alpha_n^2)\|x_n-u_n\|^2+o(\alpha_n). \end{aligned}\tag{2.17}$$

From (1.5) for all n sufficiently large we have

$$\alpha_n \leqslant k. \tag{2.18}$$

Substituting (2.18) into (2.17), we obtain

$$1-2k\alpha_n+\alpha_n^2 \leqslant 1-2k\alpha_n+k\alpha_n = 1-k\alpha_n. \tag{2.19}$$

Finally (2.17) becomes

$$\|x_{n+1}-u_{n+1}\|^2 \leqslant (1-k\alpha_n)\|x_n-u_n\|^2+o(\alpha_n) \tag{2.20}$$

with

$$a_n := \|x_n-u_n\|^2,$$

$$\lambda_n := k\alpha_n \in (0,1), \tag{2.21}$$

and using Lemma 1.2, we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\|x_n-u_n\|^2=0$, i.e.

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{2.22}$$

Suppose that $\lim_{n\to\infty} u_n = x^*$. The inequality

$$0 \leqslant \|x^*-x_n\| \leqslant \|u_n-x^*\|+\|x_n-u_n\| \tag{2.23}$$

and (2.22), imply that $\lim_{n\to\infty} x_n = x^*$. Analogously $\lim_{n\to\infty} x_n = x^*$ implies that $\lim_{n\to\infty} u_n = x^*$. $\square$

For $p=2$ we get the following result from [10].

Theorem 2.2 (Rhoades and Soltuz [10]). *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T: B \to B$ be a continuous and strongly pseudocontractive operator. Then for $u_1 = x_1 \in B$ the following are equivalent*:

(i) *the Mann iteration (1.1) converges to the fixed point of T,*
(ii) *the Ishikawa iteration (1.2) converges to the fixed point of T.*

Theorems 2.1 and 2.2 lead to the following result.

Corollary 2.3. *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T: B \to B$ be a continuous*

and strongly pseudocontractive operator. Then for $u_1 = x_1 \in B$ *the following are equivalent*:
(i) *the Mann iteration* (1.1) *converges to the fixed point of* T,
(ii) *the Ishikawa iteration* (1.2) *converges to the fixed point of* T,
(iii) *the iteration* (1.4) *converges to the fixed point of* T.

For $p = 3$, from Theorems 2.1 and 2.2, we have the following result:

Corollary 2.4. *Let* X *be a real Banach space with a uniformly convex dual and* B *a nonempty, closed, convex, bounded subset of* X. *Let* $T : B \to B$ *be a continuous and strongly pseudocontractive operator. Then for* $u_1 = x_1 \in B$ *the following are equivalent*:
(i) *the Mann iteration* (1.1) *converges to the fixed point of* T,
(ii) *the Ishikawa iteration* (1.2) *converges to the fixed point of* T,
(iii) *the Noor iteration* (1.3) *converges to the fixed point of* T.

Remark 2.5. (i) If B is not bounded then Theorem 2.1 holds only supposing that $\{x_n\}$ is bounded.

(ii) If the Mann iteration converges to a point, it is clear that this point is a fixed point of T. Thus we can omit the discussion of the existence of a fixed point in the proof of Theorem 2.1.

(iii) *If* $T(B)$ is bounded then $\{x_n\}$ is bounded.

Comments (i) and (ii) already been discussed in [10].

Proof. We prove part (iii). Let

$$M := \max\left\{\sup_{x\in B} \|Tx\|, \|x_1\|\right\}. \tag{2.24}$$

Then $\|x_1\| \leqslant M$ and supposing $\|x_n\| \leqslant M$, we have

$$\|x_{n+1}\| \leqslant (1-\alpha_n)\|x_n\| + \alpha_n M \leqslant (1-\alpha_n)M + \alpha_n M = M. \quad \square \tag{2.25}$$

3. Further equivalences

Let I denote the identity map.

Remark 3.1. Let $T, S : X \to X$, $f \in X$ given. Then
(i) A fixed point for the map $Tx = f + (I - S)x, \forall x \in X$ is a solution for $Sx = f$.
(ii) A fixed point for $Tx = f - Sx$ is a solution for $x + Sx = f$.

Remark 3.2 (Rhoades and Soltuz [10]). (i) The operator T is a (strongly) pseudocontractive map if and only if $(I - T)$ is (strongly) accretive.

(ii) If S is an accretive map then $T = f - S$ is strongly pseudocontractive map.

We consider iterations (1.1) and (1.4), with $Tx = f + (I - S)x$ and $p \geqslant 2$, $\{\alpha_n\}, \{\beta_n^i\} \subset (0, 1)$, $i = 1, \ldots, p-1$ satisfying (1.5) and (1.6)

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n(f + (I - S)u_n), \tag{3.1}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - S)y_n^1),$$

$$y_n^i = (1 - \beta_n^i)x_n + \beta_n^i(f + (I - S)y_n^{i+1}), \quad i = 1, \ldots, p - 2,$$

$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1}(f + (I - S)x_n). \tag{3.2}$$

Theorems 2.1 and 2.2, Remark 2.5(i), Remark 3.1(i), Remark 3.2(i) and Corollary 2.4 lead to the following result.

Corollary 3.3. *Let X be a real Banach space with a uniformly convex dual and $S : X \to X$ be a continuous and strongly accretive operator and let $\{x_n\}$ given by (3.2) be bounded. If $\{\alpha_n\} \subset (0, 1)$ satisfies (1.5) and $\{\beta_n^i\} \subset [0, 1)$, $i = 1, \ldots, p - 1$, satisfy (1.6) and $u_1 = x_1 \in B$, then the following are equivalent*:
(i) *the Mann iteration (3.1) converges to the solution of $Sx = f$,*
(ii) *the Ishikawa iteration (1.2) with $Tx = f + (I - S)x$, converges to the solution of $Sx = f$,*
(iii) *the iteration (3.2) converges to the solution of $Sx = f$,*
(iv) *the Noor iteration (1.3) with $Tx = f + (I - S)x$,* converges to the solution of $Sx = f$.

We consider iterations (1.1) and (1.4), with $Tx = f - Sx$ and $p \geqslant 2$, $\{\alpha_n\}, \{\beta_n^i\} \subset (0, 1)$, $i = 1, \ldots, p - 1$ satisfying (1.5) and (1.6)

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n(f - Su_n), \tag{3.3}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Sy_n^1),$$

$$y_n^i = (1 - \beta_n^i)x_n + \beta_n^i(f - Sy_n^{i+1}), \quad i = 1, \ldots, p - 2,$$

$$y_n^{p-1} = (1 - \beta_n^{p-1})x_n + \beta_n^{p-1}(f - Sx_n). \tag{3.4}$$

Theorems 2.1 and 2.2, Remark 2.5(i), Remark 3.1(ii), Remark 3.2(ii), and Corollary 2.4 lead to the following result.

Corollary 3.4. *Let X be a real Banach space with a uniformly convex dual and $S : X \to X$ be a continuous and accretive operator and let $\{x_n\}$ given by (3.4) be bounded. If $\{\alpha_n\} \subset (0, 1)$ satisfies (1.5) and $\{\beta_n^i\} \subset [0, 1)$, $i = 1, \ldots, p - 1$, satisfy (1.6) and $u_1 = x_1 \in B$, then the following are equivalent*:

(i) *the Mann iteration (3.3) converges to the solution of $x + Sx = f$,*
(ii) *the Ishikawa iteration (1.2) with $Tx = f - Sx$, converges to the solution of $x + Sx = f$,*
(iii) *the iteration (3.4) converges to the solution of $x + Sx = f$,*
(iv) *the Noor iteration (1.3) with $Tx = f - Sx$, converges to the solution of $x + Sx = f$.*

4. The equivalence between T-stabilities

All the arguments for the equivalence between T-stabilities of Mann, Ishikawa, Multistep and Noor iterations are similar to those from [13]. Let us denote by $F(T) = \{x^* \in B\colon\ x^* = T(x^*)\}$. Suppose that $x^* \in F(T)$. The following nonnegative sequences are well-defined for all $n \in \mathbb{N}$:

$$\varepsilon_n := \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n T y_n^1\|, \tag{4.1}$$

$$\delta_n := \|u_{n+1} - (1 - \alpha_n)u_n - \alpha_n T u_n\|. \tag{4.2}$$

Definition 4.1. If $\lim_{n\to\infty} \varepsilon_n = 0$, (respectively $\lim_{n\to\infty} \delta_n = 0$) implies that $\lim_{n\to\infty} x_n = x^*$, (respectively $\lim_{n\to\infty} u_n = x^*$), then (1.1) (respectively (1.4)) is said to be T-stable.

Remark 4.2 (Rhoades and Soltuz [13]). Let X be a normed space, $B \subset X$ be a nonempty, convex, closed subset and $T : B \to B$ be continuous map. If the Mann (respectively (1.4)) iteration converges, then $\lim_{n\to\infty} \delta_n = 0$ (respectively $\lim_{n\to\infty} \varepsilon_n = 0$).

Theorem 4.3. *Let X be a real Banach space with a uniformly convex dual and B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a continuous and strongly pseudocontractive operator. If $\{\alpha_n\} \subset (0, 1)$ satisfies (1.6) and $\{\beta_n^i\} \subset [0, 1)$, $i = 1, \ldots, p - 1$, satisfy (1.5) and $u_1 = x_1 \in B$, then the following are equivalent:*

(i) *the Mann iteration (1.1) is T-stable,*
(ii) *the iteration (1.4) is T-stable.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) means that $\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} \delta_n = 0$. The implication $\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} \delta_n = 0$ is obvious by setting $\beta_n^i = 0$, $i \in \{1, \ldots, p - 1\}, \forall n \in \mathbb{N}$, in (1.4) and using (4.2). Conversely, suppose that (1.1) is T-stable. Using Definition 4.1 we obtain

$$\lim_{n\to\infty} \delta_n = 0 \Rightarrow \lim_{n\to\infty} u_n = x^*. \tag{4.3}$$

Theorem 2.1 assures that $\lim_{n\to\infty} u_n = x^*$ leads us to $\lim_{n\to\infty} x_n = x^*$. Using Remark 4.2 we have $\lim_{n\to\infty} \varepsilon_n = 0$. Thus we get $\lim_{n\to\infty} \delta_n = 0 \Rightarrow \lim_{n\to\infty} \varepsilon_n = 0$. $\square$

Analogously, we can prove the equivalence between T-stabilities for the strongly accretive and accretive cases with $Tx = f + (I - S)x$, respectively $Tx = f - Sx$.

Corollary 4.4. *Let X be a real Banach space with a uniformly convex dual and $S : X \to X$ be a continuous and strongly accretive operator and let $\{x_n\}$ given by (3.2) be bounded. If $\{\alpha_n\} \subset (0, 1)$ satisfies (1.5) and $\{\beta_n^i\} \subset [0, 1)$, $i = 1, \ldots, p - 1$,* satisfy *(1.6) and $u_1 = x_1 \in B$, then the following are equivalent:*

(i) *the Mann iteration (3.1) is T-stable,*
(ii) *the iteration (3.2) is T-stable.*

Corollary 4.5. *Let X be a real Banach space with a uniformly convex dual and $S : X \to X$ be a continuous and accretive operator and let $\{x_n\}$ given by (3.4) be bounded. If $\{\alpha_n\} \subset (0,1)$ satisfies (1.5) and $\{\beta_n^i\} \subset [0,1)$, $i = 1, \ldots, p-1$, satisfy (1.6) and $u_1 = x_1 \in B$, then the following are equivalent*:

(i) *the Mann iteration (3.3) is T-stable*,
(ii) *the iteration (3.4) is T-stable.*

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