Solving inverse problems via hemicontractive maps

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ABSTRACT

We prove a "collage" theorem for hemicontractive maps and we use it for inverse problems. A numerical example is given.

1. Introduction

Let X be a real Banach space, $T : X \to X$ be an operator. The map $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}$, $\forall x \in X$, is called *the normalized duality mapping*. It is easy to see that

$$\langle y, j(x)\rangle \leq \|x\| \, \|y\|, \quad \forall x, y \in X, \forall j(x) \in J(x). \tag{1}$$

Definition 1. A map $T : X \to X$ is called hemicontractive if there exist $k \in (0, 1)$ and $q \in X$ with $q = Tq$ such that for every $x \in X$ there exists $j(x-q) \in J(x-q)$ satisfying

$$\langle Tx - Tq, j(x-q)\rangle \leq k \|x-q\|^2, \quad \forall x \in X. \tag{2}$$

Remark 2. The fixed point q in Definition 1 is uniquely determined and, sometimes, will be denoted by x_T^*.

Indeed, if $p = Tp$ is another fixed point of the hemicontractive mapping T, then

$$\begin{aligned} \|p-q\|^2 &= \langle p-q, j(p-q)\rangle \\ &= \langle Tp - Tq, j(p-q)\rangle \leq k \|p-q\|^2, \end{aligned}$$

implying $\|p-q\| = 0$, i.e., $p = q$.

It is well known that T is a contraction if there exists $k \in (0, 1)$ such that $\|Tx - Ty\| \leq k \|x-y\|$, $\forall x, y \in X$.

Remark 3. The class of contractions is a subclass of hemicontractions.

Let T be a k-contraction of the Banach space X. Then T has a unique fixed point q and

$$\begin{aligned} \langle Tx - Tq, j(x-q)\rangle &\leq \|Tx - Tq\| \, \|x-q\| \\ &\leq k \|x-q\|^2, \end{aligned}$$

for $j(x-q) \in J(x-q)$.

Remark 4. The above inclusion is proper.

Indeed, note that $T(x, y) = (-y, x)$ is not a contraction while it is hemicontractive with $q = (0, 0)$ and $k = 0.5$,

$$\begin{aligned} \langle T(x, y), (x, y)\rangle &= \langle(-y, x), (x, y)\rangle = 0 \\ &\leq (1/2)\left(x^2 + y^2\right) = 0.5\,\|(x, y)\|^2 . \end{aligned}$$

Recently, Kunze et al. (see [1–3]) have considered a class of inverse problems for ordinary differential equations and provided a mathematical basis for solving them within the framework of Banach spaces and contractions. We shall consider the same framework of Banach spaces and the larger class of hemicontractive maps.

Notation 5. *Denote by HemiLip* $:= \{T, T : X \to X, T$ *a hemicontractive map with constant* $k \in (0, 1)$, *Lipschitzian with constant* $L \geq 1$ *and* $T(X)$ *bounded*$\}$.

A typical inverse problem is the following:

Problem 6. For given $\varepsilon > 0$ and a "target" $\bar{x}$, find $T_\varepsilon \in$ *HemiLip* such that $\left\|\bar{x} - x^*_{T_\varepsilon}\right\| < \varepsilon$, where $x^*_{T_\varepsilon} = T_\varepsilon\left(x^*_{T_\varepsilon}\right)$ is the unique fixed point of the hemicontractive mapping T_ε.

According to [1], randomly selecting various maps in *HemiLip*, finding their fixed points and computing the distance from our target is an extremely tedious procedure. Consider now the following problem which we shall fit in our framework and which is very useful for practitioners.

Problem 7. Let $\bar{x} \in X$ be a target and let $\delta > 0$ be given. Find $T_\delta \in$ *HemiLip* such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$.

In other words, instead of searching for hemicontractive maps whose fixed points lie close to target $\bar{x}$, we search for hemicontractive maps that send $\bar{x}$ close to itself.

2. Main results

Theorem 8 (*Collage Theorem for Hemicontractive Maps*). *Let X be a real Banach space and T a hemicontractive map with contraction factor $k \in (0, 1)$ and fixed point $x^* \in X$. Then for any $x \in X$,*

$$\left\|x - x^*\right\| \leq \frac{1}{1-k}\,\|x - Tx\| .$$

Proof. The hemicontractive condition assures that the fixed point x^* exists and it is unique. If $x = x^*$, the above inequality holds. If $x \neq x^*$, $\forall x \in X$, then using (1) and (2) one obtains

$$\begin{aligned} \left\|x - x^*\right\|^2 &= \left\langle x - x^*, j(x - x^*)\right\rangle \\ &= \left\langle Tx - Tx^*, j(x - x^*)\right\rangle + \left\langle x - Tx, j(x - x^*)\right\rangle \\ &\leq k\left\|x - x^*\right\|^2 + \left\langle x - Tx, j(x - x^*)\right\rangle \\ &\leq k\left\|x - x^*\right\|^2 + \|x - Tx\|\left\|x - x^*\right\| . \end{aligned}$$

From which one gets the conclusion. □

The above "Collage Theorem" allows us to reformulate the inverse Problem 6 in the particular and more convenient Problem 7.

Theorem 9. *If Problem 7 has a solution, then Problem 6 has a solution too.*

Proof. Let $\varepsilon > 0$ and $\bar{x} \in X$ be given. For $\delta := (1-k)\,\varepsilon$, let $T_\delta \in$ *HemiLip* be such that $\|\bar{x} - T_\delta \bar{x}\| < \delta$. If $x^*_{T_\delta}$ is the unique fixed point of the hemicontractive mapping T_δ, then, by Theorem 8,

$$\left\|\bar{x} - x^*_{T_\delta}\right\| \leq \frac{1}{1-k}\,\|\bar{x} - T_\delta \bar{x}\| \leq \frac{1}{1-k}\delta = \varepsilon. \quad \square$$

Note that shrinking the distance between two operators, one of them from *HemiLip*, reduces the distance between their fixed points.

Proposition 10. *Let X be a real Banach space and $T_1 \in$ HemiLip with contraction factor $k_1 \in (0, 1)$ and $T_2 : X \to X$ a map such that $x_1^*, x_2^* \in X$ are distinct fixed points for T_1 and T_2. Then,*

$$\left\|x_1^* - x_2^*\right\| \leq \frac{1}{1-k_1}\sup_{x \in X}\|T_1 x - T_2 x\| .$$

Proof. Using (2) one obtains

$$\begin{aligned}\left\|x_1^*-x_2^*\right\|^2 &= \left\langle x_1^*-x_2^*, j\left(x_1^*-x_2^*\right)\right\rangle = \left\langle T_1x_1^*-T_2x_2^*, j\left(x_1^*-x_2^*\right)\right\rangle \\ &= \left\langle T_1x_1^*-T_1x_2^*, j\left(x_1^*-x_2^*\right)\right\rangle + \left\langle T_1x_2^*-T_2x_2^*, j\left(x_1^*-x_2^*\right)\right\rangle \\ &\le k_1\left\|x_1^*-x_2^*\right\|^2 + \left\|x_1^*-x_2^*\right\|\left\|T_1x_2^*-T_2x_2^*\right\| \\ &\le k_1\left\|x_1^*-x_2^*\right\|^2 + \left\|x_1^*-x_2^*\right\|\left(\sup_{x\in X}\left\|T_1x-T_2x\right\|\right),\end{aligned}$$

from which we get the conclusion. □

Theorem 11. *Let X be a real Banach space, $T : X \to X, \bar{x} = T\bar{x}$ and suppose there exists $T_1 \in HemiLip$ such that $\sup_{x\in X}\|T_1x - Tx\| \le \varepsilon$. Then*

$$\|\bar{x}-T_1\bar{x}\| \le \frac{1+L}{1-k}\varepsilon.$$

Proof. Let $x^* = T_1x^*$, and by use of Proposition 10 we obtain

$$\left\|\bar{x}-x^*\right\| \le \frac{1}{1-k}\left(\sup_{x\in X}\left\|T_1x-Tx\right\|\right).$$

Thus,

$$\begin{aligned}\|\bar{x}-T_1\bar{x}\| &\le \left\|\bar{x}-x^*\right\| + \left\|x^*-T_1\bar{x}\right\| \\ &\le \|x-x_1\| + \left\|T_1x^*-T_1\bar{x}\right\| \\ &\le (1+L)\left\|\bar{x}-x^*\right\| \\ &\le \frac{1+L}{1-k}\left(\sup_{x\in X}\left\|T_1x-Tx\right\|\right) \le \frac{1+L}{1-k}\varepsilon. \quad \square\end{aligned}$$

3. Application

Example 12. Let $A \in (0,1), B, C, D \in \mathbb{R}$ be fixed numbers and $F : [0,3]\times[0,3] \to \mathbb{R}^2$ be given by $F(x,y) = \left(Ax+Bxy-Cy, Ay-Bx^2+Cx\right)$. Then F is Lipschitzian and hemicontractive with bounded range.

Proof. It is obvious that F is Lipschitzian and has bounded range. In order to prove that it is hemicontractive, note that

$$\begin{aligned}\langle F(x,y),(x,y)\rangle &= \left\langle\left(Ax+Bxy-Cy, Ay-Bx^2+Cx\right),(x,y)\right\rangle \\ &= Ax^2+Bx^2y-Cxy+Ay^2-Bx^2y+Cxy \\ &= A\left\|(x,y)\right\|^2. \quad \square\end{aligned}$$

Set $C = 0$, to obtain our T_ε function:

Example 13. Let $A \in (0,1), B \in \mathbb{R}$ be fixed numbers and $H : [0,3]\times[0,3] \to \mathbb{R}^2$ be given by $H(x,y) = \left(Ax+Bxy, Ay-Bx^2\right)$. Then H is Lipschitzian and hemicontractive with bounded range.

Remark 14. Let $\bar{h}$ be the "target", in order to find T_ε. As for fitting, we shall look for an appropriate T_δ. Then by using Matlab (i.e. fminsearch) for min $\left\|\bar{h}-T_\delta\bar{h}\right\|$, by Theorem 11 we find the parameters which minimize the problem. Set in Example 13, $A = 0.3, B = 6, T_\varepsilon := H$ and let $\bar{h} = (\bar{x},\bar{y}) = H((\bar{x},\bar{y}))$ on $([-1,3]\times[-1,3])$ be the target generated by T_ε. Use the above algorithm with $T_\delta := F$, to obtain the H map, i.e. $(A,B,C) = (0.3000, 6.0000, 0.0000)$ starting from each point between $(0.3000, 2.0000, 2.0000)$ and $(0.9000, 8.0000, 6.0000)$.

Remark 15. In Example 13, set $A = 0, B = 0.5, T_\varepsilon := H$ and let $\bar{h} = (\bar{x},\bar{y}) = H((\bar{x},\bar{y}))$ on $([-1,3]\times[-1,3])$ be the target generated by T_ε. Use again the above algorithm with $T_\delta := F$, to obtain the H map, i.e. $(A,B,C) = (0.00, 0.50, 0.00)$ starting from each point between $(0.0000, 0.3000, 0.0000)$ and $(1.0000, 3.0000, 1.0000)$.

Remark 16. In Example 13, set $A = 0, B = 1, T_\varepsilon := H$ and let $\bar{h} = (\bar{x},\bar{y}) = H((\bar{x},\bar{y}))$ on $([-1,3]\times[-1,3])$ be the target generated by T_ε. Use fminsearch with $T_\delta := F$, to obtain the H map, i.e. $(A,B,C) = (0.00, 1.00, 0.00)$ starting from each point between $(0.2000, 0.3000, 0.2000)$ and $(0.7000, 0.5000, 0.5000)$.

Remark 17. In Example 13, set $A = 1, B = 0, T_\varepsilon := H$ and let $\bar{h} = (\bar{x}, \bar{y}) = H((\bar{x}, \bar{y}))$ on $([-1, 3] \times [-1, 3])$ be the target generated by T_ε. Note that T_ε is not strongly pseudocontractive. Use fminsearch with $T_\delta := F$, to obtain the H map, i.e. $(A, B, C) = (1.00, 0.00, 0.00)$ starting from each point between $(0.2000, 0.2000, 0.2000)$ and $(0.7000, 0.7000, 0.7000)$.

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