A SCHECHTER TYPE CRITICAL POINT RESULT IN ANNULAR CONICAL DOMAINS OF A BANACH SPACE AND APPLICATIONS

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Abstract. Using Ekeland's variational principle we obtain a critical point theorem of Schechter type for extrema of a functional in an annular conical domain of a Banach space. The result can be seen as a variational analogue of Krasnoselskii's fixed point theorem in cones and can be applied for the existence, localization and multiplicity of the positive solutions of variational problems. The result is then applied to p-Laplace equations, where the geometric condition on the boundary of the annular conical domain is established via a weak Harnack type inequality given in terms of the energetic norm. This method can be applied also to other homogeneous operators in order to obtain existence, multiplicity or infinitely many solutions for certain classes of quasilinear equations.

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1. Introduction

The bounded critical point method is a very useful tool to study the existence and localization of solutions of nonlinear equations. Some references are as follows [9], [10], [12], [15], [21]. We particularly mention Schechter's theory [25], [26] which yields critical points of a C^1 functional in a ball of a Hilbert space, by taking into account boundary conditions of Leray-Schauder type. A result of this type is the following:

Theorem 1.1 (Schechter)**.** *If X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $R>0$ and $F:X\to\mathbb{R}$ be a lower semicontinuous functional bounded from below. Then given $\varepsilon>0$, there exists a point $x\in M$ such that*

$$F(x)\leq \inf F(M)+\varepsilon,$$

$$F(x)\leq F(y)+\varepsilon d(x,y) \quad \text{for all } \ y\in M. \tag{1.1}$$

2. Main abstract result

Let X be a real Banach space, X^* its dual, $\langle\cdot,\cdot\rangle$ denotes the duality between X^* and X, and let the norms on X and X^* be denoted by the same symbol $\|\cdot\|$.

We shall denote by J the *duality mapping* corresponding to the normalization function $\varphi(t) := t^{p-1}$ $(t \in \mathbb{R}_+)$, where $p > 1$, i.e. the set-valued operator $J : X \to \mathcal{P}(X^*)$ defined by

$$Jx = \{x^* \in X^* : \ \langle x^*, x\rangle = \|x\|^p, \ \|x^*\| = \|x\|^{p-1}\}, \quad x \in X.$$

Obviously,

$$J(\lambda x) = |\lambda|^{p-2}\lambda Jx$$

for every $x \in X$ and $\lambda \in \mathbb{R}$.

It is known, see [5, Theorem 3, p. 31], that if that X^* is strictly convex, then J is single-valued and so

$$\langle Jx, x\rangle = \|x\|^p, \quad \|Jx\| = \|x\|^{p-1}.$$

Also, if in addition X is reflexive and locally uniformly convex, then J is demicontinuous and bijective and its inverse $\bar{J}$ is bounded, continuous and monotone. In what follows we shall assume that the following condition holds:

Assumption (A1): X and X^* are locally uniformly convex reflexive Banach spaces and J is locally strongly monotone, i.e., there is $\beta > 1$ such that for each $\rho > 0$ there exists a constant $a = a(\rho) > 0$ with

$$\langle Jx - Jy, x - y\rangle \geq a\|x - y\|^{\beta} \tag{2.1}$$

for all $x, y \in X$ satisfying $\|x\| \leq \rho$ and $\|y\| \leq \rho$.

Let K be a *wedge* of the Banach space X, i.e. a closed convex subset of X such that $K \neq \{0\}$ and $\lambda K \subset K$ for every $\lambda \in \mathbb{R}_+$. Notice that K can be a *cone*, i.e. may have the property $K \cap (-K) = \{0\}$, and also can be the whole space X.

We shall localize critical points x of F by means of a functional G which verifies suitable assumptions (see for instance assumption (A2)). More exactly, for two fixed numbers r, R with $0 < r < R$, we shall look for $x \in K$ such that $F'(x) = 0$ and $r \leq G(x) \leq R$. Hence we seek critical points of F in the *annular conical set*

$$K_{r,R} := \{x \in K : r \leq G(x) \leq R\}.$$

Denote by

$$N_r = \{x \in K : G(x) = r\}, \quad N_R = \{x \in K : G(x) = R\}$$

the two parts of the boundary of $K_{r,R}$ which are assumed non-void. Our second assumption is as follows:

Assumption (A2): $G : X \to \mathbb{R}$ is a C^1 functional such that G' maps bounded sets into bounded sets,

$$\text{the set} \quad \{x \in K : G(x) \leq R\} \quad \text{is bounded} \tag{2.2}$$

and

$$\inf_{x \in N_r \cup N_R} \langle G'(x), x\rangle > 0. \tag{2.3}$$

As for the functional F, we shall assume:

Assumption (A3): $F : X \to \mathbb{R}$ is a C^1 functional which is bounded from below on $K_{r,R}$, and F' maps bounded sets into bounded sets.

We introduce some auxiliary mappings:

$$D : N_R \cup N_r \to X^*, \quad D(x) := F'(x) - \frac{\langle F'(x), x\rangle}{\langle G'(x), x\rangle} G'(x)\,,$$

$$E : N_R \cup N_r \to X, \quad E(x) := \bar{J}\left(\mu J x\right) - \bar{J}\left(\mu J x - Dx\right) - \lambda x,$$

where μ will be chosen in a suitable way (see (2.9)) and λ is such that

$$\left\langle G'\left(x\right), E\left(x\right)\right\rangle = 0. \tag{2.4}$$

Lemma 2.1. *Assume that* (A1), (A2) *and* (A3) *are satisfied. For every* $x \in N_R \cup N_r$, *one has*

$$\left\langle D\left(x\right), x\right\rangle = 0 \tag{2.5}$$

and there exists $a = a\left(R\right) > 0$ *such that*

$$\langle F'(x), E(x)\rangle \geq a \left\|\bar{J}\left(\mu J x\right) - \bar{J}\left(\mu J x - Dx\right)\right\|^{\beta},$$

for all $x \in N_R \cup N_r$.

Proof. Let $x \in N_R \cup N_r$ be arbitrary. A direct computation gives

$$\langle D(x), x\rangle = \langle F'(x) - \frac{\langle F'(x), x\rangle}{\langle G'(x), x\rangle} G'(x),\ x\rangle = 0.$$

Next, from (2.4), (2.5) and (2.1),

$$\begin{aligned}\langle F'(x), E(x)\rangle &= \left\langle D\left(x\right), E\left(x\right)\right\rangle = \left\langle D\left(x\right), \bar{J}\left(\mu J x\right) - \bar{J}\left(\mu J x - Dx\right)\right\rangle \\ &\geq a \left\|\bar{J}\left(\mu J x\right) - \bar{J}\left(\mu J x - Dx\right)\right\|^{\beta}.\end{aligned}$$

Notice that (2.1) applied since both $\bar{J}\left(\mu J x\right)$, $\bar{J}\left(\mu J x - Dx\right)$ are bounded independently on $x \in N_R \cup N_r$ as a consequence of (2.2), (2.3) and of the fact that $J, \bar{J}, F'$ and G' map bounded sets into bounded sets. □

Lemma 2.2. *Assume that* (A1), (A2) *and* (A3) *are satisfied. Let* $x \in K$ *and* $z \in X$ *be such that*

$$y := x - tz \in K \quad \textit{for all} \quad t > 0 \quad \textit{sufficiently small.} \tag{2.6}$$

Then $y \in K_{r,R}$ *for* $t > 0$ *small enough, in each of the following situations:*

(a) $r < G\left(x\right) < R$;
(b) $x \in N_R$ *and* $\langle G'(x), z\rangle > 0$;
(c) $x \in N_r$ *and* $\langle G'(x), z\rangle < 0$.

Proof. In case (a), the conclusion follows from (2.6), the continuity of G, and the strict inequalities $r < G\left(x\right) < R$. Assume now that condition (b) holds. From the definition of the Fréchet derivative of G, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for each $t \in (0, \delta_\varepsilon)$ we have

$$-\varepsilon t \leq G(x - tz) - G(x) + \langle G'(x), tz\rangle \leq \varepsilon t. \tag{2.7}$$

Hence

$$R-\varepsilon t-t\langle G'(x),z\rangle \le G(x-tz)\le R+\varepsilon t-t\langle G'(x),z\rangle \quad \text{for} \quad t\in(0,\delta_\varepsilon).$$

Since $\langle G'(x),z\rangle>0$, we may take $\varepsilon:=\langle G'(x),z\rangle$ to obtain

$$R-2t\langle G'(x),z\rangle \le G(x-tz)\le R \quad \text{for} \quad t\in(0,\delta_\varepsilon).$$

Hence, for t sufficiently small such that $r\le R-2t\langle G'(x),z\rangle$ and $t\in(0,\delta_\varepsilon)$, we have $r\le G(x-tz)\le R$. This together with (2.6) shows that $y\in K_{r,R}$ for $t>0$ small enough. Finally, if (c) holds, then (2.7) gives

$$r-\varepsilon t-t\langle G'(x),z\rangle \le G(x-tz)\le r+\varepsilon t-t\langle G'(x),z\rangle \quad \text{for} \quad t\in(0,\delta_\varepsilon).$$

In this case we may take $\varepsilon:=-\langle G'(x),z\rangle$ and obtain

$$r\le G(x-tz)\le r-2t\langle G'(x),z\rangle \quad \text{for} \quad t\in(0,\delta_\varepsilon).$$

Hence, for t sufficiently small such that $r-2t\langle G'(x),z\rangle\le R$ and $t\in(0,\delta_\varepsilon)$, we have $r\le G(x-tz)\le R$, that is the desired conclusion. □

The next lemma is about the condition (2.6). It requires some compatibility conditions with respect to the wedge K.

Assumption (A4): One has

$$\bar{J}\left(Jx-F'(x)\right)\in K, \tag{2.8}$$

for all $x\in K$, and for each $\rho>0$, there exists $\mu_\rho>0$ such that if $x\in K$ and $\|x\|\le\rho$, then

$$\bar{J}\left(\mu Jx-D(x)\right)\in K \tag{2.9}$$

for some μ (depending on x) with $|\mu|\le\mu_\rho$.

Notice that (A4) is trivially satisfied in case that K is the whole space X.

Lemma 2.3. *Assume that* (A1), (A2), (A3) *and* (A4) *hold. Let* $x\in K$.

(i) *One has that* $x-t\left[x-\bar{J}\left(Jx-F'(x)\right)\right]\in K_{r,R}$ *for all* $t>0$ *sufficiently small, in each of the following conditions:*
   (i_1) $r<G(x)<R$;
   (i_2) $x\in N_R$ *and* $\langle G'(x),x-\bar{J}\left(Jx-F'(x)\right)\rangle>0$;
   (i_3) $x\in N_r$ *and* $\langle G'(x),x-\bar{J}\left(Jx-F'(x)\right)\rangle<0$.

(ii) *If* $x\in N_R$, *then for every* $\varepsilon>0$, *one has* $x-t\left(\varepsilon x+E(x)\right)\in K_{r,R}$ *for all* $t>0$ *sufficiently small.*

(iii) *If* $x\in N_r$, *then for every* $\varepsilon>0$, *one has* $x-t\left(-\varepsilon x+E(x)\right)\in K_{r,R}$ *for all* $t>0$ *sufficiently small.*

Proof. (i) First note that using (2.8), the representation

$$x-t\left[x-\bar{J}\left(Jx-F'(x)\right)\right]=(1-t)\,x+t\bar{J}\left(Jx-F'(x)\right)$$

and the convexity of K yield that $x-t\left[x-\bar{J}\left(Jx-F'(x)\right)\right]\in K$ for every $t\in(0,1)$. Then the conclusion of (i) follows from Lemma 2.2.

(ii) According to Lemma 2.2 (b), we first check that $\langle G'(x), z\rangle > 0$, where $z := \varepsilon x + E(x)$. Indeed, using (2.4) and (2.3), we can see that

$$\left\langle G'(x), z\right\rangle = \langle G'(x), \varepsilon x + E(x)\rangle = \varepsilon \left\langle G'(x), x\right\rangle \geq \varepsilon \inf_{u\in N_R} \langle G'(u), u\rangle > 0.$$

Next we need to check (2.6). One has

(2.10)

$$y := x - t(\varepsilon x + E(x)) = t\bar{J}(\mu Jx - D(x)) + \left(1 - t\mu |\mu|^{\frac{2-p}{p-1}} - t\varepsilon + t\lambda\right) x,$$

where $\mu = \mu(x)$ is as in (2.9), assumed to be nonzero. Clearly, for small t, $1 - t\mu|\mu|^{\frac{2-p}{p-1}} - t\varepsilon + t\lambda > 0$, and thus (2.10) together with (2.9) shows that $y \in K$ for all small enough $t > 0$. The conclusion now follows from Lemma 2.2 (b). The case $\mu = 0$ is investigated similarly.

(iii) We proceed as at the case (ii) and find that

$$\langle G'(x), -\varepsilon x + E(x)\rangle = -\varepsilon \left\langle G'(x), x\right\rangle \leq -\varepsilon \inf_{u\in N_R} \langle G'(u), u\rangle < 0.$$

Hence, if $z := -\varepsilon x + E(x)$, then $\langle G'(x), z\rangle < 0$. Furthermore

$$y := x - t(-\varepsilon x + E(x)) = t\bar{J}(\mu Jx - D(x)) + \left(1 - t\mu |\mu|^{\frac{2-p}{p-1}} + t\varepsilon + t\lambda\right) x,$$

and we obtain as above that $y \in K$ for all small enough $t > 0$. □

Now we are ready to state and prove our main result of this section.

Theorem 2.4. *Assume that* (A1), (A2), (A3) *and* (A4) *are satisfied. Then there exists a sequence* $(x_n) \subset K_{r,R}$ *such that*

$$F(x_n) \to \inf F(K_{r,R}) \quad \text{as} \quad n \to \infty, \tag{2.11}$$

and one of the following statements holds:

(a) $x_n - \bar{J}(Jx_n - F'(x_n)) \to 0$ *as* $n \to \infty$;

(b) *for each* $n \geq 1$, $G(x_n) = R$, $\langle G'(x_n), x_n - \bar{J}[Jx_n - F'(x_n)]\rangle \leq 0$ *and*

$$\bar{J}(\mu_n Jx_n) - \bar{J}(\mu_n Jx_n - Dx_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.12}$$

where $\mu_n = \mu(x_n)$ *is chosen accordingly to* (2.9);

(c) *for each* $n \geq 1$, $G(x_n) = r$, $\langle G'(x_n), x_n - \bar{J}[Jx_n - F'(x_n)]\rangle \geq 0$ *and* (2.12) *holds.*

If in addition, F *satisfies a Palais-Smale type compactness condition guarantying that any sequence as above has a convergent subsequence, and the following boundary conditions hold*

$$F'(x) + \eta G'(x) \neq 0 \quad \text{for all} \quad \eta > 0,\ x \in N_R, \tag{2.13}$$

$$F'(x) + \eta G'(x) \neq 0 \quad \text{for all} \quad \eta < 0,\ x \in N_r, \tag{2.14}$$

then there exists $x \in K_{r,R}$ *such that*

$$F(x) = \inf F(K_{r,R}) \quad \text{and} \quad F'(x) = 0.$$

Proof. We shall apply Ekeland's variational principle for $M := K_{r,R}$ (we use here that K is closed and G is continuous, hence $K_{r,R}$ is a closed subset of the Banach space X) endowed with the metric $d(x,y) := \|x-y\|$, for the function F (which from (A3) is C^1 and bounded from below), and for $\varepsilon := \frac{1}{n}$ $(n \in \mathbb{N} \setminus \{0\})$. It follows that there exists a sequence (x_n) in $K_{r,R}$ such that

$$F(x_n) \leq \inf F(K_{r,R}) + \frac{1}{n} \tag{2.15}$$

and

$$F(x_n) \leq F(y) + \frac{1}{n}\|x_n - y\| \quad \text{for every} \quad y \in K_{r,R}. \tag{2.16}$$

Clearly (2.15) implies (2.11).

Since (x_n) belongs to $K_{r,R}$, we distinguish three cases:

Case 1: There exists a subsequence of (x_n), still denoted by (x_n), in one of the following situations: (i_1) $r < G(x_n) < R$ for all n; (i_2) $x_n \in N_R$ and $\langle G'(x_n), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\rangle > 0$ for all n; (i_3) $x_n \in N_r$ and $\langle G'(x_n), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\rangle < 0$ for all n.

Case 2: There exists a subsequence of (x_n), still denoted by (x_n), such that $x_n \in N_R$ and $\langle G'(x_n), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\rangle \leq 0$ for all n.

Case 3: There exists a subsequence of (x_n), still denoted by (x_n), such that $x_n \in N_r$ and $\langle G'(x_n), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\rangle \geq 0$ for all n.

Assume Case 1. According to Lemma 2.3 (i), for each n, we have $y := x_n - t\left(x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right) \in K_{r,R}$, for all $t > 0$ sufficiently small. Thus we may apply (2.16) and deduce

$$-t\left\langle F'\left(x_n\right), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\rangle + o\left(t\right) + \frac{t}{n}\left\|x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\| \geq 0.$$

Divide by t and let t go to zero to obtain

$$-\left\langle F'\left(x_n\right), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\rangle + \frac{1}{n}\left\|x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\| \geq 0.$$

It follows that

$$\left\langle F'\left(x_n\right), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\rangle \leq \frac{1}{n}\left\|x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\|. \tag{2.17}$$

Then from (2.1),

$$\begin{aligned}\left\langle F'\left(x_n\right), x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\rangle &= \left\langle F'\left(x_n\right), \bar{J}Jx_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\rangle \\ &\geq a\left\|x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\right\|^{\beta}.\end{aligned}$$

Using these equality in (2.17) we deduce that

$$a\|x_n - \bar{J}\left[Jx_n - F'(x_n)\right]\|^{\beta-1} \leq \frac{1}{n}.$$

Hence $x_n - \bar{J}\left[Jx_n - F'(x_n)\right] \to 0$ as $n \to \infty$ and so, property (a) holds in Case 1.

Assume Case 2. Now Lemma 2.3 (ii) guarantees that for each n and any $\varepsilon > 0$, $y := x_n - t\left(\varepsilon x_n + E\left(x_n\right)\right) \in K_{r,R}$ for all $t > 0$ sufficiently small. Then (2.16) implies

$$\langle F'(x_n), \varepsilon x_n + E(x_n)\rangle \leq \frac{1}{n}\|\varepsilon x_n + E(x_n)\|.$$

Letting $\varepsilon \to 0$ and using Lemma 2.1 we deduce

(2.18)

$$a\left\|\bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right)\right\|^{\beta} \leq \langle F'(x_n), E(x_n)\rangle \leq \frac{1}{n}\|E(x_n)\|.$$

Let us consider the continuous linear operator

$$P_n : X \to X, \quad P_n x := x - \frac{\langle G'(x_n), x\rangle}{\langle G'(x_n), x_n\rangle}\, x_n.$$

Since $(x_n) \subset N_R$ and the level set N_R is bounded, it follows that (x_n) is a bounded sequence. By the assumption on G' it follows that $(G'(x_n))$ is also bounded. In addition

$$\langle G'(x_n), x_n\rangle \geq \inf_{u \in N_R} \langle G'(u), u\rangle := b_R > 0.$$

We have

$$\|P_n x\| \leq \|x\| + \frac{\|G'(x_n)\|\|x\|}{b_R}\,\|x_n\| = \left(1 + \frac{\|G'(x_n)\|\|x_n\|}{b_R}\right)\|x\|, \quad x \in X.$$

Hence there exists $\alpha_R > 0$ (independent on n) such that

$$\|P_n x\| \leq \alpha_R \|x\|, \quad \text{for all } x \in X \text{ and } n \geq 1.$$

For $x := \bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right)$, one has $P_n x = E(x_n)$ and thus

$$\|E(x_n)\| \leq \alpha_R \|\bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right)\|.$$

Then by (2.18),

$$a\left\|\bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right)\right\|^{\beta} \leq \frac{\alpha_R}{n}\|\bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right)\|.$$

Since $\beta > 1$, this yields $\bar{J}\left(\mu_n J x_n\right) - \bar{J}\left(\mu_n J x_n - D x_n\right) \to 0$ as $n \to \infty$, that is (2.12) holds.

Finally, in Case 3 we proceed similarly by using Lemma 2.3 (iii) and taking in (2.16) $y := x_n - t\left(-\varepsilon x_n + E\left(x_n\right)\right)$. The conclusion is the same, namely (2.12) holds.

Assume now that the additional hypotheses of the theorem are satisfied. The (PS) condition guarantees the existence of a subsequence of $\left(x_n\right)$, which is still denoted by $\left(x_n\right)$, such that $x_n \to x$ as $n \to \infty$, for some element $x \in K_{r,R}$. Clearly, (2.11) gives $F(x) = \inf F(K_{r,R})$. In case of the property (a), if we denote $y_n := x_n - \bar{J}\left(J x_n - F'\left(x_n\right)\right)$, then $y_n \to 0$ as $n \to \infty$, and from

$$J\left(x_n - y_n\right) = J x_n - F'\left(x_n\right),$$

letting $n \to \infty$ and using the continuity of F' and the demicontinuity of J, we obtain $F'(x) = 0$ and the proof is finished. Assume that the property (b) holds. Then, if we pass to the limit we obtain

$$G(x) = R, \quad \langle G'(x), x - \bar{J}\left[Jx - F'(x)\right]\rangle \leq 0 \tag{2.19}$$

and

$$\bar{J}(\mu Jx) - \bar{J}(\mu Jx - Dx) = 0, \tag{2.20}$$

where μ is the limit of some convergent subsequence of (μ_n). Notice that such a subsequence exists since according to (A4), $|\mu_n| \leq \mu_\rho$, where ρ is a bound for the sequence $(\|x_n\|)$. Next from (2.20)

$$Dx = 0,$$

that is

$$F'(x) + \eta G'(x) = 0, \tag{2.21}$$

where

$$\eta := -\frac{\langle F'(x), x\rangle}{\langle G'(x), x\rangle}.$$

In case that $\eta = 0$, (2.21) shows that $F'(x) = 0$ and we are done. Assume $\eta \neq 0$. From (2.21),

$$\left\langle F'(x), x - \bar{J}\left[Jx - F'(x)\right]\right\rangle = -\eta\left\langle G'(x), x - \bar{J}\left[Jx - F'(x)\right]\right\rangle. \tag{2.22}$$

This together with (2.19) gives

$$0 \geq \langle G'(x), x - \bar{J}\left[Jx - F'(x)\right]\rangle = -\frac{1}{\eta}\langle F'(x), \bar{J}Jx - \bar{J}\left[Jx - F'(x)\right]\rangle.$$

Since

$$\langle F'(x), \bar{J}Jx - \bar{J}\left[Jx - F'(x)\right]\rangle \geq a\left\|\bar{J}Jx - \bar{J}\left[Jx - F'(x)\right]\right\|^{\beta} \geq 0,$$

we may infer that $\eta > 0$. Then

$$x \in N_R, \quad \eta > 0 \quad \text{and} \quad F'(x) + \eta G'(x) = 0,$$

which contradicts (2.13). Thus the case $\eta \neq 0$ can not occur. The case of the property (c) is similar. □

3. Application

In this section we present an application of Theorem 2.4 for the localization in annular conical domains of the positive solutions of the two-point boundary value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)'(t) = f(u(t)), \quad t \in [0,1] \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $p > 1$, f is a continuous function on $\mathbb{R}$, which is nonnegative and nondecreasing on $\mathbb{R}_+$. Hence all possible nonnegative solutions are concave

functions on $[0,1]$. We seek *symmetric* solutions with respect to the middle of the interval $[0,1]$, that is with the property

$$u(1-t)=u(t) \quad \text{for every} \quad t\in\left[0,\frac{1}{2}\right].$$

Consider the Banach space $X:=W_0^{1,p}(0,1)$ endowed with the energetic norm $\|u\|_{1,p}=\left(\int_0^1|u'(t)|^p\,dt\right)^{1/p}$ and define the functional

$$F:W_0^{1,p}(0,1)\to\mathbb{R},\quad F(u)=\int_0^1\left(\frac{1}{p}|u'(t)|^p-g(u(t))\right)dt,$$

where $g(\tau)=\int_0^\tau f(s)\,ds$. Clearly, F is a C^1-functional and

$$F'(u)=-\left(|u'|^{p-2}u'\right)'-f(u). \tag{3.2}$$

Hence the solutions of (3.1) are critical points of F.

Let $G:W_0^{1,p}(0,1)\to\mathbb{R}$ be given by $G(u)=\frac{1}{p}\|u\|_{1,p}^p$. It is known that the functional G is continuously Fréchet differentiable on $W_0^{1,p}(0,1)$ and $G'(u)=-\left(|u'|^{p-2}u'\right)'$. The operator $-\left(|u'|^{p-2}u'\right)'$ is in fact the duality mapping $J:W_0^{1,p}(0,1)\to W^{-1,q}(0,1)$ $(1/p+1/q=1)$ corresponding to the normalization function $\varphi(t)=t^{p-1}$, $t\in\mathbb{R}_+$ (see, e.g. [4, Theorem 7 and Theorem 9, pp. 348-350]). Hence, in our case

$$G'(u)=J(u)=-\left(|u'|^{p-2}u'\right)'.$$

In this specific case, assumption (A1) holds. Indeed, it is well-known that $W_0^{1,p}(0,1)$ and its dual $W^{-1,q}(0,1)$ are locally uniformly convex reflexive Banach spaces, while the second requirement in (A1) is a consequence of the following result due to Glowinski and Marrocco [11] (which also holds in higher dimension):

(i) If $p\in(1,2]$, then

$$\left(\|u\|_{1,p}+\|v\|_{1,p}\right)^{2-p}\langle Ju-Jv,\ u-v\rangle\geq\|u-v\|_{1,p}^2 \quad \text{for all } u,v\in W_0^{1,p}(0,1);$$

(ii) *If* $p\in(2,\infty)$, then there exists a constant $c(p)>0$ such that

$$\langle Ju-Jv,\ u-v\rangle\geq c(p)\|u-v\|_{1,p}^p \quad \text{for all } u,v\in W_0^{1,p}(0,1).$$

Thus (2.1) is true for $a(\rho)=(2\rho)^{p-2}$ and $\beta=2$, if $p\leq 2$, and for $a(\rho)=c(p)$ and $\beta=p$, if $p>2$.

Let us consider the cone of all nonnegative functions in $W_0^{1,p}(0,1)$ which are symmetric with respect to the middle of the interval $[0,1]$, namely

$$K:=\left\{u\in W_0^{1,p}(0,1):u\geq 0,\ u(t)=u(1-t)\ \text{for all } t\in[0,1/2]\right\}.$$

We can immediately see that the assumption (A2) holds. As concerns assumption (A3), note that F is bounded from below on the intersection of K with each ball of $W_0^{1,p}(0,1)$. Indeed, if $u \in K$ and $\|u\|_{1,p} \leq \rho$, then

(3.3)
$$0 \leq u(t) = \int_0^t u'(s)\,ds \leq \left(\int_0^1 1^q ds\right)^{1/q} \left(\int_0^1 |u'(s)|^p\,ds\right)^{1/p} = \|u\|_{1,p} \leq \rho$$

for all $t \in [0,1]$, where $1/p + 1/q = 1$. Next, since f is nonnegative on $\mathbb{R}_+$, g is nondecreasing on $\mathbb{R}_+$ and thus

$$F(u) \geq -\int_0^1 g(u(t))\,dt \geq -g(\rho).$$

Hence the assumption (A3) also holds.

In order to check assumption (A4), we first show that the condition (2.8) is satisfied. Indeed, if $u \in K$ and we let $v := \bar{J}(Ju - F'(u))$, then $Jv = Ju - Ju + f(u)$, that is $Jv = f(u)$. Since $f(u) \geq 0$, one has $v \geq 0$. On the other hand, the symmety of u with respect to $1/2$ is obviously passed to $f(u)$, and then to v. The last assertion follows from the fact that if h is symmetric with respect to $1/2$, and $v(t)$ solves $Jv = h$, then by a direct computation, we have that $v(1-t)$ also solves it. Then, the uniqueness of the solution yields $v(t) = v(1-t)$, i.e. v is symmetric with respect to $1/2$. Therefore $v \in K$ as desired.

Next we show that the condition (2.9) holds for

$$\mu = 1 + \eta, \quad \text{where} \quad \eta = \eta(u) := -\frac{\langle F'(u), u\rangle}{\langle G'(u), u\rangle}.$$

Indeed, if $u \in K$ and we denote $v := \bar{J}(\mu Ju - D(u))$, then

(3.4)
$$\begin{aligned} Jv &= \mu Ju - D(u) = \mu Ju - F'(u) - \eta Ju \\ &= \mu Ju - Ju + f(u) - \eta Ju = f(u). \end{aligned}$$

which as above yields the conclusion $v \in K$. On the other hand, for each $\rho > 0$, there is $c(\rho) > 0$ with $|\eta(u)| \leq c(\rho)$ for every $u \in K$ with $\|u\|_{1,p} \leq \rho$. Then $|\mu| = |1 + \eta(u)| \leq 1 + c(\rho) =: \mu_\rho$ for all $u \in K$ with $\|u\|_{1,p} \leq \rho$. Thus, assumption (A4) is satisfied.

Before we state and proof the main result of existence and localization for the problem (3.1), we give the *weak Harnack type inequality* for p-superharmonic symmetric functions on $[0,1]$, which is essential for the estimations from below on the part $G(u) = r$ of the boundary of $K_{r,R}$.

Lemma 3.1. *For every function $u \in K$ with $Ju \in C([0,1];\mathbb{R}_+)$ nondecreasing, the following inequality holds*

(3.5)
$$u(t) \geq t(1-2t)^{\frac{1}{p-1}} \|u\|_{1,p}$$

for all $t \in (0, 1/2)$.

Proof. Let $u \in K$ with $Ju \in C([0,1];\mathbb{R}_+)$, and let $t \in (0,1/2)$ be any number. From $Ju \geq 0$ on $[0,1]$, one has that u is concave and so u' is decreasing, while from $Ju \in C[0,1]$, we obtain $u \in W^{2,\infty}(0,1)$. Now the symmetry of u guarantees $u'(s) \geq 0$ on $[0,1/2]$ and $u'(1/2) = 0$. Furthermore,

$$u(t) = \int_0^t u'(s)\,ds \geq tu'(t), \tag{3.6}$$

and it is not difficult to prove the following inequality

$$u'(t) \geq (1-2t)^{\frac{1}{p-1}} u'(0). \tag{3.7}$$

Indeed, if we let $\phi(s) = u'(s)^{p-1} - (1-2s)\,u'(0)^{p-1}$ for $s \in [0,1/2]$, then

$$\begin{aligned} \phi'(s) &= \left(\left|u'(s)\right|^{p-2} u'(s)\right)' + 2u'(0)^{p-1} \\ &= -(Ju)(s) + 2u'(0)^{p-1}. \end{aligned}$$

Hence ϕ' is decreasing, and consequently ϕ is concave. In addition, $\phi(0) = \phi(1/2) = 0$. Hence $\phi(s) \geq 0$ for all $s \in [0,1/2]$, i.e. (3.7) is true. Another remark is that

$$\|u\|_{1,p}^p = \int_0^1 \left|u'(s)\right|^p ds = 2\int_0^{\frac{1}{2}} u'(s)^p\,dt \leq u'(0)^p,$$

whence

$$u'(0) \geq \|u\|_{1,p}. \tag{3.8}$$

Now (3.6), (3.7) and (3.8) give (3.5). □

Now we are ready to state the main existence and localization result for the problem (3.1).

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, nonnegative and nondecreasing on $\mathbb{R}_+$. Assume that there are numbers $0 < r < R$ and $a \in (0,1/2)$ such that*

$$f\left(a(1-2a)^{\frac{1}{p-1}}(pr)^{\frac{1}{p}}\right) \geq \frac{(pr)^{\frac{p-1}{p}}}{a(1-2a)^{\frac{p}{p-1}}}, \tag{3.9}$$

$$f\left((pR)^{\frac{1}{p}}\right) \leq (pR)^{\frac{p-1}{p}}. \tag{3.10}$$

Then (3.1) has a positive, concave and symmetric solution u which minimizes F on the set of all functions $v \in K$ satisfying $(pr)^{1/p} \leq \|v\|_{1,p} \leq (pR)^{1/p}$.

Proof. We shall apply Theorem 2.4. As shown before, the assumptions (A1)-(A4) hold. Thus it remains to check the boundary conditions (2.13), (2.14) and the Palais-Smale type compactness condition.

First we check (2.13). Assume that (2.13) does not hold. Then there is $u \in K$ with $\|u\|_{1,p}^p/p = R$ and $\eta > 0$ such that $F'(u) + \eta G'(u) = 0$. Hence $f(u) = -(1+\eta)|u'|^{p-2}u'$, that is

$$-\left(|u'|^{p-2}u'\right)'(t) = \frac{1}{1+\eta} f(u(t)) \quad \text{on } [0,1] \text{ and } u(0) = u(1) = 0. \tag{3.11}$$

If we multiply by $u(t)$, we integrate over $[0,1]$, and we take into account (3.3) and the monotony of f, we obtain

$$\begin{aligned} pR &= \|u\|_{1,p}^p = \frac{1}{1+\eta}\int_0^1 f(u(t))\,u(t)\,dt < \int_0^1 f(u(t))\,u(t)\,dt \\ &\le f\left(\|u\|_{1,p}\right)\|u\|_{1,p} = f\left((pR)^{\frac{1}{p}}\right)(pR)^{\frac{1}{p}}, \end{aligned}$$

which contradicts (3.10). Hence (2.13) holds.

Next assume that the boundary condition (2.14) does not hold. Then, for some $u \in K$ with $\|u\|_{1,p}^p/p = r$ and $\eta < 0$, we have $F'(u)+\eta G'(u) = 0$, that is $f(u) = -(1+\eta)\left(|u'|^{p-2}u'\right)'$. The case $1+\eta \le 0$ is not possible since it would imply that u is convex, whence since $u(0) = u(1) = 0$, $u = 0$, which is excluded by $\|u\|_{1,p}^p/p = r > 0$. Hence $1+\eta > 0$, and we have (3.11), where this time $\frac{1}{1+\eta} > 1$. As above, after multiplication and integration, we obtain

$$\begin{aligned} pr &= \|u\|_{1,p}^p = \frac{1}{1+\eta}\int_0^1 f(u(t))\,u(t)\,dt > \int_0^1 f(u(t))\,u(t)\,dt \\ &= 2\int_0^{\frac{1}{2}} f(u(t))\,u(t)\,dt \ge 2\int_a^{\frac{1}{2}} f(u(t))\,u(t)\,dt \\ &\ge 2\left(\frac{1}{2}-a\right)u(a)\,f(u(a)). \end{aligned}$$

This together of (3.5) implies

$$pr > a(1-2a)^{\frac{p}{p-1}}(pr)^{\frac{1}{p}} f\left(a(1-2a)^{\frac{1}{p-1}}(pr)^{\frac{1}{p}}\right),$$

that is

$$f\left(a(1-2a)^{\frac{1}{p-1}}(pr)^{\frac{1}{p}}\right) < \frac{(pr)^{\frac{p-1}{p}}}{a(1-2a)^{\frac{p}{p-1}}},$$

which contradicts (3.9). Thus, the conditions (2.13) and (2.14) hold.

Finally, we have to check the Palais-Smale compactness condition. The key property is the complete continuity of the operator $\bar{J}$ from $C[0,1]$ to $W_0^{1,p}(0,1)$. Assume that the sequence (u_n) guaranteed by Theorem 2.4 is in Case (a), i.e. $v_n := u_n - \bar{J}(Ju_n - F'(u_n)) \to 0$ as $n \to \infty$. Since $F'(u_n) = Ju_n - f(u_n)$, we have $u_n = v_n + \bar{J}f(u_n)$. Being the sequences (v_n) and $\left(\bar{J}f(u_n)\right)$ relatively compact, it follows that the sequence (u_n) is relatively compact too. Hence the Palais-Smale type condition holds in case

(a). Assume now that (u_n) satisfies one of the cases (b) and (c). Hence, passing to the limit for $n \to \infty$, we have

$$\bar{J}(\mu_n J u_n) - \bar{J}(\mu_n J u_n - D u_n) = \bar{J}(\mu_n J u_n) - \bar{J} f(u_n) \to 0, \tag{3.12}$$

where

$$\mu_n = 1 - \frac{\langle F'(u_n), u_n \rangle}{\langle G'(u_n), u_n \rangle}.$$

Since (μ_n) is bounded, passing eventually to a subsequence, we may assume that $\mu_n \to \mu$ as $n \to \infty$. The case $\mu = 0$ is not possible. Indeed, otherwise, $\frac{\langle F'(u_n), u_n \rangle}{\langle G'(u_n), u_n \rangle} \to 1$, whence $\langle f(u_n), u_n \rangle \to 0$. However, using the behavior of u_n, the monotonicity of f and (3.5), we have

$$\begin{aligned} \langle f(u_n), u_n \rangle &= \int_0^1 f(u_n(t)) u_n(t)\, dt \geq \int_a^{1/2} f(u_n(t)) u_n(t)\, dt \\ &\geq \left(\frac{1}{2} - a \right) f(u_n(a)) u_n(a) \geq c > 0, \end{aligned}$$

where c depends only on r and R, respectively, being independent on n. It follows the contradiction $0 \geq c > 0$. Hence $\mu \neq 0$. Since from (3.12), $\left(\bar{J}(\mu_n J u_n)\right)$ is compact, and $\bar{J}(\mu_n J u_n) = |\mu_n|^{\frac{2-p}{p-1}} \mu_n u_n$, we derive that (u_n) is compact as desired.

Therefore all the assumptions of Theorem 2.4 hold. □

In the next corollary we give conditions on the function f which assure the existence of the numbers r and R having the properties (3.9), (3.10).

Corollary 3.3. *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function, nonnegative and nondecreasing on $\mathbb{R}_+$. If for some $a \in (0, 1/2)$,*

$$\limsup_{\tau \to 0} \frac{f(\tau)}{\tau^{p-1}} > \frac{1}{a^p (1-2a)^{\frac{1}{p-1}}}, \tag{3.13}$$

$$\liminf_{\tau \to \infty} \frac{f(\tau)}{\tau^{p-1}} < 1, \tag{3.14}$$

then (3.1) has at least one nontrivial positive, concave and symmetric solution.

Proof. From (3.13), we can find a number $r > 0$ sufficiently small, such that (3.9) holds. Also, from (3.14), it can be found a large enough $R > r$ with the property (3.10). Thus we can apply Theorem 3.2. □

Finally we note that Theorem 2.4 in the abstract setting and Theorem 3.2 for the considered concrete application, immediately yield multiplicity results of solutions if their hypotheses are satisfied for several finitely or infinitely many pairs of numbers r, R. Thus, Theorem 3.2 gives the following multiplicity result for (3.1).

Theorem 3.4. *Assume that $f:\mathbb{R}\to\mathbb{R}_+$ is a continuous function, nonnegative and nondecreasing on $\mathbb{R}_+$.*

(i) *Let $(r_j)_{1\le j\le k}, (R_j)_{1\le j\le k}$ $(k\le\infty)$ be increasing finite or infinite sequences with $r_j<R_j<r_{j+1}<R_{j+1}$ for $1\le j\le k-1$, and let $(a_j)_{1\le j\le k}$ with $a_j\in(0,1/2)$ for all j. If*

$$f\left(a_j\left(1-2a_j\right)^{\frac{1}{p-1}}\left(pr_j\right)^{\frac{1}{p}}\right)\ge\frac{\left(pr_j\right)^{\frac{p-1}{p}}}{a_j\left(1-2a_j\right)^{\frac{p}{p-1}}}, \tag{3.15}$$

$$f\left(\left(pR_j\right)^{\frac{1}{p}}\right)\le\left(pR_j\right)^{\frac{p-1}{p}}, \tag{3.16}$$

for all j, then (3.1) has k (respectively, when $k=\infty$, an infinite sequence of) distinct positive, concave and symmetric solutions u_j $(1\le j\le k)$, such that for each j, u_j minimizes F on the set of all functions $v\in K$ satisfying $\left(pr_j\right)^{1/p}\le\|v\|_{1,p}\le\left(pR_j\right)^{1/p}$.

(ii) *Let $(r_j)_{j\ge1}, (R_j)_{j\ge1}$ be decreasing infinite sequences such that $R_{j+1}<r_j<R_j$ for $j\ge1$, and let $(a_j)_{j\ge1}$ be a sequence of numbers from the interval $(0,1/2)$ such that the conditions (3.15), (3.16) hold for all j. Then (3.1) has an infinite sequence of distinct positive, concave and symmetric solutions u_j $(j\ge1)$, such that for each j, u_j minimizes F on the set of all functions $v\in K$ satisfying $\left(pr_j\right)^{1/p}\le\|v\|_{1,p}\le\left(pR_j\right)^{1/p}$.*

The existence of two infinite sequences $(r_j)_{j\ge1}, (R_j)_{j\ge1}$ as in Theorem 3.4 is guaranteed for nonlinearities f which oscillate toward infinity, or zero. More exactly we have the following result for which the sequence $(a_j)_{j\ge1}$ is a constant one.

Corollary 3.5. *Let $f:\mathbb{R}\to\mathbb{R}_+$ be a continuous function, nonnegative and nondecreasing on $\mathbb{R}_+$, and let $a\in(0,1/2)$.*

(i) *If*

$$\operatorname{limsup}_{\tau\to\infty}\frac{f(\tau)}{\tau^{p-1}}>\frac{1}{a^p\left(1-2a\right)^{\frac{1}{p-1}}}\quad\text{and}\quad\operatorname{liminf}_{\tau\to\infty}\frac{f(\tau)}{\tau^{p-1}}<1,$$

then (3.1) has an infinite sequence $(u_j)_{j\ge1}$ of distinct positive, concave and symmetric solutions, with $\|u_j\|_{1,p}\to\infty$.

(ii) *If*

$$\operatorname{limsup}_{\tau\to0}\frac{f(\tau)}{\tau^{p-1}}>\frac{1}{a^p\left(1-2a\right)^{\frac{1}{p-1}}}\quad\text{and}\quad\operatorname{liminf}_{\tau\to0}\frac{f(\tau)}{\tau^{p-1}}<1,$$

then (3.1) has an infinite sequence $(u_j)_{j\ge1}$ of distinct positive, concave and symmetric solutions, with $\|u_j\|_{1,p}\to0$.

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