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MEASURE OF NONCOMPACTNESS AND SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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REZUMAT. — Măsura de necompactitate şi ecuaţiile diferenţiale de ordinul al doilea cu argument modificat. În această lucrare este studiată existenţa şi unicitatea soluţiei problemei lui Dirichlet (1.1), (1.2) într-un spaţiu Banach. Această problemă este privită ca un caz particular al problemei lui Dirichlet (3.2) pentru ecuaţia funcţional-diferenţială (3.1). Principalul rezultat referitor la existenţa soluţiei problemei (3.1), (3.2) este conţinut în Teorema 1, în care aplicaţiei din membrul drept al ecuaţiei (3.1) i se cere să satisfacă o condiţie mai slabă decît aceea de a fi compactă. Această condiţie se exprimă cu ajutorul măsurii de necompactitate a lui Kuratowski.

1. Introduction. This paper deals with the boundary value problem

$$u''(t) = f(t, u(t), u'(t), u(g_1(t)), \ldots, u(g_m(t))), \quad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \ t \in I' \setminus \text{int } I, \tag{1.2}$$

in a real Banach space X, where $I = [a, b]$, $I' = [a', b']$, $a' \leqslant a < b \leqslant b'$, f is a continuous mapping from $I \times X^{m+2}$ into X, $g_i (i = 1, \ldots, m)$ are continuous functions from I into I' and φ is a continuous function from $I' \setminus$ int I into X.

By a solution to problem (1.1), (1.2) we mean a function $u \in C^2(I; X) \cap C(I'; X)$ satisfying conditions (1.1) and (1.2).

For $u \in C(I; X)$ let us denote by u^i $(i = 1, \ldots, m)$ the function from I into X, $u^i(t) = u(g_i(t))$ if $g_i(t) \in I$ and $u^i(t) = \varphi(g_i(t))$ otherwise.

Let us consider the mapping $h: I \times X \times X \times C(I; X) \to X$,

$$h(t, x, y; u) = f(t, x, y, u^1(t), \ldots, u^m(t)), \tag{1.3}$$

for $t \in I$, $x, y \in X$ and $u \in C(I; X)$.

A function u is a solution to (1.1), (1.2) if and only if $u \in C^2(I; X)$ and satisfies

$$u''(t) = h(t, u(t), u'(t); u), \quad t \in I, \tag{1.4}$$

$$u(a) = \varphi(a), \ u(b) = \varphi(b). \tag{1.5}$$

In the particular case when $g_i(I) \subset I$, $i = 1, \ldots, m$, by the continuity of f it follows that h is also continuous. In this case, the existence of solutions to problem (1.4), (1.5) was established by K. Schmitt and R. Thompson [17] assuming in addition the compactness of h and also by us [14] under more general additional conditions on h.

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Generally, for $g_i(I) \subset I'$, $i = 1, \ldots, m$, the mapping (1.3) may be not continuous on $I \times X \times X \times C(I; X)$. Nevertheless, its restriction to the subset $I \times X \times X \times C_b$, where

$$C_b = \{u \in C(I; X) : u(a) = \varphi(a),\ u(b) = \varphi(b)\},$$

is continuous. The main result on the existence of solutions to (1.4), (1.5), Theorem 1, requires that h be α– Lipschitz (α being the Kuratowski measure of noncompactness). The proof of Theorem 1 uses the topological transversality theorem (Lenay-Schauder's alternative) for condensing mappings, which has been proved in [14] without using the topological degree. In addition, we make use of a priori bounds technique. Similar methods have been used by K. Schmitt and R. Thompson [17], R. Thompson [18], A. Granas, R. Guenther and J. Lee [8]. Theorem 1 may be compared with the results obtained by V. Lakshmikantham [11] and J. Chandra, V. Lakshmikantham, A. Mitchell [5].

In particular, sufficient conditions for that the problem (1.1), (1.2) have solutions are given. These conditions are relaxed in case $X = \mathbf{R}^n$.

The existence theorems are stated in Section 3 and the main result, Theorem 1, is proved in Section 4. In Section 5 a uniqueness theorem is given.

2. **Preliminaires.** Let X be a real Banach space, X^* its dual. We shall denote both the norm in X and its dual norm in X^* by $|\cdot|$. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x^*, x). In case $X = \mathbf{R}^n$ the bilinear functional $(.,.)$ stands for the scalar product.

Denote $\|u\| = \max(|u(t)| : t \in I)$ for $u \in C = C(I; X)$, $\|u\|_1 = \max(\|u\|, \|u'\|)$ for $u \in C^1 = C^1(I; X)$ and $\|u\|_2 = \max(\|u\|, \|u'\|, \|u''\|)$ for $u \in C^2 = C^2(I; X)$.

Let J be the duality mapping of X, i.e. $J : X \to 2^{X^*}$,

$$Jx = \{x^* \in X^* : (x^*, x) = |x|^2 = |x^*|^2\},\ x \in X.$$

Recall that

$$(x^*, y - x) \leqslant \frac{1}{2}|y|^2 - \frac{1}{2}|x|^2, \tag{2.1}$$

for all $x, y \in X$ and $x^* \in Jx$.

Let us denote by α the Kuratowski measure of noncompactness; for each bounded subset A of a Banach space one has

$$\alpha(A) = \inf\{\delta > 0 : A \text{ can be convered by finitely many sets of diameter} \leqslant \delta\}.$$

With a view to avoid any confusion we will denote by α_n the Kuratowski measure of noncompactness on the Banach space $C^n(I; X)$ endowed with the norm $\|u\|_n = \max(\|u\|, \ldots, \|u^{(n)}\|)$. Only for the space $C(I; X)$ the Kuratowski measure of noncompactness will be simply denoted by α instead of α_0.

Clearly, if A is a bounded subset of $C(I; X)$ then $\alpha(A(t)) \leqslant \alpha(A)$ for all $t \in I$, where $A(t) = \{u(t) : u \in A\} \subset X$. Moreover, we have

LEMMA 1. *If A is a bounded equicontinuous subset of $C(I; X)$ then*

$$\alpha(A) = \alpha(A(I)) = \sup(\alpha(A(t)) : t \in I), \tag{2.2}$$

where $A(I) = \{u(t) : u \in A,\ t \in I\} \subset X$.

This result has been proved by A. Ambrosetti [2] and generalizes the classical Ascoli—Arzelà Theorem.

If A is a bounded subset of $C^n(I;X)$ $(n \geqslant 1)$ then

$$\alpha_n(A) = \max\,(\alpha(A), \alpha(A'), \ldots, \alpha(A^{(n)})), \tag{2.3}$$

where

$$A^{(i)} = \{u^{(i)} : u \in A\} \subset C(I;X),\ i = 0, 1, \ldots, n.$$

Also

$$\alpha_{i-1}(A) \leqslant \alpha_i(A), \tag{2.4}$$

for $i = 1, \ldots, n$.

Let Y be a closed convex subset of X and Z an arbitrary subset of X. A continuous mapping $F: Z \to Y$ is said to be (α, ρ) — *Lipschitz*, $\rho \geqslant 0$, if for every bounded subset A of Z, $F(A)$ is bounded and

$$\alpha(F(A)) \leqslant \rho\alpha(A).$$

F is called α — *Lipschitz* if there exists $\rho \geqslant 0$ such that F be (α, ρ) — Lipschitz. F is said to be *condensing* if for every bounded subset A of Z, $F(A)$ is bounded and if $\alpha(A) > 0$ then

$$\alpha(F(A)) < \alpha(A).$$

Let $U \subset Y$ be bounded and open in Y and let $\mathcal{A}(\bar{U};Y)$ be the set of all condensing mappings $F: \bar{U} \to Y$ which are fixed point free on the boundary ∂U of U. A mapping $F \in \mathcal{A}(\bar{U};Y)$ is said to be *essential* if each mapping of $\mathcal{A}(\bar{U};Y)$ which coincides with F on ∂U has at least one fixed point in U.

In this connection, the following lemma will be used latter (for the proof see Lemma 2.1 in [14]).

LEMMA 2. *For each fixed $x_0 \in U$ the mapping $F: \bar{U} \to Y$, $Fx = x_0$ for all $x \in \bar{U}$, is essential.*

Two mappings $F_0, F_1 \in \mathcal{A}(\bar{U};Y)$ are said to be *homotopic* if there exists $H: [0,1] \times \bar{U} \to Y$ such that $H_\lambda = H(\lambda, .) \in \mathcal{A}(\bar{U};Y)$ for all $\lambda \in [0,1]$, $H_0 = F_0$, $H_1 = F_1$ and $H(., x): [0,1] \to Y$ is continuous uniformly with respect to $x \in \bar{U}$.

We also note the following topological transversality theorem (Leray—Schauder's alternative) for condensing mappings.

LEMMA 3. *Let $F_0, F_1 \in \mathcal{A}(\bar{U};Y)$ be two homotopic mappings. Then F_0 is essential if and only if F_1 is essential.*

The proof of this lemma can be found in [14]. It reproduces with some specifical changes that of the topological transversality theorem for completely continuous mappings (see [6] and [7]).

For other properties of measures of noncompactness and other results on condensing mappings we refer to the book of R. R. Ahmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii [1].

3. Existence theorems. Let us consider the problem

$$u''(t) = h(t, u(t), u'(t); u), \quad t \in I, \tag{3.1}$$

$$u(a) = r, \ u(b) = s, \tag{3.2}$$

where r and s are two fixed elements of X.

Let $C_b = \{u \in C(I;X) : u(a) = r, \ u(b) = s\}$, $C_b^1 = C_b \cap C^1$ and $C_b^2 = C_b \cap C^2$. We shall consider on C_b, C_b^1 and C_b^2 the topologies induced by those of C, C^1 and C^2, respectively.

The main existence result is

THEOREM 1. *Assume that*

(i) $h \in C(I \times X \times X \times C_b^1; X)$ *and h is uniformly continuous on* $I \times A_1 \times A_2 \times A_3$ *whenever A_1, A_2 are bounded subsets of X and $A_3 \subset C_b^2$ is bounded in C^1.*

(ii) *There exists ρ such that*

$$0 \leqslant \rho < \min\,(8/(b-a)^2, 2/(b-a), 1) \tag{3.3}$$

and

$$\alpha(h(t, A_1, A_2, A_3)) \leqslant \rho \max\,(\alpha(A_1), \alpha(A_2), \alpha_1(A_3)), \tag{3.4}$$

whenever $t \in I$, A_1 and A_2 are bounded subsets of X and $A_3 \subset C_b^2$ is bounded in C^1.

(iii) *For each $x \in X$ satisfying $|x| > M \geqslant \max\,(|r|, |s|)$ there exists $x^* \in Jx$ such that*

$$(x^*, h(t, x, y; z)) > 0, \tag{3.5}$$

for all $t \in \,]a, b[$, $y \in X$ satisfying $(x^*, y) = 0$ and $z \in C_b^2$ with $\|z\| = |x|$.

(iv) *There exists a nondecreasing function*

$$\Psi : [0, +\infty[\, \to \,]0, +\infty[$$

such that

$$\liminf_{t \to +\infty} t^2/\Psi(t) > 4M \tag{3.6}$$

and

$$|h(t, x, y; z)| \leqslant \Psi(|y|), \tag{3.7}$$

for all $t \in I$, $x, y \in X$ and $z \in C_b^2$ satisfying $|x| \leqslant \|z\| \leqslant M$.

Then equation (3.1) *has at least one solution $u \in C_b^2$.*

Remark 1. If $h : I \times X \times X \times C_b^1 \to X$ is completely continuous then condition (i) in Theorem 1 holds, condition (3.4) holds with $\rho = 0$ and (3.5) may be relaxed as follows:

$$(x^*, h(t, x, y; z)) \geqslant 0. \tag{3.8}$$

Indeed, in this case, for $1/n < \min\,(8/(b-a)^2, 2/(b-a), 1)$ the mapping $h_n(t, x, y; z) = h(t, x, y; z) + (1/n)x$ satisfies the hypothesis of Theorem 1 with $\rho = 1/n$ and $\Psi + M$ instead of Ψ. In consequence, by Theorem 1, the equation

$$u''(t) = h_n(t, u(t), u'(t); u), \ t \in I \tag{3.9}$$

has at least one solution $u_n \in C_b^2$. Let $\{u_n\}_{n \geq n_0}$ be a sequence of solutions to (3.9), where $1/n_0 < \min\,(8/(b-a)^2, 2/(b-a), 1)$. As follows from the proof of Theorem 1, the set $\{u_n\}_{n \geq n_0}$ is bounded in C^2. Hence $\{u_n\}_{n \geq n_0}$ is equicontinuous in C^1. On the other hand, if $G_n : I \times I \to \mathbf{R}$ is the Green's function associated with the scalar problem $y'' - (1/n)y = f(t)$, $y(a) = y(b) = 0$ and u_b^n is the unique solution from C_b^2 to the equation $u'' - (1/n)\,u = 0$, then we must have

$$u_n(t) = -\int_a^b G_n(t, \xi)\, h(\xi, u_n(\xi), u_n'(\xi)\,;\, u_n)d\xi + u_b^n(t). \tag{3.10}$$

Whence, using the compactness of h we obtain that the sets $\{u_n(t)\}$ and $\{u_n'(t)\}$ are precompact in X for each $t \in I$. Thus, by the Ascoli—Arzelà Theorem the sequence $\{u_n\}$ has a subsequence which converges in C^1; its limit is a solution to (3.1) as follows also by (3.10).

As regards the existence of solutions to (1.1), (1.2) we have the following result

COROLLARY 1. *Suppose that*

(i) *f is uniformly continuous on each bounded subset of $I \times X^{m+2}$, $g_i \in C(I \cdot I')$, $i = 1, \ldots, m$ and $\varphi \in C(I' \setminus \text{int}\, I\,; X)$.*

(ii) *There exists ρ satisfying condition* (3.3) *and*

$$\alpha(f(t, A_1, A_2, \ldots, A_{m+2})) \leqslant \rho \max\,(\alpha(A_i) : i = 1, 2, \ldots, m+2), \tag{3.11}$$

whenever $t \in I$ and A_i, $i = 1, 2, \ldots, m+2$ are bounded subsets of X.

(iii) *For each $x \in X$ satisfying $\|x\| > M \geqslant \max\,(\|\varphi(t)\| : t \in I' \setminus \text{int}\, I)$ there exists $x^* \in \mathfrak{J}x$ such that*

$$(x^*, f(t, x, y, x^1, \ldots, x^m)) > 0, \tag{3.12}$$

for all $t \in\,]a, b[$ and all $x, y, x^1, \ldots, x^m \in X$ satisfying $(x^*, y) = 0$ and $\|x^i\| \leqslant \leqslant \|x\|$, $i = 1, \ldots, m$.

(iv) *There exists a nondecreasing function $\Psi : [0, +\infty[\, \to\,]0, +\infty[$ satisfying condition* (3.6) *and*

$$\|f(t, x, y, x^1, \ldots, x^m)\| \leqslant \Psi(\|y\|), \tag{3.13}$$

for all $t \in I$ and all $x, y, x^1, \ldots, x^m \in X$ with $\|x\| \leqslant M$ and $\|x^i\| \leqslant M$, $i = = 1, \ldots, m$.

Then problem (1.1), (1.2) *has at least one solution.*

Remark 2. Theorem 1 and Corollary 1 remain true if we consider certain other measures of noncompactness instead of the Kuratowski measure of noncompactness.

On finite-dimensional spaces some requirements of the hypothesis of Theorem 1 may be lessened as follows.

THEOREM 2. *Assume that*

(i) *The mapping $h : I \times \mathbf{R}^n \times \mathbf{R}^n \times C_b \to \mathbf{R}^n$ is continuous.*

(ii) *There exists $M \geqslant \max\,(\|r\|, \|s\|)$ such that*

$$(x, h(t, x, y\,; z)) \geqslant 0, \tag{3.14}$$

for all $t \in I$, $x \in \mathbf{R}^n$ *with* $\|x\| > M$, $y \in \mathbf{R}^n$ *with* $(x, y) = 0$ *and* $z \in C_b^2$ *satisfying* $\|z\| = \|x\|$.

(iii) *For each* $j \in \{1, \ldots, n\}$ *and each* $M' > 0$ *there is a function* $\Psi_{j,M'}$: $[0, +\infty[\to]0, +\infty[$ *such that* $t/\Psi_{j,M'}(t)$ *is locally integrable on* $[K, +\infty[$, *where* $K = \|r - s\|/(b - a)$,

$$\int_K^\infty t / \Psi_{j,M'}(t)\, dt > 2M \tag{3.15}$$

and

$$|h_j(t, x, y; z)| \leqslant \Psi_{j,M'}(|y_j|), \tag{3.16}$$

whenever $t \in I$, $x \in \mathbf{R}^n$ *and* $z \in C_b^2$ *satisfy* $\|x\| \leqslant \|z\| \leqslant M$ *and* $y \in \mathbf{R}^n$, $y = (y_1, \ldots, y_n)$ *satisfies* $|y_i| \leqslant M'$ *for all* $i \leqslant j - 1$.

Then the system (3.1) *has at least one solution* $u \in C_b^2$.

This result may be compared with Theorem 2.4 in [8], chap. V. Its proof follows easily by that of Theorem 1 if we take into account Lemma 5.6 in [8], chap. II and Remark 1.

As a consequence of Theorem 2 we have

COROLLARY 2. *Let the following conditions hold*

(i) $f \in C(I \times (\mathbf{R}^n)^{m+2}; \mathbf{R}^n)$, $g_i \in C(I; I')$, $i = 1, \ldots, m$ *and*

$$\varphi \in C(I' \setminus \operatorname{int} I; \mathbf{R}^n).$$

(ii) *There exists* $M \geqslant \max(\|\varphi(t)\| : t \in I' \setminus \operatorname{int} I)$ *such that*

$$(x, f(t, x, y, x^1, \ldots, x^m)) \geqslant 0, \tag{3.17}$$

for all $t \in I$, $x \in \mathbf{R}^n$ *with* $\|x\| > M$, $y \in \mathbf{R}^n$ *with* $(x, y) = 0$ *and all* $x^i \in \mathbf{R}^n$ *satisfying* $\|x^i\| \leqslant \|x\|$, $i = 1, \ldots, m$.

(iii) *For each* $j \in \{1, \ldots, n\}$ *and each* $M' > 0$ *there is a function* $\Psi_{j,M'}: [0, +\infty[\to]0, +\infty[$ *such that* $t/\Psi_{j,M'}(t)$ *is locally integrable on* $[K, +\infty[$, *satisfies condition* (3.15) *and*

$$|f_j(t, x, y, x^1, \ldots, x^m)| \leqslant \Psi_{j,M'}(|y_j|), \tag{3.18}$$

for every $t \in I$, $x, x^1, \ldots, x^m \in \mathbf{R}^n$ *satisfying* $\|x\| \leqslant M$, $\|x^i\| \leqslant M$, $i = 1, \ldots, m$ *and any* $y \in \mathbf{R}^n$ *with* $|y_i| \leqslant M'$ *for all* $i \leqslant j - 1$.

Then the problem (1.1), (1.2) *has at least one solution.*

4. **Proofs.** For the proof of Theorem 1 we need some lemmas referring to the a priori bounds on solutions of equation (3.1).

LEMMA 4. *Assume that conditions* (i) *and* (iii) *from Theorem* 1 *hold. Then any solution* $u \in C_b^2$ *of equation* (3.1) *satisfies the inequality*

$$\|u\| \leqslant M. \tag{4.1}$$

Proof. Let $u \in C_b^2$ be a solution of (3.1) and let $t_0 \in I$ be such that $\|u\| = \|u(t_0)\|$. If $t_0 = a$ or $t_0 = b$ then (4.1) follows by $M \geqslant \max(\|r\|, \|s\|)$.

Let $t_0 \in]a, b[$. Then we have $(x_0^*, u'(t_0)) = 0$ for any $x_0^* \in \mathfrak{J}u(t_0)$.

Assume, a contrario, that $\|u(t_0)\| > M$. Then, by (3.5), there exists $x_0^* \in \mathfrak{J}u(t_0)$ such that

$$(x_0^*, h(t_0, u(t_0), u'(t_0) ; u)) > 0.$$

Since h is continuous there is $\delta > 0$ such that

$$(x_0^*, h(t_0 + \lambda, u(t_0 + \lambda), u'(t_0 + \lambda) ; u)) > 0,$$

whenever $|\lambda| < \delta$ and $t_0 + \lambda \in I$. This implies that

$$(x_0^*, u''(t_0 + \lambda)) > 0 \text{ for } |\lambda| < \delta \text{ with } t_0 + \lambda \in I.$$

Hence, by using the Taylor's formula

$$u(t_0 + \lambda) - u(t_0) = \lambda u'(t_0) + (\lambda^2/2)u''(t_0 + \mu),$$

where $\mu = \mu(\lambda)$ lies between t_0 and $t_0 + \lambda$, we deduce that

$$(x_0^*, u(t_0 + \lambda) - u(t_0)) > 0 \text{ for } |\lambda| < \delta,\ \lambda \neq 0 \text{ with } t_0 + \lambda \in I.$$

On the other hand, since $x_0^* \in \mathfrak{J}u(t_0)$, by (2.1), we must have

$$(x_0^*, u(t_0 + \lambda) - u(t_0)) \leqslant \frac{1}{2}\|u(t_0 + \lambda)\|^2 - \frac{1}{2}\|u(t_0)\|^2 \leqslant 0,$$

which contradicts the previous inequality. Therefore $\|u(t_0)\| \leqslant M$ and the proof is complete.

The next lemma is due to K. Schmitt and R. Thompson [17] and it will be used to derive a priori bounds on derivatives of solutions of equation (3.1).

LEMMA 5. *Let* $\Psi : [0, +\infty[\to]0, +\infty[$ *be a nondecreasing function satisfying condition* (3.6) *and let* M *be a positive number. Then there exists a positive constant* M_1 *(depending only on* Ψ *and* M*) such that, if* $u \in C^2(I ; X)$ *is such that* $\|u\| \leqslant M$ *and* $\|u''\| \leqslant \Psi(\|u'\|)$, *then*

$$\|u'\| \leqslant M_1. \tag{4.2}$$

Let $G : I \times I \to \mathbf{R}$ be the Green's function associated with the scalar boundary value problem $y'' = f(t)$, $y(a) = y(b) = 0$. We have

$$G(t, \xi) = \frac{(\xi - a)(b - t)}{b - a} \text{ for } \xi \leqslant t$$
$$= \frac{(t - a)(b - \xi)}{b - a} \text{ for } \xi > t.$$

Define the linear integral operator $N : C \to C^2$,

$$(Nu)(t) = -\int_a^b G(t, \xi)\, u(\xi)\, d\xi, \quad t \in I.$$

We have

$$\|Nu\|_2 \leqslant \max\left((b - a)^2/8,\ (b - a)/2,\ 1\right), \tag{4.3}$$

for all $u \in C$ having $\|u\| \leqslant 1$.

If we assume that $h: I \times X \times X \times C_b \to X$ is continuous we may also define the operator $F: C_b^2 \to C$,

$$(Fu)(t) = h(t, u(t), u'(t) ; u), \quad t \in I.$$

Let u_b be the unique solution from C_b^2 to the equation $u'' = 0$ and define the operator $T: C_b^2 \to C_b^2$,

$$Tu = NFu + u_b, \; u \in C_b^2. \tag{4.4}$$

LEMMA 6. *If the mapping $NF: C_b^2 \to C^2$ is condensing and if there exists $\bar{M} > 0$ such that $\|u\|_2 < \bar{M}$ for any solution $u \in C_b^2$ to the equation*

$$u''(t) = \lambda h(t, u(t), u'(t) ; u), \qquad t \in I \tag{4.5}$$

and for all $\lambda \in [0, 1]$, then the problem (3.1), (3.2) *has at least one solution.*

Proof. A function u is a solution to problem (3.1), (3.2) if and only if

$$u(t) = -\int_a^b G(t, \xi)\, h(\xi, u(\xi), u'(\xi) ; u) d\xi + u_b(t), \quad t \in I$$

or, equivalently, if and only if it is a fixed point of T, i.e.

$$u = NFu + u_b. \tag{4.6}$$

Similarly, $u \in C_b^2$ is a solution to (4.5) if and only if

$$u = \lambda NFu + u_b. \tag{4.7}$$

Let $U = \{u \in C_b^2 : \|u\|_2 < \bar{M}\}$. Clearly C_b^2 is a convex closed subset of the Banach space C^2 and U is open in C_b^2. By Lemma 2 the mapping $H_0: \bar{U} \to C_b^2$, $H_0 u = u_b$ for all $u \in \bar{U}$, is essential. Also, if we define $H_\lambda: \bar{U} \to C_b^2$, $H_\lambda u = \lambda NFu + u_b$ we see that $H_\lambda \in \mathcal{a}(\bar{U} ; C_b^2)$ for all $\lambda \in [0, 1]$. Moreover, since NF is condensing and $\bar{U}$ is bounded we have that $NF(\bar{U})$ is a bounded subset of C^2 and in consequence the mapping $H(. ; u): [0, 1] \to C_b^2$ is continuous uniformly with respect to $u \in \bar{U}$. Thus H_0 and H_1 are homotopic and by Lemma 3 it follows that H_1 is also essential. Therefore $T(= H_1)$ has at least one fixed point, as desired.

Proof of Theorem 1. We will prove first that the mapping $F: C_b^2 \to C$ is (α, ρ) — Lipschitz.

First of all let us show that by (i) and (ii) we have

$$\alpha(h(I, A_1, A_2 ; A_3)) \leqslant \rho \max (\alpha(A_1), \alpha(A_2), \alpha_1(A_3)), \tag{4.8}$$

whenever A_1, A_2 are bounded in X and $A_3 \subset C_b^2$ is bounded in C^1. To this end, let $\varepsilon > 0$ be arbitrary fixed. Then, by the uniform continuity of h assumed in (i), it follows that for each $\bar{t} \in I$ there is a neighbourhood $V(\bar{t} ; \varepsilon)$ of $\bar{t}$ such that

$$|h(t, x, y ; z) - h(\bar{t}, x, y ; z)| < \varepsilon$$

for all $t \in V(\bar{t} ; \varepsilon)$, $x \in A_1, y \in A_2$ and $z \in A_3$. Consequently

$$\alpha(h(V(\bar{t} ; \varepsilon), A_1, A_2 ; A_3)) \leqslant \alpha(h(\bar{t}, A_1, A_2 ; A_3)) + 2\varepsilon.$$

This, by (3.4) and the compactness of I, yields

$$\alpha(h(I, A_1, A_2; A_3)) \leqslant \rho \max (\alpha(A_1), \alpha(A_2), \alpha_1(A_3)) + 2\varepsilon.$$

Now letting $\varepsilon \to 0$ we get (4.8) as desired.

The continuity of F follows easily by that of h.

Let D be an arbitrary bounded subset of C_b^2. If we apply (4.8) to $A_1 = D(I)$, $A_2 = D'(I)$ and $A_3 = D$ we see that $F(D)$ is bounded. Further we will show that

$$\alpha(F(D)) \leqslant \rho\alpha_2(D). \tag{4.9}$$

Since D is bounded in C^2, the sets D and D' are equicontinuous families of functions. Hence, by Lemma 1, we have

$$\alpha(D) = \sup (\alpha(D(t)) : t \in I), \ \alpha(D') = \sup (\alpha(D'(t)) : t \in I). \tag{4.10}$$

Moreover, the equicontinuity of D and D' together with the uniform continuity of h assumed in (i), implie that $F(D)$ is also an equicontinuous family of functions. Thus

$$\alpha(F(D)) = \sup (\alpha(F(D)(t)) : t \in I). \tag{4.11}$$

But, by (3.4), (2.3) and (2.4), we have

$$\alpha(F(D)(t)) = \alpha(\{h(t, u(t), u'(t) ; u) : u \in D\}) \leqslant \rho \max (\alpha(D(t)), \alpha(D'(t)), \alpha_1(D)) = \rho\alpha_1(D) \leqslant \rho\alpha_2(D),$$

for all $t \in I$. Whence, (4.9) follows by (4.11).

Therefore the mapping F is (α, ρ) — Lipschitz as claimed.

Further, by (4.3) and (4.9), we get

$$\alpha_2(NF(D)) \leqslant \rho \max ((b - a)^2/8, (b - a)/2, 1)\alpha_2(D),$$

whence we may claim that NF is condensing.

Now, according to Lemma 6, we have only to prove the boundedness in C^2 of the set of solutions to equation (4.5).

For each $\lambda \in]0, 1]$ the function λh satisfies the hypothesis of Lemma 4. Thus $\|u\| \leqslant M$ for any solution $u \in C_b^2$ to equation (4.5) and for all $\lambda \in]0,1]$. In addition, since $M \geqslant \max (|r|, |s|)$ we see that u_b, the unique solution in C_b^2 to equation (4.5) for $\lambda = 0$, also satisfies $\|u_b\| \leqslant M$.

Further, according to assumption (iv) and Lemma 5, there exists a constant M_1 such that $\|u'\| \leqslant M_1$ for any solution $u \in C_b^2$ to (4.5) and all $\lambda \in [0, 1]$.

Finally, if we put

$D = \{u \in C_b^2 : u$ is a solution to (4.5) for a certain $\lambda \in [0, 1]\}$

and we apply (4.8) to $A_1 = D(I)$, $A_2 = D'(I)$, $A_3 = D$, then we obtain that the set $D''(I)$ is bounded in X. Hence, there exists a constant M_2 such that $\|u''\| \leqslant M_2$ for any solution $u \in C_b^2$ to equation (4.5) and all $\lambda \in [0, 1]$. The proof of Theorem 1 is now complete.

5. **Uniqueness.** We will establish the uniqueness of solution to equation

$$u''(t) = h(t, u(t), u'(t)) + A(t, u), \quad t \in I, \tag{5.1}$$

together with the boundary conditions (3.2), where h maps $I \times X \times X$ into X and A maps $I \times C_b^2$ into X.

The uniqueness is established under some monotonicity conditions.

THEOREM 3. *Suppose that the following conditions are satisfied:*

(i) $$(x^*, h(t, x + x^1, y + y^1) - h(t, x^1, y^1)) > 0 \quad (\geqslant 0), \tag{5.2}$$

for all $t \in]a, b[$, $x, x^1 \in X$ *with* $x \neq 0$, $x^* \in \mathfrak{I}x$ *and all* $y, y^1 \in X$ *satisfying* $(x^*, y) = 0$.

(ii) $$(x^*, A(t_0, u_1) - A(t_0, u_2)) \geqslant 0 \quad (> 0), \tag{5.3}$$

for all $u_1, u_2 \in C_b^2$, $u_1 \neq u_2$, $t_0 \in]a, b[$ *such that* $\| u_1(t_0) - u_2(t_0) \| = \| u_1 - u_2 \|$ *and all* $x^* \in \mathfrak{I}(u_1(t_0) - u_2(t_0))$.

Then problem (5.1), (3.2) *has at most one solution* $u \in C_b^2$.

Proof. Let u_1 and u_2 be two solutions to (5.1), (3.2) and let $u = u_1 - u_2$. If $u_1 \neq u_2$ then it whould exist $t_0 \in]a, b[$ such that $\| u(t_0) \| = \| u \| > 0$. This would imply that $(x^*, u'(t_0)) = 0$ and

$$(x^*, u''(t_0)) \leqslant 0, \tag{5.4}$$

for all $x^* \in \mathfrak{I}u(t_0)$. On the other hand, by (5.2) and (5.3) we should have

$$(x^*, u''(t_0)) = (x^*, h(t_0, u_1(t_0), u_1'(t_0)) - h(t_0, u_2(t_0), u_2'(t_0))) + \\ + (x^*, A(t_0, u_1) - A(t_0, u_2)) > 0,$$

which would contradict (5.4). Thus $u_1 = u_2$ and the theorem is proved.

As an application we will establish the uniqueness of solution to problem (1.1), (1.2) in the particular case when equation (1.1) has the form

$$u''(t) = h(t, u(t), u'(t)) + q(t)\, u(t) + \sum_{i=1}^{m} q_i(t)\, u(g_i(t)), \tag{5.5}$$

where q and q_i, $i = 1, \ldots, m$ are real functions defined on I.

COROLLARY 3. *Suppose that h satisfies condition* (i) *from Theorem 2 and*

$$q_i(t) \leqslant 0, \quad q(t) + \sum_{i=1}^{m} q_i(t) \geqslant 0 \quad (> 0), \tag{5.6}$$

for all $t \in]a, b[$.

Then problem (5.5), (1.2) *has at most one solution.*

Proof. Apply Theorem 3 to

$$A(t, u) = q(t)\, u(t) + \sum_{i=1}^{m} q_i(t)\, u^i(t).$$

Note that Theorem 3 and Corollary 3 generalize Theorem 6 and Theorem 7 in [15].

REFERENCES

1. Ahmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E., Sadovskii, B.N., *Measure of noncompactness and condensing operators* (Russian), Novosibirsk, Nauka, 1986.
2. Ambrosetti, A., *Un theorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova, 39 (1967), 349–361.
3. Bernfeld, S. R., Lakshmikantham, V., *An introduction to nonlinear boundary value problems*, Academic Press, New York, London, 1974.
4. Bernstein, S. N., *Sur les équations du calcul des variations*, Ann. Sci. Ecole Norm. Sup. 29 (1912), 431–485.
5. Chandra, J., Lakshmikantham, V., Mitchell, A., *Existence of solutions of boundary value problems for nonlinear second order systems in a Banach space*, J. Nonlinear Analysis 2 (1978), 157–168.
6. Dugundji, J., Granas, A., *Fixed point theory*, vol. 1., Warszawa, 1982.
7. Granas, A., *Sur la méthode de continuité de Poincaré*, C. R. Acad. Sci., Paris 282 (1976), 983–985.
8. Granas, A., Guenther, R., Lee, J., *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Mathematicae, CCXIV, Warszawa, 1985.
9. Hartman, P., *On boundary value problem for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. 96 (1960), 493–509.
10. Hartman, P., *Ordinary differential equations*, John Wiley & Sons, Inc., New York, 1964.
11. Lakshmikantham, V., *Abstract boundary value problems*, Nonlinear Equations in Abstract Spaces (Ed. V. Lakshmikantham), Academic Press, New York, San Francisco, London, 1978, 117–123.
12. Lasota, A., Yorke, J. A., *Existence of solutions of two-point boundary value problems for nonlinear systems*, J. Differential Equations 11 (1972), 509–518.
13. Mawhin, J., *Nonlinear boundary value problems for ordinary differential equations from Schauder theorem to stable homotopy*, Université Catholique de Louvain, Rapport No. 86 (1976).
14. Precup, R., *Nonlinear boundary value problems for infinite systems of second-order functional differential equations*, Univ. Babeş-Bolyai, Cluj-Napoca, Preprint Nr. 8, 1988, 17–30.
15. Rus, A. I., *Maximum principles for some nonlinear differential equations with deviating arguments*, Studia Univ. „Babeş-Bolyai" 32, No. 2 (1987), 53–57.
16. Scorza-Dragoni, G., *Sul problema dei valori ai limiti per i sistemi di equazioni diferenziali del secondo ordine*, Bull. U.M.I. 14 (1935), 225–230.
17. Schmitt, K., Thompson, R., *Boundary value problems for infinite systems of second-order differential equations*, J. Differential Equations 18 (1975), 277–295.
18. Thompson, R. C., *Differential inequalities for infinite second order systems and an application to the method of lines*, J. Differential Equations 17 (1975), 421–434.