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GENERALIZED TOPOLOGICAL TRANSVERSALITY AND MAPPINGS OF MONOTONE TYPE

RADU PRECUP*

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REZUMAT. — Transversalitate topologică generalizată şi aplicaţii de tip monoton. În lucrare se demonstrează o teoremă de existenţă de tip Browder [2]. Noutatea constă în faptul că în locul condiţiei de coercivitate se impune o condiţie de semn, mai generală. Demonstraţia se bazează pe teorema de transversalitate topologică generalizată, obţinută în [4]. Această notă constituie un addendum la lucrările [4] şi [5].

In this paper a Browder's type result [2] is proved by using our generalized topological transversality theorem given in [4] (see also [5]). We show that the coercivity condition assumed by Browder can be replaced by a more general sign condition. This note is an addendum to our previous papers [4] and [5].

1. **The generalized topological transversality principle.** Let K be a normal topological space, X and A two proper closed subsets of K, $A \subset X$, $A \neq X$ and consider a nonvoid class of mappings.

$$\mathcal{A}_A(X, K) \subset \{f: X \to K\,;\ \mathrm{Fix}\,(f) \cap A = \emptyset\}, \tag{1}$$

where $\mathrm{Fix}(f)$ stands for the set of all fixed points of f. The mappings in $\mathcal{A}_A(X,K)$ are said to be *admissible.*

An admissible mapping f is said to be *essential* if

$$f' \in \mathcal{A}_A(X, K),\ f|_A = f'|_A \text{ imply } \mathrm{Fix}(f') \neq \emptyset. \tag{2}$$

Otherwise, f is said to be *inessential.*

Also consider an equivalence relation $\sim$ on $\mathcal{A}_A(X, K)$ and assume that the following conditions are satisfied for f and f' in $\mathcal{A}_A(X, K)$:

(a) *if* $f|_A = f'|_A$ *then* $f \sim f'$;

(h) *if* $f \sim f'$ *then there is* $h: [0, 1] \times X \to K$ *such that* $h(0, .) = f'$, $h(1, .) = f$, $\mathrm{cl}(\bigcup\{\mathrm{Fix}(h(t, .))\,;\ t \in [0, 1]\}) \cap A = \emptyset$ *and* $h(\eta(.), .)$ *is admissible for any* $\eta \in C(X\,;\ [0, 1])$ *satisfying* $\eta(x) = 1$ *for all* $x \in A$.

We now state the generalized topological transversality theorem.

PROPOSITION 1. *If f and f' are admissible mappings and $f \sim f'$, then f and f' are both essential or both inessential.*

* *University of Cluj-Napoca, Department of Mathematics, 3400 Cluj-Napoca, Romania*

The next proposition is useful in order to establish the essentiality of certain admissible mappings. It is formulated in terms of fixed point structures.

By a *fixed point structure* on a certain space K we mean a pair (S, M) where S is a class of nonempty subsets of K and M is a mapping attaching to each $D \in S$ a family $M(D)$ of mappings from D into D having, each of them, at least one fixed point.

PROPOSITION 2. *Let (S, M) be a fixed point structure on the normal topological space K and let $f_0 \in \mathcal{A}_A(X, K)$. If for every $f \in \mathcal{A}_A(X, K)$ satisfying $f|_A = f_0|_A$, there exist $D_f \in S$ and $\tilde{f} \in M(D_f)$ such that*

$$f|_{X \cap D_f} = \tilde{f}|_{X \cap D_f}$$

and

$$\mathrm{Fix}\,(\tilde{f}) \setminus X = \emptyset,$$

then f_0 is essential.

The proofs of Proposition 1 and Proposition 2 and some applications can be found in the papers 4] and [5].

The aim of this paper is to give another application of Proposition 1.

2. **The fixed point structure.** Now we describe the fixed point structure which will be used in the next section.

Let E be a real reflexive Banach space which is normed so that E and its dual E^* are locally uniformly convex and let $J : E \to E^*$ be the duality mapping. Set

$$S = \{D\,;\ S \text{ is a nonvoid bounded closed convex subset of } E\} \tag{3}$$

and for each $D \in S$,

$$M(D) = \{(J + T)^{-1}(J - N) : D \to D(T)\,;\ T \subset D \times E^* \text{ is maximal monotone in } E \times E^* \text{ and } N : D \to E^* \text{ is pseudomonotone, bounded and demicontinuous}\}. \tag{4}$$

Recall that a mapping $N : D \to E^*$ is said to be *pseudomonotone* if, for any sequence (x_n) in D for which $x_n \rightharpoonup x$ and $\limsup \langle N(x_n), x_n - x\rangle \leqslant 0$, we have $\langle N(x), x - y\rangle \leqslant \liminf \langle N(x_n), x_n - y\rangle$ for all $y \in D$. Also, N is said to be of *type* (S_+) if for any sequence (x_n) in D for which $x_n \rightharpoonup x$ and $\limsup \langle N(x_n), x_n - x\rangle \leqslant 0$, it follows $x_n \to x$.

LEMMA 1. *The pair (S, M) given by (3) and (4) is a fixed point structure on E.*

This statement is just a Browder's result [2] (see also [6, Theorem 32 A]). Nevertheless, we will insert here its proof.

Proof of Lemma 1. We have to show that each mapping in $M(D)$ has at least one fixed point, i.e., there exists at least one solution to

$$x_0 \in D(T),\ 0 \in N(x_0) + T(x_0). \tag{5}$$

We will first solve (5) under the assumption that N is of type (S_+): In view of the maximal monotonicity of T in $E \times E^*$, (6) is equivalent to:

$$x_0 \in D, \ \langle x^* + N(x_0), \ x - x_0 \rangle \geqslant 0 \tag{6}$$

for all $(x, x^*) \in T$.
For any finite — dimensional subspace Y of E with $Y \cap D \neq \emptyset$, we look for a solution y to

$$y \in Y \cap D, \ \langle x^* + N(y), \ x - y \rangle \geqslant 0 \tag{7}$$

for all $(x, x^*) \in T$ with $x \in Y$.
Since N is demicontinuous, a solution to (7) exists in view of Debrunner-Flor's lemma (see [6, Proposition 2.17]). Thus, the set

$$V_Y = \{(y, \ -N(y)) \in D \times E^*; \ \langle x^* + N(y), \ x - y \rangle \geqslant 0 \ \text{ for all } (x, x^*) \in T \text{ with } x \in Y\}$$

is nonempty. Clearly, the family $\{V_Y\}$ has the finite intersection property; thus the family of weak-compact sets $\{\text{w}-\text{cl}(V_Y)\}$, $Y \cap D \neq \emptyset$, has a nonvoid intersection. Let (x_0, x_0^*) an element of its intersection.

Note that, due to the maximal monotonicity of T, there exists $(z_0, z_0^*) \in T$ such that

$$\langle z_0^* - x_0^*, \ z_0 - x_0 \rangle \leqslant 0. \tag{8}$$

Now, for an arbitrary $(x, x^*) \in T$, we choose Y such that x, x_0 and z_0 belong to Y and we take a sequence $(y_n, \ -N(y_n))$ in V_Y such that $y_n \rightharpoonup x_0$ and $-N(y_n) \rightharpoonup x_0^*$. We have

$$\langle z^* + N(y_n), \ z - y_n \rangle \geqslant 0, \tag{9}$$

for all $(z, z^*) \in T$ with $z \in Y$.
From (9) we get

$$\begin{aligned} \langle N(y_n), \ y_n - w \rangle &= \langle N(y_n), \ y_n - z \rangle + \langle N(y_n), \ z - w \rangle \leqslant \\ &\leqslant \langle z^*, \ z - y_n \rangle + \langle N(y_n), \ z - w \rangle \end{aligned} \tag{10}$$

for all $(z, z^*) \in T$ with $z \in Y$ and $w \in E$.
Taking $w = x_0$, $z = z_0$, $z^* = z_0^*$ we obtain

$$\langle N(y_n), \ y_n - x_0) \rangle \leqslant \langle z_0^*, \ z_0 - y_n \rangle + \langle N(y_n), \ z_0 - x_0 \rangle,$$

whence, letting $n \to \infty$ and taking into account (8), we get

$$\limsup \langle N(y_n), \ y_n - x_0 \rangle \leqslant 0.$$

This, since N is supposed of type (S_+), implies that $y_n \to x_0$. Consequently, $x_0^* = -N(x_0)$ and passing to limit in (10) with $w = z = x$ and $z^* = x^*$ we obtain just (6). This proves the solvability of (5) in case N is of type (S_+).

Finally, for N pseudomonotone, use the fact that $N + \varepsilon J$ is of type (S_+) for each $\varepsilon > 0$, in order to deduce the existence of an y_ε, solution to

$$0 \in N(x_\varepsilon) + \varepsilon J(x_\varepsilon) + T(x_\varepsilon)$$

and letting $\varepsilon \to 0$, find a solution to (5). This step is well known and we omit the details. The lemma is thus proved.

The existence of a solution to (5) is known even if D is unbounded, but under the additional hypothesis that N is coercive with respect to 0 (see [2, p. 92] or [6, Theorem 32. A]). In what follows we shall prove, via Proposition 1, that the coercivity of N may be replaced by a more general sign condition.

3. **Application of the generalized transversality theorem.** The main result of this note is the following proposition.

THEOREM 1. *Let E be a real reflexive Banach space, K an unbounded closed convex subset of E, $T \subset K \times E^*$ maximal monotone in $E \times E^*$ with $(0, 0) \in T$ and let $N : K \to E^*$ be a bounded demicontinuous pseudomonotone mapping such that there exists $r > 0$ so that*

$$\langle N(x), x\rangle \geqslant 0 \text{ for all } x \in K \text{ with } ||x|| = r. \tag{11}$$

Then there exists $x \in D(T)$ a solution to

$$0 \in N(x) + T(x).$$

Remark. Condition (11) is less restrictive than the coercivity condition:

$$\langle N(x), x\rangle > 0 \text{ for all } x \in K \text{ with } ||x|| \geqslant r.$$

Under the coercivity condition on N, Theorem 1 was proved in [2, p. 92].

Proof of Theorem 1. The same argument as in the proof of Lemma 1, allows us, setting $N + \varepsilon J (\varepsilon > 0)$ in place of N, to assume that N is of type (S_+) and in addition, that the inequality in (11) is strict.

We shall succed two steps:

1) *Application of Proposition* 1. Consider the class

$$\mathcal{A}_A(K_r, K) = \{(J + T)^{-1} \circ \eta_\lambda (J - N) : K_r \to D(T) \ ; \ \eta_\lambda \in C(K_r; [0,1]), \ \eta_\lambda(x) = \lambda \text{ for } x \in A\}$$

where $A = \{x \in K ; \ ||x|| = r\}$ and for each $R > 0$ we denote

$$K_R = \{x \in K ; \ ||x|| \leqslant R\}.$$

Note that the mappings in $\mathcal{A}_A(K_r, K)$ can not have fixed points in A because, in view of (11), the inclusion

$$(\lambda - 1) J(x) - \lambda N(x) \in T(x)$$

is false for all $x \in A$.

Also define an equivalence relation on $\mathcal{A}_A(K_r, K)$ by setting

$$(J + T)^{-1} \circ \eta_\lambda (J - N) \sim (J + T)^{-1} \circ \eta_{\lambda'} (J - N)$$

if and only if

$$\lambda = \lambda' \text{ or } \{\lambda, \lambda'\} = \{0, 1\}, \text{ in case } J - N \not\equiv 0 \text{ on } A$$
$$\text{always, in case } J - N \equiv 0 \text{ on } A.$$

Since $(J+T)^{-1}$ is one-to-one, condition (a) is satisfied. In order to ver condition (h), set

$$h(t,\ .) = (J+T)^{-1} \circ [(1-t)\eta_{\lambda'} + t\eta_\lambda](J-N).$$

Clearly, $h(\eta(.),\ .) \in \mathcal{A}_A(K_r,\ K)$ for each $\eta \in C(K_r;\ [0,\ 1])$ satisfying $\eta(x) =$ for all $x \in A$. Also, by (11), the sets A and $Z = \cup\{\mathrm{Fix}\,(h(t,\ \cdot))\,;\ t \in [0,\ 1$ are disjoint. It remains only to show that Z is closed. For this, let (x_n) be sequence in Z such that $x_n \to x_0$. We have $h(t_n,\ x_n) = x_n$ for some $t_n \in [0,\ 1$ We may assume $t_n \to t_0$. Setting

$$\mu_n = (1-t_n)\eta_{\lambda'}(x_n) + t_n\eta_\lambda(x_n) \text{ and } \mu_0 = (1-t_0)\eta_{\lambda'}(x_0) + t_0\eta_\lambda(x_0),$$

we thus have

$$\langle -(1-\mu_n)J(x_n) - \mu_n N(x_n) - x^*,\ x_n - x\rangle \geqslant 0$$

for all $(x,\ x^*) \in T$. Letting $n \to \infty$ and using the demicontinuity of J and we get

$$\langle -(1-\mu_0)J(x_0) - \mu_0 N(x_0) - x^*,\ x_0 - x\rangle \geqslant 0$$

for all $(x,\ x^*) \in T$, i.e., $h(t_0,\ x_0) = x_0$, as desired.

Therefore, Proposition 1 can be applied. But

$$(J+T)^{-1}(J-N) \sim (J+T)^{-1} \circ 0(J-N) \equiv 0.$$

Hence, in order that the mapping $(J+T)^{-1}(J-N)$ have a fixed point, it i sufficient to prove that the null operator is essential in $\mathcal{A}_A(K_r,\ K)$.

2) *Use of the fixed point structure.* We shall now prove that the null operator is essential, i.e., each mapping

$$f = (J+T)^{-1} \circ \eta_\lambda(J-N)$$

satisfying $f \equiv 0$ on A, has at least one fixed point. Remark that if $J - N \not\equiv 0$ on A, then λ must be zero, while if $J - N \equiv 0$ on A, then λ is any number in $[0,\ 1]$. To do this, for any fixed $R \geqslant r$ we consider the mapping

$$f_R = (J+T_R)^{-1} \circ \tilde\eta(J-N)\ :\ K_R \to K_R,$$

where $T_R \subset K_R \times E^*$ is maximal monotone in $E \times E^*$ and $T|_{K_R} \subset T_R$ (see [1, Theorem 1.4])
and

$$\tilde\eta(x) = \begin{cases} \eta_\lambda(x), & \text{if } x \in K_r \\ 0, & \text{if } x \in K_R \setminus K_r. \end{cases} \qquad (12)$$

Clearly, $K_R \in S$. We shall prove that $f_R \in M(K_R)$, i.e., the mapping

$$\tilde N\ :\ K_R \to E^*,\ \tilde N = J + \tilde\eta(N - J) \qquad (13)$$

is pseudomonotone, bounded and demicontinuous. The last two properties are immediate. To prove its pseudomonotonicity, consider any sequence (x_n) in K_R such that $x_n \rightharpoonup x$ and

$$\limsup \langle N(x_n),\ x_n - x\rangle \leqslant 0. \tag{14}$$

According to (13), we have

$$\min\{\langle J(x_n),\ x_n - x\rangle, \langle N(x_n),\ x_n - x\rangle\} \leqslant \langle \tilde{N}(x_n),\ x_n - x\rangle. \tag{15}$$

Now, from (14) and (15) and since J and N are both of type (S_+), it easily follows that $x_n \to x$. Hence, $\tilde{N}$ is of type (S_+) and since $\tilde{N}$ is also demicontinuous, it follows that $\tilde{N}$ is pseudomonotone (see [6, Proposition 27.6]). Therefore, $f_R \in M(K_R)$ and according to Lemma 1, there exists a fixed point $x_R \in K_R$ for f_R. Moreover, by (12), $x_R \in K_r$. Since $f_R(x_R) = x_R$,

$$\langle z^* + \tilde{N}(x_R),\ z - x_R\rangle \geqslant 0 \tag{16}$$

for all $(z, z^*) \in T_R$ and in particular, for all $(z, z^*) \in T$, with $z \in K_R$. Now let (R_n) be an increasing sequence such that $R_n \to \infty$ and denote $x_n = x_{R_n}$. We may assume

$$x_n \rightharpoonup x_0 \in K \text{ and } \tilde{N}(x_n) \rightharpoonup x_0^* \in E^*.$$

Choose a pair $(z_0, z_0^*) \in T$ such that

$$\langle z_0^* + x_0^*,\ z_0 - x_0\rangle \leqslant 0. \tag{17}$$

Now for an arbitrary pair $(x, x^*) \in T$, there is n_0 such that $x_0, z_0, x \in K_{R_n}$ for all $n \geqslant n_0$. From (16), we get

$$\langle \tilde{N}(x_n),\ x_n - w\rangle \leqslant \langle z^*,\ z - x_n\rangle + \langle \tilde{N}(x_n),\ z - w\rangle \tag{18}$$

for all $(z, z^*) \in T$ with $z \in K_{R_n}$ and $w \in E$.
Taking $w = x_0$, $z = z_0$, $z^* = z_0^*$, letting $n \to \infty$ and using (17), we get

$$\limsup \langle \tilde{N}(x_n),\ x_n - x_0\rangle \leqslant 0$$

whence, since $\tilde{N}$ is of type (S_+), $x_n \to x_0$ and
$x_0^* = \tilde{N}(x_0)$. Clearly, $x_0 \in K_r$ and
$\tilde{N}(x_0) = J(x_0) + \eta_\lambda(x_0)(N(x_0) - J(x_0))$. Finally, passing to limite in (18) with $w = z = x$ and $z^* = x^*$, we obtain

$$\langle x^* + J(x_0) + \eta_\lambda(x_0)(N(x_0) - J(x_0)),\ x - x_0\rangle \geqslant 0.$$

Consequently, x_0 is a fixed point of f and the proof is complete.

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