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GENERALIZED TOPOLOGICAL TRANSVERSALITY AND EXISTENCE THEOREMS

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§1. Introduction

In this paper Granas' topological transversality theorem on compact mappings is generalized in order to obtain, in an unitary manner, several known existence results concerning mappings of a number of classes as: compact, condensing, of class $(S)_+$, or mappings of the form $T+f$ with T maximal monotone and f of class $(S)_+$. No references to degree theory are made. Although the mappings in our examples are all single-valued, the theory applies to multivalued mappings, too. Example 3 is notable for the manner in which the topological transversality theorem is applied to a minimal class of admissible mappings attached to a fixed mapping of a certain type.

§2. Generalized topological transversality

Let X be a normal topological space, A a proper closed subset of X, Y a set and B a proper subset of Y. Consider a nonempty class of mappings

$$\mathcal{A}_A^B(X,Y) \subset \{F: X \to Y; F^{-1}(b) \cap A = \emptyset\}.$$

The mappings of this class are called <u>admissible</u>. An admissible mapping F is said to be <u>essential</u> if for each admissible mapping F' having the same restriction to A as F, i.e. $F'|_A = F|_A$, one has $F'^{-1}(B) \neq \emptyset$. An admissible mapping which is not essential is called <u>inessential</u>.

Let us denote by $\sim$ an equivalence relation on $\mathcal{A}_A^B(X,Y)$ such that

(A) if $F'|_A = F|_A$ then $F' \sim F$.

We are interested in the case when the equivalence classes contain either only essential mappings or only inessential mappings. The following condition will be sufficient to have such a case:

(H) If $F' \sim F$ then there is $H:[0,1] \times X \to Y$ such that $H(0,.) = F'$, $H(1,.) = F$, $cl(\cup\{H(t,.)^{-1}(B); t \in [0,1]\}) \cup A = \emptyset$ and $H(\eta(.),.) \in \mathcal{A}_A^B(X,Y)$ for any continuous $\eta: X \to [0,1]$ satisfying $\eta(x) = 1$ for all $x \in A$.

In what follows we shall assume that conditions (A) and (H) are satisfied.

LEMMA 1. An admissible mapping F is inessential if and only if there exists an admissible mapping F' such that $F \sim F'$ and $F'^{-1}(B) = \emptyset$.

Proof. Our arguments are similar with those in [4].

The necessity part is a consequence of the definition of inessential mappings and of condition (A).

Conversely, let F' be an admissible mapping such that $F \sim F'$ and $F'^{-1}(B) = \emptyset$. Then, by the symmetry of relation $\sim$, we have $F' \sim F$. Now let H be a mapping associated with F' and F as in condition (H). Denote $Z = \cup\{H(t,.)^{-1}(B);\ t \in [0,1]\}$. If $Z = \emptyset$, then $H(1,.)^{-1}(B) = \emptyset$ and so $F = H(1,.)$ is essential. Thus, we may assume without loss of generality that $Z \neq \emptyset$. By condition (H), the nonempty closed subsets A and $cl(Z)$ of the normal topological space X, are disjoint. Consequently, according to Urysohn's theorem (see [3], Theorem VII.4.1), there is a continuous function $\eta: X \to [0,1]$ such that $\eta(x) = 0$ for all $x \in cl(Z)$ and $\eta(x) = 1$ for all $x \in A$. By condition (H), the mapping $F^* = H(\eta(.),.)$ is admissible. Moreover, it is easily seen that $F^*|_A = F|_A$ and $F^{*-1}(b) = \emptyset$. This shows that F is inessential. The proof is complete.

Now we state the topological transversality theorem under our general assumptions.

THEOREM 1. Let F and F' be two admissible mappings such that $F \sim F'$. Then F and F' are both essential or both inessential.

<u>Proof</u>. Assume that F is inessential. Then, by Lemma 1, there exists an admissible mapping F'' such that $F \sim F''$ and $F''^{-1}(B) = \emptyset$. By the transitivity of $\sim$ we get $F' \sim F''$ whence, again by Lemma 1, it follows that F' is inessential too and the proof is complete.

§3. Topological transversality and fixed point theory

Let E be a normal topological space, X and A two proper closed subsets of E such that $A \subset X$, $A \neq X$. Consider a nonempty class of mappings.

$$\mathcal{A}_A(X,E) \subset \{f : X \to E;\ \mathrm{Fix}(f) \cap A = \emptyset\},$$

where we have denoted by $\mathrm{Fix}(f)$ the set of all fixed points of f. The mappings in $\mathcal{A}_A(X,E)$ are called admissible. An admissible mapping f is said to be essential if for each admissible mapping f' with $f'|_A = f|_A$ one has $\mathrm{Fix}(f') \neq \emptyset$. Otherwise, f is called inessential.

Let us consider an equivalence relation $\sim$ on $\mathcal{A}_A(X,E)$ which is assumed to satisfy the following two conditions:

(a) $\qquad$ if $f'|_A = f|_A$ then $f' \sim f$

and

(b) if $f' \sim f$ then there is $h : [0,1] \times X \to E$ such that $h(0,.) = f'$, $h(1,.) = f$, $cl(\cup \{\mathrm{Fix}(h(t,.));\ t \in [0,1]\}) \cap A = \emptyset$ and $h(\eta(.),.)$ is admissible for any continuous function $\eta : X \to [0,1]$ satisfying $\eta(x) = 1$ for all $x \in A$.

The following theorem is a consequence of Theorem 1.

PROPOSITION 1. <u>Let f and f' be two mappings in $\mathcal{A}_A(X,E)$ such that $f \sim f'$. Then f and f' are both essential or both inessential.</u>

<u>Proof</u>. Take $Y = X \times E$, $B = \{(x,x) : x \in X\}$, $\mathcal{A}_A{}^B(X,Y) = \{F : X \to Y;$ there is $f \in \mathcal{A}_A(X,E)$ such that $F(x) = (x, f(x))$ for all $x \in X\}$ and say that $F \sim F'$ if $f \sim f'$, where $F(x) = (x, f(x))$ and $F'(x) = (x, f'(x))$. Obviously F is essential (inessential) if and only if its associated f is essential (inessential). Also, condition (a) is

equivalent with condition (A). Moreover, condition (h) on $\sim$ implies condition (H) as follows immediately if we put $H(t,x)=(x,h(t,x))$ for all $t\in[0,1]$ and $x\in X$. Thus, we may apply Theorem 1 and Proposition 1 is proved.

Remark 1. Another proof of Proposition 1 can be done if, in addition, E is a real linear space. In this case, we may take $Y=E$, $B=\{0\}$, $\mathcal{A}_A^B(X,Y)=\{F:X\to E:F(x)=x-f(x)$ for all $x\in X$, $f\in\mathcal{A}_A(X,E)\}$, define $F\sim F'$ if $f\sim f'$, where $F(x)=x-f(x)$ and $F'(x)=x-f'(x)$ and put $H(t,x)=x-h(t,x)$, in order to apply Theorem 1.

Remark 2. A similar result to Proposition 1 can be stated, more generally, for multivalued mappings from X into $P(E)$ (the family of all nonempty subsets of E) in a class of admissible mappings

$$\mathcal{A}_A(X,P(E))\subset\{f:X\to P(E);\ \mathrm{Fix}(f)\cap A=\emptyset\}.$$

In this case, the mapping h in condition (h) is multivalued. In the proof, we take $Y=X\times P(E)$, $B=\{(x,E_1);\ x\in X, E_1\in P(E), x\in E_1\}$ and we apply Theorem 1, again.

A result as that in Proposition 1 is useful in applications expecially when typical examples of essential mappings are known. In the following we shall propose a test of essentiality in terms of fixed point structures.

Here, by a fixed point structure (see [12]) on a certain space E we mean a pair (S,M), where S is a class of nonempty subsets of E $(S\subset P(E))$ and M is a map which attaches to each $D\in S$ a family $M(D)$ of mappings from D into D having at least one fixed point. Obviously, in this respect we may speak about a fixed point structure on a space E whenever we refer to a fixed point theorem for a certain class of mappings which map a subset of a certain kind into itself. Thus, by Schauder's fixed point structure on a Banach space E, we mean the pair (S,M), where S is the class of all nonempty, bounded closed convex subsets of E and for each $D\in S$, one considers the family $M(D)$ of all completely continuous mappings from D into itself; by Sadovskii's fixed point structure on a Banach space E, we mean the pair (S,M), where S is also the class of all nonempty bounded closed convex subsets of E and for each $D\in S$, $M(D)$ is the set of all γ-condensing mappings from D into itself. Here we have denoted by γ, Kuratowski's or the ball measure of noncompactness on E (see [2], 2.7.3, 2.9.1 and 2.9.3). We shall also use Mönchs fixed point structure on a Banach space E. Here, S is the class of all nonempty closed convex subsets of E and for each $D\in S$, $M(D)$ is the set of all continuous mappings $f:D\to D$ for which there is some $x_0\in D$ such that if $C\subset D$ is countable and $C\subset \overline{\text{conv}}(\{x_0\}\cap f(C))$ then $\overline{C}$ is compact (see [2], 5.18.2).

PROPOSITION 2. *Let (S, M) be a fixed point structure on the normal topological space E and let $f_0\in \mathcal{A}_A(X,E)$. If for every $g\in \mathcal{A}_A(X,E)$ satisfying $g|_A=f_0|_A$ there exist $D_g\in S$ and $\tilde g\in M(D_g)$ such that*

$$\tilde g|_{X\cap D_g}=g|_{X\cap D_g} \tag{1}$$

and

$$\text{Fix}(\tilde g)\setminus X=\emptyset, \tag{2}$$

then f_0 is essential.

Proof. Since $\tilde g\in M(D_g)$, there exists an $x\in D_g$ such that $\tilde g(x)=x$. Then, by (2), $x\in X$. Hence $x\in X\cap D_g$ and by (1) we have $g(x)=x$. Thus, each $g\in \mathcal{A}_A(X,E)$ satisfying $g|_A=f_0|_A$ has a fixed point, which proves that f_0 is essential.

4. *Examples and applications.* In this section we shall give some examples of classes of admissible mappings endowed with a prescribed equivalence relation satisfying conditions (a) and (b) and some known theorems of Leray-Schauder type will be derived from our Proposition 1, via Proposition 2, as applications.

1) *Topological transversality theorem for compact mappings.* A special

case of our Proposition 1 is a result of A. Granas (see [4] which was at the origin of our theory).

Let E be a Banach space, Ω a nonempty open subset of E, $X=\overline{\Omega}$ the closure of Ω and $A=\partial\Omega$ its boundary. We deal here with compact mappings $f:\overline{\Omega}\to E$, i.e. continuous mappings having $cl(f(\overline{\Omega}))$ compact in E. Let us denote

$$\mathcal{A}_{\partial\Omega}(\overline{\Omega};E)=\{f:\overline{\Omega}\to E;\ f \text{ compact and } \mathrm{Fix}(f)\cap\partial\Omega=\emptyset\}, \tag{3}$$

and consider the equivalence relation $\sim$ on $\mathcal{A}_{\partial\Omega}(\overline{\Omega};X)$:

(h_1) $f'\sim f$ iff there is a compact $h:[0,1]\times\overline{\Omega}\to E$ such that $h(0,.)=f'$, $h(1,.)=f$ and $\mathrm{Fix}(h(t,.))\cap\partial\omega=\emptyset$ for all $t\in[0,1]$.

It is easily seen that condition (a) is satisfied (if $f'|_{\partial\Omega}=f|_{\partial\Omega}$ we take $H(t,.)=(1-t)f'+tf)$ and also that condition (h_1) implies (h).

With this choice of admissible mappings and of the equivalence relation $\sim$, Proposition 1 is the topological transversality theorem itself which is given by A. Granas in [4], Theorem II.4.7.

Note that the constant mapping $f_o(x)=x_o$ for $x\in\overline{\Omega}$, where $x_o\in\Omega$, is essential. This follows from Proposition 2, where we use Schauder's fixed point structure on E and we take $D_g=\overline{\mathrm{conv}}(g(\overline{\Omega}))$, $\bar{g}(x)=g(x)$ if $x\in\overline{\Omega}$ and $\bar{g}(x)=x_o$ if $x\notin\overline{\Omega}$. (See also [4], Theorem II.4.9.)

As an application, we mention a Leray-Schauder's result:

COROLLARY 1. <u>Let E be a Banach space, Ω an open subset of E and $f:\overline{\Omega}\to E$ a compact mapping. If for some $x_o\in\Omega$ one has</u>

$$t(f(x)-x_o)\neq x-x_o \ \underline{\text{for all}}\ t\in\,]0,1]\ \underline{\text{and}}\ x\in\partial\Omega, \tag{4}$$

<u>then f has at least one fixed point in Ω.</u>

<u>Proof</u>. By (4), the essential constant mapping $f_o(x)=x_o$ for $x\in\overline{\Omega}$, is equivalent with f in the sense of (h_1) (use $h(t,.)=(1-t)x_o+tf)$. Thus, by

Proposition 1, f is also essential in class of mappings (3) and in particular $\mathrm{Fix}(f) \neq \emptyset$.

2) Topological transversality theorem for condensing mappings. Let E be a Banach space, Ω a nonempty open bounded subset of E, $X = \bar{\Omega}$ and $A = \partial\Omega$. We shall consider γ-condensed mappings $f:\bar{\Omega} \to E$, i.e. continuous mappings for which $\gamma(f(B)) < \gamma(B)$ whenever $B \subset \bar{\Omega}$ and $\gamma(B) > 0$. Denote

$$\mathcal{A}_{\partial\Omega}(\bar{\Omega}, E) = \{f:\bar{\Omega} \to E;\ f\ \gamma\text{-condensing and } \mathrm{Fix}(f) \cap \partial\Omega = \emptyset\} \tag{5}$$

and consider an equivalence relation $\sim$ on $\mathcal{A}_{\partial\Omega}(\bar{\Omega}, E)$, namely

(h_2) $f' \sim f$ iff there is a mapping $h:[0,1] \times \bar{\Omega} \to E$ such that $h(0,.) = f'$, $h(1,.) = f$, $h(t,.) \in \mathcal{A}_{\partial\Omega}(\bar{\Omega}, E)$ for all $t \in]0,1[$ and $\{h(.,x); x \in \bar{\Omega}\}$ is equicontinuous.

If $f', f \in \mathcal{A}_{\partial\Omega}(\bar{\Omega}, E)$ and $f'|_{\partial\Omega} = f|_{\partial\Omega}$ then taking $h(t,.) = (1-t)f' + tf$ we immediately see that $f' \sim f$. Thus, condition (a) is satisfied.

In order to prove that condition (h_2) implies (h), let us consider two admissible mappings f' and f such that $f' \sim f$ and a mapping h as in condition (h_2). Since $h(t,.) \in \mathcal{A}_{\partial\Omega}(\bar{\Omega}, E)$ for all $t \in [0,1]$, we have that $Z = \cup\{\mathrm{Fix}(h(t,.)); t \in [0,1]\} \subset \Omega$. In addition, the equicontinuity of the set $\{h(.,x); x \in \bar{\Omega}\}$ implies that Z is closed. Hence $cl(\cup\{\mathrm{Fix}(h(t,.));\ t \in [0,1]\}) \cup \partial\Omega = \emptyset$. Thus, we have to show only that $h(\eta(.),.)$ is admissible for every continuous $\eta: X \to [0,1]$ with $\eta(x) = 1$ for all $x \in \partial\Omega$. Obviously, $h(\eta(.),.)$ is continous and since $\mathrm{Fix}(f) \cap \partial\Omega = \emptyset$, one has $\mathrm{Fix}(h(\eta(.),.)) \cap \partial\Omega = \emptyset$. Now let $B \subset \bar{\Omega}$ be such that $\gamma(B) > 0$. We wish to show that $\gamma(h(\eta(B) \times B)) < \gamma(B)$. To do so, let $\epsilon(t) > 0$ be such that $\gamma(h(t,B)) < \gamma(B) - 3\epsilon(t)$ and let $V(t)$ be a neighborhood of t such that $\| h(s,x) - h(t,x) \| \leq \epsilon(t)$ for all $s \in V(t)$ and $x \in B$. It follows that $\gamma(h(V(t) \times B)) < \gamma(B)$. Let $\{V(t_i); i = 1,\dots,n\} \subset \{V(t); t \in [0,1]\}$ be a finite convering of the interval $[0,1]$. Then $\gamma(h([0,1] \times B)) \leq \max\{\gamma(h(V(t_i) \times B)); i = 1,\dots,n\} < \gamma(B)$ and consequently, $\gamma(h(\eta(B) \times B)) < \gamma(B)$. Hence (h_2) implies (h) and Proposition 1 is true.

The constant mappings $f_0(x) = x_0$ for all $x \in \overline{\Omega}$, with $x_0 \in \Omega$, are essential. This follows easily by Proposition 2 where we use Sadovskii's fixed point structure on E and we take D_g and $\tilde{g}$ as in Example 1.

As an application we have the following result (see [2]):

COROLLARY 2. Let E be a Banach space, Ω an open bounded subset of E and $f: \overline{\Omega} \to E$ a γ-condensing mapping. If for some $x_0 \in \Omega$ one has

$$t(f(x) - x_0) \neq x - x_0 \text{ for all } t \in [0,1] \text{ and } x \in \partial\Omega,$$

then f has at least one fixed point in Ω.

Proof. Observe that f is equivalent in the sense of (h_2), with the essential constant mapping $f_0(x) = x_0$ $(x \in \overline{\Omega})$ and apply Proposition 1.

3) A Mönch's fixed point theorem. The following generalization of the result in Corollary 2 is due to H. Mönch (see [7] and [2], Theorem 5.18.1). We shall deduce it from Proposition 1 via Proposition 2, by using Mönch's fixed point structure.

COROLLARY 3. Let E be a Banach space, $\Omega \subset E$ open and $f: \overline{\Omega} \to E$ continuous. Suppose that for some $x_0 \in \Omega$

$$t(f(x) - x_0) \neq x - x_0 \text{ for all } t \in [0,1] \text{ and } x \in \partial\Omega \tag{6}$$

and

$$\text{if } C \subset \overline{\Omega} \text{ is countable and } C \subset \overline{\text{conv}}(\{x_0\} \cap f(C)), \text{ then } C \text{ is compact.} \tag{7}$$

Then f has at least on fixed point in Ω.

Proof. In order to apply Propositions 1 and 2, let us set $X = \overline{\Omega}, A = \partial\Omega$,

$$\mathcal{A}_{\partial\Omega}(\overline{\Omega}, E) = \{g; g(x) = x_0 + \theta_\lambda(x)(f(x) - x_0) \text{ for } x \in \overline{\Omega},$$
$$\text{where } \lambda \in [0,1], \theta_\lambda \in C(\overline{\Omega}; [0,1]), \theta_\lambda(x) = \lambda \text{ for all } x \in \partial\Omega\} \tag{8}$$

and define an equivalence relation on this class of mappings by

(h_3) $\quad g' \sim g$ iff $\lambda' = \lambda$ or $\{\lambda', \lambda\} = \{0,1\}$, in case $f(x) \equiv x_0$ on $\partial\Omega$;

$$g'|_{\partial\Omega} = g|_{\partial\Omega}, \text{ otherwise.}$$

Here $g(x)=x_0+\theta_\lambda(x)(f(x)-x_0)$ and $g'(x)=x_0+\theta_{\lambda'}(x)(f(x)-x_0)$.

It is easy to see that this relation satisfies the above conditions (a) and (b). Indeed, if $g'|_{\partial\Omega}=g|_{\partial\Omega}$, then obviously $g'\sim g$ in case that $f(x)\equiv x_0$ on $\partial\Omega$. In the opposite case, one deduces that $\lambda'=\lambda$ and hence $g'\sim g$ as well. Thus, condition (a) holds. On the other hand, if $g'\sim g$, and we consider $h(t,x)=(1-t)g'(x)+tg(x)=x_0+[(1-t)\theta_{\lambda'}(x)+t\theta_\lambda(x)](f(x)-x_0)$, we see that condition (b) also holds. Consequently, Proposition 1 is applicable and since f is clearly equivalent, in the sense of (h_3), with the constant mapping x_0, we have only to prove that the constant mapping x_0 is essential in class (8). To do this, let g be any element in class (8) satisfying $g|_{\partial\Omega}\equiv x_0$. Assume

$$g(x)=x_0+\theta(x)(f(x)-x_0) \text{ for } x\in\overline{\Omega}, \tag{9}$$

where θ is a certain element of $C(\Omega;[0,1])$ constant on $\partial\Omega$. Let

$$D_g=\overline{\mathrm{conv}}(g(\Omega)) \text{ and } \tilde{g}:D_g\to D_g,$$

$\tilde{g}(x)=g(x)$ for $x\in\overline{\Omega}$ and $\tilde{g}(x)=x_0$ if $x\notin\Omega$. Let us observe that $\tilde{g}\in M(D_g)$, where M arises from Mönch's fixed point structure. Indeed, let $C\subset D_g$ be countable and

$$C\subset\overline{\mathrm{conv}}(\{x_0\}\cup\tilde{g}(C))=\overline{\mathrm{conv}}(\{x_0\}\cap g(C\cup\overline{\Omega})).$$

By (9), we have $\overline{\mathrm{conv}}(\{x_0\}\cup g(C\cap\overline{\Omega}))\subset\overline{\mathrm{conv}}(\{x_0\}\cup f(C\cap\overline{\Omega}))$. Consequently, $C\subset\overline{\mathrm{conv}}(\{x_0\}\cup f(C\cup\overline{\Omega}))$ and using (7) we find that $\overline{C}\cap\overline{\Omega}$ is compact. Further, the continuity of f implies that $f(\overline{C}\cap\overline{\Omega})$ is compact and Mazur's lemma that $\overline{\mathrm{conv}}(\{x_0\}\cup f(C\cap\overline{\Omega}))$ is also compact. It follows that C is compact, too, and so $\tilde{g}\in M(D_g)$. Since the conditions (1) and (2) in Proposition 2 are clearly satisfied, it follows that the constant mapping x_0 is essential as claimed and the proof is complete.

<u>Remark 3</u>. One can prove Corollary 1 and Corollary 2 in the smae way as Corollary 3, i.e. by showing that f is essential in the smaller class of mappings (8), endowed with the equivalence relation (h_3). But in case of Corollary 3, we can not prove the essentiality of f in the larger class of all

continuous mappings satisfying conditions (6) and (7). It is because we don't know if for two such mappings g and g', the mapping $(1-t)g+tg'$ also has property (7).

4) Topological transversality and mappings of monotone type. Let E be a Banach space and E^* the conjugate space of E. We denote by $<.,.>$ the duality between E^* and E and we shall use the symbol $\to$ for strong convergence and $\rightharpoonup$ for weak convergence. Let X be a subset of E and f mapping from X into E^*. Then f is said to be monotone if

$$< f(x)-f(y), x-y > \geq 0 \text{ for all } x,y \in X,$$

and is said to be of class $(S)_+$ if for any sequence (x_j) in X for which $x_j \rightharpoonup x$ in E and

$$\overline{\lim} < f(x_j), x_j - x > \leq 0,$$

we have $x_j \to x$. The mapping f is called hemicontinuous if $f(x+ty) \rightharpoonup f(x)$ as $t \to 0$ and demicontinuous if $y \to x$ implies $f(y) \rightharpoonup f(x)$.

Any monotone hemicontinuous mapping $f: E \to E^*$ is maximal monotone (see [8], Corollary III.2.3) in the sense that its graph is a maximal element in the set of all $T \subset E \times E^*$ satisfying

$$< x^* - y^*, x-y > \geq 0 \text{ for every } [x,x^*], [y,y^*] \in T,$$

ordered by inclusion.

If E is a reflexive Banach space then it can be renormed so that E and E^* are both locally uniformly convex. Then there exists a unique mapping $J: E \to E^*$ (the duality mapping) such that $< J(x), x > = \|x\|^2 = \|J(x)\|^2$ for all $x \in E$. Moreover, J is bijective, bicontinuous, monotone and of class $(S)_+$ (see [1], Proposition 8, or [8], III.2.6). In this case, the maximal monotonicity of a monotone mapping f is equivalent with the surjectivity of $J+f$. Moreover, if f is maximal monotone then $J+f$ is even bijective and $(J+f)^{-1}: E^* \to E$ is demicontinuous (see [8], III.2.11).

In what follows we shall give a topological transversality theorem for mappings of the form $(J+T)^{-1}(J-f)$ with f demicontinuous and of class $(S)_+$, T being fixed monotone hemicontinuous mapping. We shall apply this theorem to establish the existence of solutions to equation

$$(T+f)(x)=0.$$

In contrast to paper [1] of F. Browder, these results are proved without using the degree theory.

Let E be a reflexive Banach space which is normed so that E and E^* are both locally uniformly convex, $T:E \to E^*$ a hemicontinuous monotone mapping and $\Omega \subset E$ a nonempty open bounded subset of E. Let us consider the following class of admissible mappings:

$$\mathcal{A}_{\partial\Omega}(\overline{\Omega},E)=\{g; g=(J+T)^{-1}(J-f),\ f \text{ demicontinuous, of class } (S)_+ \text{ and } 0\notin (T+f)(\partial\Omega)\}\ . \tag{10}$$

If $g', g \in \mathcal{A}_{\partial\Omega}(\overline{\Omega},E)$, $g'=(J+T)^{-1}(J-f')$ and $g=(J+T)^{-1}(J-f)$ we set

(h_4) $g' \sim g$ if there exists $h^*:[0,1]\times\overline{\Omega} \to E^*$ such that $h^*(0,.)=f'$, $h^*(1,.)=f$, $0\notin (T+h^*(t,.))(\partial\Omega)$ for every $t\in]0,1[$ and for any sequence (x_j) in Ω with $x_j \rightharpoonup x$ and any sequence (t_j) in $[0,1]$ with $t_j \to t$ for which $\overline{\lim} < h^*(t_j,x_j), x_j - x > \,\le 0$, we have $x_j \to x$ and $h^*(t_j,x_j) \rightharpoonup h^*(t,x)$.

The relation $\sim$ defined in this way is an equivalence relation on class (10).

Suppose that $g'|_{\partial\Omega}=g|_{\partial\Omega}$. Then, by the injectivity of $(J+T)^{-1}$, one has $f'|_{\partial\Omega}=f|_{\partial\Omega}$. Now if we set $h^*(t,.)=(1-t)f'+tf$, then it is easily seen (see [1], Proposition 12) that $g' \sim g$. Thus, condition (a) is satisfied.

Denoting by $h=(J+T)^{-1}(J-h^*)$ we shall prove that (h_4) implies (h).

Clearly, since $0 \notin (T + h^*(t,.))(\partial\Omega)$ for any $t \in [0,1]$, we have that $Z = \cup\{\text{Fix}(h(t,.)); t \in [0,1]\}$ and $\partial\Omega$ are disjoint. In addition, Z is closed. Indeed, let (x_j) be a sequence in Z such that $x_j \to x$. then, there is a sequence (t_j) in $[0,1]$ such that $h(t_j, x_j) = x_j$, or equivalently, $h^*(t_j, x_j) = 0$. Passing, if necessary, to a subsequence, we may assume that $t_j \to t$ and by (h_4) we obtain $h^*(t,x) = 0$, that is $h(t,x) = x$. So, $x \in Z$, too. Hence $cl(Z) \cap \partial\Omega = \emptyset$. Let $\eta: \bar{\Omega} \to [0,1]$ be continuous and $\eta(x) = 1$ for all $x \in \partial\Omega$. We wish to show that $h(\eta(.),.)$ is admissible in class (10). First, since $0 \notin (T + f)(\partial\Omega)$, we see that $h(\eta(.),.)$ is fixed point free on $\partial\Omega$. Next, let (x_j) be a sequence in Ω such that $x_j \rightharpoonup x$ and

$$\overline{\lim} < h^*(\eta(x_j), x_j), x_j - x > \leq 0.$$

We may suppose, without loss of generality, that $\eta(x_j) \to \lambda$ in $[0,1]$. Then by (h_4), $x_j \to x$, $h^*(\eta(x_j), x_j) \rightharpoonup h^*(\lambda, x)$ and since η is continuous, $\lambda = \eta(x)$. Hence $h^*(\eta(.),.)$ is of class $(S)_+$ and demicontinuous. Thus, (h_4) implies (h), as claimed.

Therefore, Proposition 1 is applicable and so we may speak about a topological transversality theorem for mappings of class (10). As an application, we have

COROLLARY 4. Let E be a reflexive Banach space which is normed so that E and E^* are locally uniformly convex, $T: E \to E^*$ a hemicontinuous monotone mapping with $0 = T(0)$, $\Omega \subset E$ an open bounded subset with $0 \in \Omega$ and $f_0: \bar{\Omega} \to E^*$ a demicontinuous mapping of class $(S)_+$. Assume that

$$(1-t)J(x) + tf_0(x) + T(x) \neq 0 \text{ for all } t \in [0,1] \text{ and } x \in \partial\Omega. \tag{11}$$

Then there exists at least one solution $x \in \Omega$ to equation $(T + f_0)(x) = 0$.

Proof. Observe that $0 = (J+T)^{-1}(J-J) \sim (J+T)^{-1}(J-f_0)$. Indeed, the mapping h^*, $h^*(t,.) = (1-t)J + tf_0$ for $t \in [0,1]$, satisfies the requirements of condition (h_4). Hence, taking into account Proposition 1, one has only to show that the constant mapping $0 = (J+T)^{-1}(J-J)$ is essential in class (10). To do this, let g be any admissible mapping satisfying $g|_{\partial\Omega} = 0$. We have $g = (J+T)^{-1}(J-f)$, where f is demicontinuous, of class $(S)_+$, $0 \notin (T+f)(\partial\Omega)$ and $f(x) = J(x)$ for all $x \in \partial\Omega$. We want to show that g has a fixed point, i.e. there exists $x \in \Omega$ such that $(Tf)(x) = 0$. We shall proceed as in the proof of Theorem 4 in [1]. Let Δ be the partially ordered set of finite-dimensional subspaces E_λ of E ordered by inclusion and denote by φ_λ^* the projection mapping of E^* onto E_λ^*. For each λ, the mapping $f_\lambda = \varphi_\lambda^*(T+f):\overline{\Omega}_\lambda \to E_\lambda^*$, where $\Omega_\lambda = \Omega \cap E_\lambda$ is continuous and $t(x - f_\lambda(x)) \neq x$ for all $t \in [0,1]$ and $x \in \partial\Omega_\lambda$. Indeed, if we assume that $t(x - f_\lambda(x)) = x$ for some $t \in [0,1]$ and $x \in \partial\Omega_\lambda$, then since $f(x) = J(x)$ we obtain that $t < T(x), x > + \| x \|^2 = 0$. But $< T(x), x > \geq 0$ because $0 = T(0)$ and T is monotone. Hence $x = 0$, which is impossible because $x \in \partial\Omega$ and $0 \in \Omega$. Now by Corollary 1, the mapping $I - f_\lambda$ from $\overline{\Omega}_\lambda$ to $E_\lambda^* \equiv E_\lambda$ has a fixed point $x_\lambda \in \Omega_\lambda$, that is $f_\lambda(x_\lambda) = 0$, or equivalently, $< (T+f)(x_\lambda), y > = 0$ for all $y \in E_\lambda$. Thus, the set $V_\lambda = \{x \in \Omega;\ < (T+f)(x), x > \leq 0,\ < (T+f)(x), y > = 0 \text{ for all } y \in E_\lambda\}$ is nonempty. Clearly, the fammily $\{V_\lambda\}$ has the finite intersection property. It follows that the family of weakly compact sets $\{w - cl(V_\lambda)\}$ has nonempty intersection. Let x_o be a point of this intersection. For an arbitrary point $y \in E$ choose E_λ in Δ such that $x_o,\ y \in E_\lambda$. Let (x_j) be a sequence in V_λ such that $x_j \rightharpoonup x_o$. Since $x_j \in V_\lambda$, we have

$$< (T+f)(x_j), x_j > \leq 0, \quad < (T+f)(x_j), x_o > = 0 \text{ and } < (T+f)(x_j), y > = 0 \tag{12}$$

It follows that $< (T+f)(x_j), x_j - x_o > \leq 0$, or equivalently

$$< f(x_j), x_j - x_o > \ \leq - < T(x_j), x_j - x_o >$$

Since $< T(x_j), x_j - x_o > \geq < T(x_o), x_j - x_o >$ and $< T(x_o), x_j - x_o > \to 0$, we deduce that $\overline{\lim} < f(x_j), x_j - x_o > \leq 0$ and since f is of class $(S)_+$ and demicontinuous, we infer that $x_j \to x_o$ and $f(x_j) \rightharpoonup f(x_o)$. By (12) we also deduce $< T(x_j), x_o > \to - < f(x_o), x_o >$ and $< T(x_j), y > \to - < f(x_o), x >$. Now, from $0 \leq < T(x_j) - T(y), x_j - y > = < T(x_j) - T(y), x_o - y >$ $+ < T(x_j), x_j > - < T(x_j), x_o > - < T(y), x_j - x_o > \ \leq\ < T(x_j) - T(y), x_o - y > - < f(x_j), x_j >$

$-\langle T(x_j), x_0\rangle - \langle T(y), x_j - x_0\rangle$; passing to limit $j\to\infty$, we obtain

$$0 \le \langle -f(x_0) - T(y), x_0 - y\rangle .$$

Since y was arbitrary and T is maximal monotone, we may conclde that $-f(x_0) = T(x_0)$, as we wished.

Remark 4. In the same terms of essential mappings one can prove the existence of a solution for $0 \in (T+f)(x)$, where T is more generally a multivalued maximal monotone mapping (see Remark 2).

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