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NOTE ON AN ABSTRACT CONTINUATION THEOREM

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REZUMAT. -Notă asupra unei teoreme abstracte de continuare. În această notă se arată că teorema de transversalitate topologică a lui Granas şi proprietatea de invarianţă la omotopie a indexului de punct fix, pot fi deduse dintr-o teoremă abstractă de continuare demonstrată în [6], [8]. Aceste aplicaţii ca şi aplicaţia privind proprietatea de omotopie a aplicaţiilor zero-epi în sensul lui Furi-Martelli-Vignoli [9] arată că teorema noastră de continuare permite o abordare unitară a unor principii de bază ale analizei neliniare. În context, teorema de transversalitate topologică a lui Granas pentru aplicaţii cu valori într-o submulţime inchisă şi convexă a unui spaţiu Banach E, este extinsă la cazul aplicaţiilor cu valori într-un retract al lui E.

1. Introduction. In this note our general continuation theorem given in [6], [8] is used in order to derive two useful principles in nonlinear analysis, namely, the topological transversality theorem of A.Granas [2] and the homotopy invariance property of the fixed point index. These applications, together with that in [9] concerning the homotopy property of zero-epi maps in the sense of Furi-Martelli-Vignoli, show that our continuation theorem permits an unified approach to some basic principles in nonlinear analysis. In context, Granas transversality theorem for maps with values into a closed convex subset of a Banach space E is extended to maps with values into a retract of E.

For other consequences of the abstract continuation theorem, several applications and related topics we send to [3], [4], [5] and [7].

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2. The abstract continuation theorem. Let X be a normal topological space, A a proper closed subset of X, Y a set, B a proper subset of Y,

$$H: [0,1] \times X \to Y$$

a map and let d be a certain function which is defined at least on the following family of subsets of X:

$$\{H(a(.),.)^{-1}(B);\ a \in C(X;[0,1]) \text{ constant on } A\} \cup \{\phi\}.$$

The nature of the values of d is not important.

THEOREM 1. *Assume that the following conditions are satisfied:*

(i) $cl(\cup\{H(t,.)^{-1}(B);\ t \in [0,1]\} \cap A = \phi$;

(ii) *the map* $F = H(0,.)$ *satisfies*

$$d(H(a(.),.)^{-1}(B)) = d(F^{-1}(B)) \neq d(\phi) \tag{1}$$

for any function $a \in C(X;[0,1])$ *constant on* A *and such that*

$$H(a(.),.)|_A = F|_A.$$

Then there exists at least one $x \in X \setminus A$ *solution to* $H(1,x) \in B$. *Moreover,* $F = H(1,.)$ *also satisfies condition* (1) *and*

$$d(H(1,.)^{-1}(B)) = d(H(0,.)^{-1}(B)). \tag{2}$$

The proof of the first part was given in [6]. For the last part, formula (2), we use the same argument as in the proof of the second part of Theorem 1 in [8].

The meaning of Theorem 1 is that property (1), which is stronger that $F^{-1}(B) \neq \phi$, is invariable to homotopy. In applications, Theorem 1 ensures the solvability of the inclusion $H(1,x) \in B$ when it is known that $H(0,.)$ satisfies (1).

A map f in the class of all maps of the form $H(a(.),.)$, where $a \in C(X;[0,1])$ is constant on A, is said to be ***d*-essential** if it satisfies condition (1). Therefore, Theorem 1 saies that the d-essentiality property is invariable to homotopy.

3. Applications. Let E be a real Banach space and K a retract of E (this means that there is a continuous map $R: E \to K$ such that $R(x) = x$ on K). All topological notions referring to subsets of K will be understood with respect to the topology induced on K.

Let U be an open bounded subset of K and $h: [0,1] \times \overline{U} \to K$ be compact.

a) A first application depends upon the concept of **fixed point index.** For a compact map $f: \overline{U} \to K$ such that $\mathrm{Fix}(f) \cap \partial U = \phi$ the fixed point index is the integer number $D_{LS}(I - fR, R^{-1}(U), 0)$ where D_{LS} is the Leray-Schauder degree. We shall denote it by $i(f,U,K)$ (see [1, pp 238]).

COROLLARY 1 (Leray-Schauder). *Assume that the following conditions are satisfied:*

(i) $h(t,x) \neq x$ *for all* $t \in [0,1]$ *and* $x \in \partial U$;

(ii) $i(h(0,.),U,K) \neq 0$.

Then there exists at least one fixed point of $h(1,.)$ *in* U. *Moreover,*

$$i(h(1,.),U,K) = i(h(0,.),U,K).$$

Proof. Apply Theorem 1 to: $X = \overline{U}$, $A = \partial U$, $Y = E$, $B = \{0\}$, $H(t,x) = x - h(t,x)$, $d(\phi) = 0$,

$$d(H(a(.),.)^{-1}(B)) = i(h(a(.),.),U,K).$$

In this case, condition (1) is satisfied and its equality part expresses just the boundary value dependence of the degree.

Remark. Condition (ii) in Corollary 1 clearly holds if $h(0,x) = x_0$ for all $x \in \bar{U}$ (recall $x_0 \in U$).

b) The next application depends upon the concept of **essential map**. A compact map $f: \bar{U} \to K$ is said to be admissible if it is fixed point free on ∂U. An admissible map is essential if each admissible extention g of $f|_{\partial U}$ has at least one fixed point in U.

COROLLARY 2 (Granas). *Assume that the following conditions are satisfied:*

(i) $h(t,x) \neq x$ *for all* $t \in [0,1]$ *and all* $x \in \partial U$;

(ii) $h(0,.)$ *is essential*.

Then there exists at least one fixed point of $h(1,.)$ *in* U. *Moreover, the map* $h(1,.)$ *is essential too.*

Proof. The conclusion follows from Theorem 1 if for each admissible extension g of $h(1,.)|_{\partial U}$ (in particular for $g = h(1,.)$) we set: $X = \bar{U}$, $A = \partial U$, $Y = E$, $B = \{0\}$,

$$H(t,x) = x - h(2t,x) \quad \text{for } t \in [0,1/2]$$

$$= x - 2(1-t)h(1,x) - (2t-1)g(x) \quad \text{for } t \in [1/2,1]$$

and $d(\phi) = 0$, $d(C) = 1$ for $C \neq \phi$.

Remark 2. Condition (ii) in Corollary 2 also holds for $h(0,x) = x_0$, $x \in \bar{U}$. This follows by Schauder's fixed point theorem.

Remark 3. Recall that every closed convex subset is a retract and that every retract is closed but not necessarily convex; for instance, $\partial B_1(0)$ is a retract of E if $\dim E = \infty$.

Remark 4. There are examples of compact maps having null index but which are essential. Here is one from [10]: Let E be a real Hilbert space, U a bounded open subset of E with $0 \in U$ and let $f: \bar{U} \to E$ be compact such that $f(x) \neq x$ on ∂U and

$$(f(x), x) \geq (x, x) \quad \text{for all } x \in \partial U.$$

If E is infinite dimensional, one has

$$D_{LS}\ (I - f,\ U,\ 0) = 0$$

and f is essential. Therefore, such a map can stand for $h(0,.)$ in Corollary 2, but not in Corollary 1 if E is infinite dimensional.

Remark 5. The main ingredient in the proof of Theorem 1 is Urysohn's extension theorem in normal (T_4) topological spaces. Using the extension argument in a way adequated to the separation properties, we are able to prove Theorem 1 even for more general T_n spaces ($n < 4$); in particular, for Hausdorff locally convex spaces.

Remark 6. Note that the properties in Corollary 1 and Corollary 2 can be derived from Theorem 1 even for more general maps.

REFERENCES

1. K.Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
2. A.Granas, *Homotopy extension theorem in Banach spaces and some of its applications to the theory of non-linear equations*, Bull. Acad. Polon. Sci. 7(1959), 387-394.
3. R.Precup, *Topological transversality, perturbation theorems and second order differential equations*, Preprint Nr. 3(1989), "Babeş-Bolyai" Univ., Fac. of Math., 149-164.
4. R.Precup, *Generalized topological transversality and mappings of monotone type*, Studia Universitatis Babeş-Bolyai 35(1990), 44-50.
5. R.Precup, *Generalized topological transversality and existence theorems*, Libertas Mathematica 11(1991), 65-79.
6. R.Precup, *Sur un principe topologique*, Preprint Nr. 6(1991), "Babeş-Bolyai" Univ., Fac. of Math., 163-168.
7. R.Precup, *Topological transversality and boundary problems for second order functional differential equations*, in Differential Equations and Control Theory (V.Barbu ed.), Longman Scientific & Technical, 1991, 283-288.
8. R.Precup, *On the topological transversality principle*, Nonlinear Analysis, Theory, Methods & Applications, 20(1993), in print.

9. R.Precup, *On a topological principle II*, Preprint Nr. **6**(1992), "Babeş Bolyai" Univ., Fac. of Math., in print.
10. V.Lakshmikantham, Yong Sun, *A theorem on the existence of two fixed points*, Journal of Mathematical and Physical Sciences **25**(1991), 281-286.