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ON THE REVERSE OF THE KRASNOSELSKII-BROWDER BOUNDARY INEQUALITY

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Dedicated to Professor P Szilágyi on his 60[th] anniversary

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REZUMAT. - Asupra contrarei inegalităţii pe frontieră a lui Krasnoselskii-Browder. Fie G o submulţime deschisă şi mărginită a spaţiului Banach X cu $0 \in G$ şi fie f o aplicaţie de la $\bar{G}$ în spaţiul dual X^*. Inegalitatea pe frontieră a lui Krasnoselskii-Browder: $(x,f(x)) \geq 0$ pentru orice $x \in \partial G$, este pentru anumite tipuri de aplicaţii, suficientă pentru existenţa unei soluţii $x \in \bar{G}$ a ecuaţiei $f(x) = 0$. Prezentul articol se ocupă de contrara acestei inegalităţi, anume: $(x,f(x)) \leq 0$ pentru orice $x \in \partial G$. Arătăm că dacă X este un spaţiu Hilbert infinit-dimensional, $f = I - g$ unde I este identitatea lui X şi $g : \bar{G} \to X$ este complet continuu, atunci inegalitatea $(x,f(x)) \leq 0$ nu are loc pentru toţi $x \in \partial G$. În consecinţă, două teoreme de punct fix demonstrate în [4] nu au obiect fiindcă ipotezele lor nu pot fi satisfăcute. Apoi punem problema dacă un rezultat negativ de tipul celui de mai sus, este valabil şi pentru aplicaţii f de tip monoton, mai generale La această întrebare se dă un răspuns parţial

1. Abstract. Let G be a bounded open subset of a Banach space X with $0 \in G$ and let f be a map from $\bar{G}$ into the dual X^*. The following Krasnoselskii-Browder boundary inequality: $(x,f(x)) \geq 0$ for all $x \in \partial G$ is for some types of maps sufficient for the existence of solutions $x \in \bar{G}$ for equation $f(x) = 0$ This article deals with the reverse of the above inequality, namely

$$(x,f(x)) \leq 0 \text{ for all } x \in \partial G$$

We prove that if X is an infinite-dimensional Hilbert space, $f = I - g$ where I is the identity on X and g $\bar{G} \to X$ is completely continuous, then the inequality $(x,f(x)) \leq 0$ can

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not be true for all $x \in \partial G$ Consequently, two existence theorems proved in [4] have no content since their assumptions are never satisfied We then ask if such a negative result holds true even for more general maps of monotone type. A partial answer is finally given

1. Introduction. Let us start with the definition of a general concept of degree of map

DEFINITION 1 ([1], [2]) Let X and Y be topological spaces Let $\mathcal{O}$ be a class of open subsets of X For each G in $\mathcal{O}$ one considers a family of maps $f.\bar{G} \to Y$; the collection of all such maps for the various G of $\mathcal{O}$ is denoted by $\mathcal{F}$. For each G in $\mathcal{O}$ consider a family of homotopies $\{ f_t \ \ 0 \le t \le 1 \}$ of maps in $\mathcal{F}$; all having the common domain $\bar{G}$; denote by $\mathcal{H}$ the collection of all such homotopies for the various G in $\mathcal{O}$, Then by a *degree* function on the family $\mathcal{F}$ which is invariant with respect to the homotopies in $\mathcal{H}$ and which is normalized by a given map f_0 from X into Y, one means an integer - valued function $d(f,G,y)$ which is defined for all $G \in \mathcal{O}$, $f \in \mathcal{F}$, $f \cdot \bar{G} \to Y$, $y \in Y \setminus f(\partial G)$ such that the following three conditions are satisfied·

(a) (normalization) If $d(f,G,y) \neq 0$, then $y \in f(G)$ For each G in $\mathcal{O}$, $f_0|_{\bar{G}} \in \mathcal{F}$ and if $y \in f_0(G)$, then $d(f_0|_{\bar{G}}, G, y) = +1$.

(b) (additivity on domain). If $f \in \mathcal{F}$, $f \cdot \bar{G} \to Y$ and $G_1, G_2 \in \mathcal{O}$ are a pair of disjoint open subsets of G such that $y \notin f\left(\bar{G} \setminus (G_1 \cup G_2)\right)$, then $f|_{\bar{G}_1}$ and $f|_{\bar{G}_2}$ both lie in $\mathcal{F}$ and $d(f,G,y) = d\left(f|_{\bar{G}_1}, G_1, y\right) + d\left(f|_{\bar{G}_2}, G_2, y\right)$

(c) (invariance under homotopy). If $\{f_t\,; 0 \le t \le 1\}$ is a homotopy in $\mathcal{H}$ with fixed domain $\bar{G}$ and if $\{y_t\,; 0 \le t \le 1\}$ is a continuous curve in Y such that for all t in [0,1], $y_t \notin f_t(\partial G)$, then $d(f_t, G, y_t)$ is constant on [0,1].

The following proposition is a variant of Proposition 1 and Proposition 3 in [1] and shows how a boundary inequality is useful for existence results

PROPOSITION 1. *Assume a degree function exists as in Definition* 1 *and* Y *is a linear topological space. Let* G *be a set in* $\mathcal{O}$, y *a given element in* $f_0(G)$ *and let* $f: \bar{G} \to Y$ *lies in* $\mathcal{F}$. *Suppose that* $\mathcal{H}$ *includes the affine homotopy* $f_t = (1-t)f_0|_{\bar{G}} + tf$. *If for each* $x \in \partial G$ *there exists a linear functional* w *on* Y *such that*

$$(w,y) = 0,\ (w,f_0(x)) > 0 \text{ and } (w,f(x)) \geq 0, \tag{1}$$

then $y \in f(\bar{G})$

Proof Suppose $y \notin f(\partial G)$. Since $y \in f_0(G)$, from (a) one has $d(f_0|_{\bar{G}}, G, y) = +1$ To show that $d(f,G,y) = d(f_0|_{\bar{G}}, G, y) = +1$, it suffices to see that $d(f_t, G, y)$ is constant in t, where $\{f_t,\ 0 \leq t \leq 1\}$ is the affine homotopy which joins $f_0|_{\bar{G}}$ and f This follows from (c) if we can verify that $y \notin f_t(\partial G)$ for all t in $[0,1]$. Suppose, however, that for some $x \in \partial G$ and some $t \in [0,1)$, we have $y = f_t(x) = (1-t)f_0(x) + tf(x)$. Then

$$\begin{aligned} 0 = (w,y) &= (w,f_t(x)) = (1-t)(w,f_0(x)) + t(w,f(x)) \geq \\ &\geq (1-t)(w,f_0(x)) > 0 \end{aligned}$$

which is a contradiction

In the particular case $X = Y$ is a Hilbert space, $\mathcal{O}$ in the class of all bounded open nonempty subsets of X; $\mathcal{F}$ is the family of continuous maps $f.\bar{G} \to X$ with $G \in \mathcal{O}$ and $(I-f)(\bar{G})$ relatively compact in X; $\mathcal{H}$ is the family of continuous homotopies $\{f_t,\ 0 \leq t \leq 1\}$ in $\mathcal{F}$ with a common domain $\bar{G}$ such that there is a compact subset K of X with $(I-f_t)(\bar{G}) \subset K$ for all $t \in [0,1]$, f_0 is the identity of X and $y = 0$, condition (1) is satisfied provided that

$$(x, f(x)) \geq 0 \quad \text{for all} \quad x \in \partial G \tag{2}$$

Condition (2) is just the well-known Krasnoselskii boundary inequality Thus, if we set $f =$

$I - g$, we obtain the fixed point theorem of Krasnoselskii:

PROPOSITION 2 (Krasnoselskii) *Let G be a bounded open subset of a real Hilbert space X with $0 \in G$. Suppose that the completely continuous map g from $\bar{G}$ into X satisfies*

$$(x, g(x)) \leq |x|^2 \quad \text{for all } x \in \partial G. \tag{3}$$

Then g has at least one fixed point in $\bar{G}$.

An obvious question is what happens if the inequality in (2), equivalently in (3), is reversed This question was asked by Lakshmikantham and Sun in [4] where the following answer was given

PROPOSITION 3 (Lakshmikantham & Sun). *Let X be a real Hilbert space of infinite dimension and G a bounded open set of X with $0 \in G$. Suppose that the completely continuous map g from $\bar{G}$ into X satisfies*

$$(x, g(x)) \geq |x|^2 \quad \text{for all } x \in \partial G. \tag{4}$$

Then g has at least one fixed point in $\bar{G}$.

An other statement in [4] is the following:

PROPOSITION 4 (Lakshmikantham & Sun). *Let X be a real Hilbert space of infinite dimension and G_1, G_2 two bounded open sets of X such that $0 \in G_1$ and $\bar{G}_1 \subset G_2$. Suppose that the completely continuous map g from $\bar{G}_2$ into X satisfies the following conditions:*

$$(x, g(x)) \geq |x|^2 \quad \text{and } g(x) \neq x \text{ for any } x \in \partial G_1; \tag{5}$$

$$(x, g(x)) \leq |x|^2 \text{ for any } x \in \partial G_2. \tag{6}$$

Then g has at least two fixed points in $\bar{G}_2$.

Proposition 3 is true if X is a space of finite dimension This can be proved by Brouwer's degree ([4], Remark 1) since

$$d(I - g, G, 0) = (-1)^{\dim X} \neq 0$$

Indeed, if g satisfies (4) and $g(x)\neq x$ for any $x\in\partial G$, then g is homotopic to θI for any $\theta > 1$, the homotopy being

$$g_t(x) = (1-t)\theta x + tg(x), \quad 0 \le t \le 1$$

It is easily seen that $g_t(x) \neq x$ for all $x \in \partial G$ and $t \in [0,1]$ Hence,

$$\begin{aligned} d(I-g,G,0) &= d(I-g_1,G,0) = \\ &= d(I-g_0,G,0) = d((1-\theta)I,G,0) = \\ &= d(-I,G,0) = (-1)^{\dim X}. \end{aligned}$$

Let us remark that Proposition 4 is true if X is a space of odd finite dimension. Indeed, under conditions (5), (6) and $g(x) \neq x$ (for all $x \in \partial G_2$), by additivity property of the degree, one has

$$\begin{aligned} d(I-g,G_2\setminus\bar{G}_1,0) &= d(I-g,G_2,0) - \\ &- d(I-g,G_1,0) = 1-(-1)^{\dim X} = 2 \end{aligned}$$

If X is a space of even finite dimension, Proposition 4 is not true as shows the following example:

Example 1. Let $X = \mathbb{R}^{2n}$, $G_1 = \{x \in \mathbb{R}^{2n}, |x| < 1\}$, $G_2 = \{x \in \mathbb{R}^{2n}, |x| < 2\}$, $g : \bar{G}_2 \to \mathbb{R}^{2n}$ where

$$g(x)_{2k-1} = 2x_{2k-1}/\left(|x|^2+1\right) + x_{2k}, \quad k = 1,2,..,n$$

$$g(x)_{2k} = 2x_{2k}/\left(|x|^2+1\right) - x_{2k-1}, \quad k = 1,2,..,n.$$

For $|x| = 1$ we have $g(x)_{2k-1} = x_{2k-1} + x_{2k}$, $g(x)_{2k} = x_{2k} - x_{2k-1}$ and so $(x,g(x)) = |x|^2 = 1$ and $g(x) \neq x$ Hence g satisfies (5) For $|x|=2$ we have $(x,g(x)) = \frac{2}{5}|x|^2 = \frac{8}{5} < 4 = |x|^2$, which shows that (6) also holds Nevertheless, the unique fixed point of g is 0 Indeed, if $g(x) = x$, then

$$\begin{aligned} 2x_{2k-1}/\left(|x|^2+1\right)+x_{2k} &= x_{2k-1} \\ 2x_{2k}/\left(|x|^2+1\right)-x_{2k-1} &= x_{2k} \end{aligned}, \quad k = 1, 2, \ldots, n \tag{7}$$

If we multiply each of these equalities by x_{2k-1} and x_{2k}, respectively, and add, we obtain

$$2|x|^2/\left(|x|^2+1\right) = |x|^2,$$

which is possible only if $|x| = 1$ or $|x| = 0$. In case $|x| = 1$, from (7) we find $x_{2k-1}+x_{2k} = x_{2k-1}, x_{2k}-x_{2k-1} = x_{2k}$, $k = 1, 2, \ldots, n$, whence $x=0$, a contradiction Thus, $x=0$ is the unique fixed point of g

3. In spaces of infinite dimension the complete continuity and the reverse of the Krasnoselskii inequality are incompatible.

THEOREM 1. *Let X be a real Hilbert space of infinite dimension, G a bounded open set of X with $0 \in G$ and let g be a completely continuous map from $\bar{G}$ into X. Then, there exists $x \in \partial G$ such that*

$$(x, g(x)) < |x|^2. \tag{8}$$

Proof Suppose otherwise Then

$$(x, g(x)) \geq |x|^2 \text{ for all } x \in \partial G. \tag{9}$$

Since G is open bounded and $0 \in G$ there exist r_1, r_2 such that

$$0 < r_1 \leq |x| \leq r_2 \quad \text{for all } x \in \partial G. \tag{10}$$

On the other hand, the compactness of g implies thet $K = g(\overline{\partial G})$ is compact and

$$|g(x)| \leq R \quad \text{for all } x \in \partial G,$$

where $R > 0$ From (9) and (10), we have

$$r_1|x| \leq |x|^2 \leq (x, g(x)) \leq |x|\,|g(x)| \quad \text{for any } x \in \partial G$$

Hence

$$0 < r_1 \le |g(x)| \le R \quad \text{for all} \quad x \in \partial G$$

Again from (9) and (10), we have

$$(x, g(x)) = |x|\,|g(x)| \cos \sphericalangle(x, g(x)) \ge |x|^2 \ge r_1|x|$$

It follows that

$$\cos \sphericalangle(x, g(x)) \ge r_1/|g(x)| \ge r_1/R \quad (x \in \partial G).$$

Thus

$$\sphericalangle(x, g(x)) \le \alpha = \arccos(r_1/R) < \pi/2 \quad \text{for all } x \in \partial G$$

Now, for each $y \in K$ define a subset U_y of X by

$$U_y = \{x \in X\backslash\{0\},\ \sphericalangle(x,y) < \pi/2 - \alpha\}.$$

Clearly, the family $\{U_y\,;\ y \in K\}$ is an open cover of K Since K is compact there is a finite subcover of K, say

$$U_{y_1},\ U_{y_2},\ \ ,\ U_{y_m}$$

For any x in ∂G, there exists $i \in \{1, 2, \ ., m\}$ such that $g(x) \in U_{y_i}$

Since $\sphericalangle(x, g(x)) \le \alpha$ and $\sphericalangle(g(x), y_i) < \pi/2 - \alpha$, we obtain $\sphericalangle(x, y_i) < \pi/2$ and so, $(x, y_i) > 0$ On the other hand, X being of infinite dimension, we may find an element x_0 on ∂G such that $(x_0, y_j) \doteq 0$ for all $j \in \{1, 2, \ \ , m\}$ Therefore we reach a contradiction. Theorem 1 is thus proved

Remark 1 Here is an equivalent statement for Theorem 1

Let X be a real Hilbert space of infinite dimension, G a bounded open set of X with $0 \in G$ and g a completely continuous map from $\bar{G}$ into X. Denote $f = I - g$ Then

$$\sup\{(x, f(x)),\ x \in \partial G\} > 0 \tag{11}$$

Remark 2 Proposition 3 and Proposition 4 have no contents Indeed, by Theorem 1, the assumptions (4) and (5) never hold

4. The reverse of the Krasnoselskii-Browder boundary inequality and maps of monotone type. The question we ask is if (11) holds true even for more general maps of monotone type The answer we give is only a partial one.

Let us first recall some definitions and standard notations of nonlinear functional analysis. If X is a real Banach space and X^* its dual, we denote by (x,w) the pairing between x in X and w in X^* We use the symbol $\to$ for strong convergence and $\rightharpoonup$ for weak convergence If D is a subset of X and f a map from D into X^*, f is said to be *demicontinuous* if it is continuous from the strong topology of X on D to the weak topology of X^*; f is said to be of *class* $(S)_+$ if it is demicontinuous and if for any sequence (x_j) in D with $x_j \rightharpoonup x$ for some $x \in X$ for which $\overline{\lim}\,(x_j - x, f(x_j)) \leq 0$, we have $x_j \to x$; f is said to be *pseudo-monotone* if it is demicontinuous and if for any sequence (x_j) in D with $x_j \rightharpoonup x$ for some $x \in X$ for which $\overline{\lim}\,(x_j - x, f(x_j)) \leq 0$, we have $(x_j - x, f(x_j)) \to 0$, while if $x \in D$, then $f(x_j) \rightharpoonup f(x)$. A multi-valued map T of X into the subsets of X^* is *monotone* if for any elements $[x,u]$ and $[y,v]$ of its graph, $(x-y,\ u-v) \geq 0$, T is *maximal monotone* if it is maximal in the sense of inclusion of graphs among monotone maps of X into the subsets of X^*

Note that if X is a Hilbert space and g is a completely continuous map from D into X, then the map $f = I - g$ is of class $(S)_+$

The classes of maps we shall deal with include the bounded maps of type $(S)_+$, the bounded pseudo-monotone maps and the sums of maximal monotone maps and bounded maps which are of class $(S)_+$ or pseudo-monotone. For all these classes of maps a degree function is known (see [1], [2]) and with respect to the corresponding degree theory, Krasnoselskii-Browder boundary inequality is a "good" condition in the sense of an existence theorem like Proposition 1 What we can prove for the moment is that, unlike this, the reverse of the

Krasnoselskii-Browder inequality is a "bad" condition for the degree theory

Let X_0 be a finite-dimensional subspace of X, G an open subset of X such that $G \cap X_0 = G_0$ is nonempty and let $f\ \bar{G} \to X^*$ be a given map. Then the Galerkin approximant from f is the map $f_0\ \bar{G} \to X_0(=X_0^*)$, $f_0(x) = \varphi^*(f(\varphi(x)))$, where φ is the injection map of X_0 into X and φ^* the corresponding projection of X^* onto X_0^*.

Let Λ be the partially ordered set of finite-dimensional subspaces X_λ of X, ordered by inclusion For each λ, denote by φ_λ the injection map of X_λ into X and by φ_λ^* the corresponding projection of X^* onto X_λ^*

LEMMA 1 (Browder) *Let X be a real reflexive Banach space, G a bounded open subset of X with $0 \in G$ and let $f \cdot \bar{G} \to X^*$ be a bounded map of class $(S)_+$ such that $0 \notin f(\partial G)$ Then, there exists λ_0 in Λ such that for all $\lambda > \lambda_0$, $0 \notin f_\lambda(\partial G_\lambda)$ and Brouwer's degree $d(f_\lambda, G_\lambda, 0)$ is independent of λ (where $G_\lambda = G \cap X_\lambda$ and $f_\lambda = \varphi_\lambda^* f \varphi_\lambda$).*

The common value of $d(f_\lambda, G_\lambda, 0)$ for $\lambda > \lambda_0$ is denoted by $d(f, G, 0)$ and is called Browder's degree of f (see [1]).

The main result of this section is the following theorem

THEOREM 2 *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$ Suppose that the bounded map f from $\bar{G}$ into X^* is of class $(S)_+$ and satisfies the following condition*

$$(x, f(x)) \le 0 \quad \textit{for any } x \in \partial G \tag{12}$$

Then f has at least one zero on ∂G

Proof Suppose that the assertion were false Then $0 \notin f(\partial G)$ and by Lemma 1, there would exist λ_0 in Λ such that for all $\lambda > \lambda_0$, $0 \notin f_\lambda(\partial G_\lambda)$ and Brouwer's degree $d(f_\lambda, G_\lambda, 0)$ is independent of λ But, from (12), we have

$$(x, f_\lambda(x)) = (x, f(x)) \le 0 \quad \text{for any } x \in \partial G_\lambda$$

Hence, $d(f_\lambda, G_\lambda, 0) = (-1)^{\dim \lambda}$ and so the degree $d(f_\lambda, G_\lambda, 0)$ could not be independent of λ Thus, the assertion in Theorem 2 is true.

Remark 3 By Theorem 2, the following proposition is true

Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$ and f a bounded map of class $(S)_+$ from $\bar{G}$ into X^*. Then

$$\sup\{(x, f(x)), x \in \partial G\} \ge 0 \tag{13}$$

and in case that $\sup\{(x, f(x)), x \in \partial G\} = 0$, there exists $x \in \partial G$ such that $f(x) = 0$

Our question is does the strict inequality hold in (13)? As we have already seen (Remark 1), the answer is positive for maps of the form $f = I - g$ with g completely continuous in Hilbert spaces For the broader class of maps of type $(S)_+$, this is an open problem

COROLLARY 1. *Let X be a real reflexive Banach space of infinite dimension and G a bounded open subset of X with $0 \in G$ Suppose that $f . \bar{G} \to X^*$ is bounded pseudo-monotone Then inequality* (13) *holds*

Proof If $0 \in \overline{f(\partial G)}$, inequality (13) obviously holds Thus we may assume $0 \notin \overline{f(\partial G)}$ Suppose that (13) were false Then there would exist a positive number ε_0 such that

$$(x, f(x)) + \varepsilon_0 |x|^2 \le 0 \quad \text{for any } x \in \partial G \tag{14}$$

For each $0 < \varepsilon \le \varepsilon_0$ the map $f_\varepsilon = f + \varepsilon J$ (J is the duality map of X) is of class $(S)_+$ and bounded By (14), f_ε satisfies condition (12) It follows that there exists $x_\varepsilon \in \partial G$ such that $f_\varepsilon(x_\varepsilon) = 0$, i e

$$f(x_\varepsilon) + \varepsilon J(x_\varepsilon) = 0$$

Letting $\varepsilon \searrow 0$ we find that $0 \in \overline{J(\partial G)}$, which contradicts our assumption Thus (13) holds.

COROLLARY 2. *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with* $0 \in G$. *Let T be a maximal monotone map of X into the subsets of* X^* *with* $0 \in T(0)$ *and let f be a bounded map of* $\bar{G}$ *into* X^* *of class* $(S)_+$. *Suppose that there exists a sequence* (ε_j), $0 < \varepsilon_j$, $\varepsilon_j \to 0$, *such that*

$$(x, (T_{\varepsilon_j} + f)(x)) \leq 0 \quad \text{for all } x \in \partial G, \ j = 1, 2, \ldots, \tag{15}$$

where $T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}$ *is the Yosida approximant of T. Then there exists at least one x* $\in \partial G$ *such that*

$$0 \in f(x) + T(x)$$

Proof The map $T_\varepsilon + f$ $(\varepsilon > 0)$ is bounded and of class $(S)_+$ So, by Theorem 1, for each j there exists $x_j \in \partial G$ such that

$$T_{\varepsilon_j}(x_j) + f(x_j) = 0 .$$

Denote $y_j = T_{\varepsilon_j}(x_j) = -f(x_j)$ Since G and f are bounded, we may suppose that we have

$$x_j \rightharpoonup x_0 \ \text{ and } \ y_j \rightharpoonup y_0$$

From $y_j = T_{\varepsilon_j}(x_j)$, we see that

$$y_j \in T(x_j - \varepsilon_j J^{-1}(y_j))$$

Hence, for any $x \in D(T)$ and any $y \in T(x)$, we have

$$(x_j - \varepsilon_j J^{-1}(y_j) - x, y_j - y) \geq 0$$

Thus,

$$(x_j - x, y_j - y) \geq (\varepsilon_j J^{-1}(y_j), y_j - y) \geq$$

$$\geq -(\varepsilon_j J^{-1}(y_j), y) \geq -\varepsilon_j \|y_j\| \, \|y\|$$

Since (y_j) is bounded, we deduce

$$\underline{\lim} \, (x_j - x, y_j - y) \geq 0 \tag{16}$$

If $\overline{\lim}\ (x_j - x_0, f(x_j)) \leq 0$, then since f is of class $(S)_+$, we get $x_j \to x_0$ and $y_0 = -f(x_0)$ Next, from (16) we obtain

$$(x_0 - x, -f(x_0) - y) \geq 0$$

Since T is maximal monotone and $x \in D(T)$, $y \in T(x)$ were arbitrar, we deduce that $x_0 \in D(T)$ and $-f(x_0) \in T(x_0)$, which finishes the proof It remains only to show that $\overline{\lim}\ (x_j - x_0, f(x_j)) \leq 0$ Suppose otherwise, i.e $\overline{\lim}\ (x_j - x_0, f(x_j)) > 0$ Then $\underline{\lim}\ (x_j - x_0, y_j) < 0$ and so, $\underline{\lim}\ (x_j, y_j) < (x_0, y_0)$ On the other hand, by (16)

$$\underline{\lim}\ (x_j, y_j) \geq (x_0 - x, y) + (x, y_0)$$

Therefore

$$(x_0 - x, y_0 - y) > 0 \text{ for all } x \in D(T) \text{ and } y \in T(x).$$

Since T is maximal monotone, it follows that $x_0 \in D(T)$ and $y_0 \in T(x_0)$. Thus the above strict inequality must also be true for $x = x_0$ and $y = y_0$, which is absurd The proof is now complete.

Remark 4 Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$, T a maximal monotone map of X into the subsets of X^* with $0 \in T(0)$ and let f be a bounded map of $\overline{G}$ into X^* of class $(S)_+$. Then there exists $\varepsilon_0 > 0$ such that

$$\sup\left\{(x, (T_\varepsilon + f)(x)),\ x \in \partial G\right\} > 0 \quad \text{for any } 0 < \varepsilon \leq \varepsilon_0, \tag{17}$$

or there exists $x \in \partial G$ such that $0 \in f(x) + T(x)$

We conjecture that there is $\varepsilon_0 > 0$ such that the strict inequality (17) always holds

COROLLARY 3 *Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0 \in G$, T a maximal monotone map of X into the subsets of X^* with $0 \in T(0)$ and let f be a bounded pseudo-monotone map of $\overline{G}$ into X^* Then*

$$\sup\left\{(x,(T_\varepsilon+f)(x)),\ x\in\partial G\right\}\geq 0 \quad \textit{for any } \varepsilon>0 \tag{18}$$

Proof Suppose otherwise Then for some $\varepsilon>0$ there would exist a positive number δ_0 such that

$$(x,(T_\varepsilon+f)(x))+\delta_0\|x\|^2\leq 0 \quad \text{for any } x\in\partial G \tag{19}$$

For each $0<\delta\leq\delta_0$ the map $T_\varepsilon+f+\delta J$ is bounded of class $(S)_+$ and satisfies (12) By Theorem 2 there exists x_δ on ∂G such that

$$T_\varepsilon(x_\delta)+f(x_\delta)+\delta J(x_\delta)=0$$

If we set $x=x_\delta$ in (19) we get $(\delta_0-\delta)\|x_\delta\|^2\leq 0\quad(\delta<\delta_0)$, a contradiction

5. Concluding remarks. Let X be a real reflexive Banach space of infinite dimension, G a bounded open subset of X with $0\in G$, f a bounded map of $\bar{G}$ into X^* and T a maximal monotone map of X into the subsets of X^* We have established the following results

1) If X is a Hilbert space and $f=I-g$, g completely continuous, then

$$\sup\left\{(x,f(x)),\ x\in\partial G\right\}>0$$

2) If f is of class $(S)_+$, then

$$0\notin f(\partial G)\Rightarrow\sup\left\{(x,f(x)),\ x\in\partial G\right\}>0$$

3) If f is pseudo-monotone, then

$$\sup\left\{(x,f(x)),\ x\in\partial G\right\}\geq 0$$

4) If f is of class $(S)_+$, then

$$0\notin(T+f)(\partial G)\Rightarrow\sup\left\{(x,(T_\varepsilon+f)(x)),x\in\partial G\right\}>0 \quad \text{for any } 0<\varepsilon\leq\varepsilon_0$$

5) If f is pseudo-monotone, then

$$\sup\left\{(x,(T_\varepsilon+f)(x)),\ x\in\partial G\right\}\geq 0 \quad \text{for any } \varepsilon>0$$

We conjecture that in cases 2) and 4) the strict inequalities on "sup" also hold if $0\in$

$f(\partial G)$ and $0 \in (T+f)(\partial G)$, respectively We recall that the conditions $0 \notin f(\partial G)$ and $0 \notin (T+f)(\partial G)$ are required by the definition of Browder's degree Thus, we have shown that for bounded maps of type $(S)_+$, the reverse of the Krasnoselskii-Browder inequality, namely

$$\sup \{(x, f(x)); x \in \partial G\} \leq 0,$$

implies $0 \in f(\partial G)$, i e it is a bad condition in the degree theory Also, for sums $T+f$ with bounded maps of type $(S)_+$, the reverse of a Krasnoselskii-Browder type condition, i e

$$\sup \left\{(x, (T_{\varepsilon_j}+f)(x)),\ x \in \partial G\right\} \leq 0 \text{ for some sequence } \varepsilon_j \searrow 0,$$

implies $0 \in (T+f)(\partial G)$ Hence it is also a bad condition for the degree theory

We conclude with an application to Leray-Lions maps Let $\Omega \subset \mathbb{R}^n$ be open bounded, $1<p<\infty$, $F\ \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the Caratheodory conditions, i e. $F(\cdot, s, \xi)$ is measurable for all s, ξ and $F(x, \cdot, \cdot)$ is continuous a e $x \in \Omega$ Also assume

$$|F(x,s,\xi)| \leq C(x) + c_1|s|^{p-1} + c_2|\xi|^{p-1}$$

and

$$(\xi - \xi^*, F(x,s,\xi) - F(x,s,\xi^*)) \geq 0,$$

for all ξ, ξ^*, s and a e $x \in \Omega$, where $c_1, c_2 \in \mathbb{R}_+$, $C(x) \geq 0$, $C(x) \in L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$ It is known that the map

$$f\ W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega),\ f(u) = -\operatorname{grad} F(x, u, \operatorname{grad} u)$$

is bounded, continuous and pseudo-monotone

By Corollary 1, for each $R > 0$, we have

$$\sup\left\{(u, f(u)) = \int_\Omega F(x, u, \operatorname{grad} u) \operatorname{grad} u\, dx,\ \|u\|_{W_0^{1,p}} = R\right\} \geq 0$$

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