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ON THE TOPOLOGICAL TRANSVERSALITY PRINCIPLE

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0. INTRODUCTION

IN THIS paper we propose a generalization of the topological transversality theorem of Granas [1] (see also [2]). We show the implication of this theorem in the axiomatic theory of the topological degree due to Amann and Weiss [3] (see also [4]) and its connection with the fixed point theory. Finally, as an application, we prove a continuation theorem for equations of the form

$$L(x) = N(x)$$

where L is a linear Fredholm mapping of index zero and N is a nonlinear mapping satisfying a compactness-like condition of Mönch type. This theorem extends some continuation results due to Mawhin [5] (see also [6]) and Volkmann [7] and in a particular case reduces to a fixed point theorem of Mönch [8].

1. GENERALIZED TOPOLOGICAL TRANSVERSALITY

Let X be a normal topological space, A a proper closed subset of X, Y a set and B a proper subset of Y. Consider a nonvoid class of mappings

$$\mathcal{Q}_A^B(X, Y) \subset \{F: X \to Y;\ F^{-1}(B) \cap A = \varnothing\}$$

whose elements are called *admissible* mappings and let

$$d: \{F^{-1}(B);\ F \in \mathcal{Q}_A^B(X, Y)\} \cup \{\varnothing\} \to \Delta$$

be any mapping with values in a nonempty set Δ.

For each admissible mapping F, the value $d(F^{-1}(B))$ stands as a "measure" of the set $F^{-1}(B)$ of all solutions $x \in X$ of

$$F(x) \in B. \tag{1}$$

Denote $\theta = d(\varnothing)$. An admissible mapping F is said to be *d-essential* if

$$d(F^{-1}(B)) = d(F'^{-1}(B)) \neq \theta$$

for any admissible mapping F' having the same restriction to A as F, i.e. $F|_A = F'|_A$. In the opposite case F is said to be *d-inessential*. Thus, F is d-inessential if $d(F^{-1}(B)) = \theta$ or there exists an admissible mapping F' such that

$$F|_A = F'|_A \quad \text{and} \quad d(F^{-1}(B)) \neq d(F'^{-1}(B)).$$

Also consider an equivalence relation $\approx$ on $\mathfrak{A}_A^B(X, Y)$ such that

(A) if $F|_A = F'|_A$ then $F \approx F'$.

We are interested in the case when the equivalence classes contain only d-essential mappings or only d-inessential mappings. A sufficient condition to have such a case is the following one:

(H) if $F \approx F'$ then there is $H\colon [0, 1] \times X \to Y$ such that $H(1, \cdot) = F$, $H(0, \cdot) = F'$, $\mathrm{cl}(\bigcup\{H(t, \cdot)^{-1}(B); t \in [0, 1]\}) \cap A = \varnothing$ and $H(\eta(\cdot), \cdot) \in \mathfrak{A}_A^B(X, Y)$ for any continuous function $\eta\colon X \to [0, 1]$ satisfying $\eta(x) = 1$ for all $x \in A$.

In what follows we shall assume that conditions (A) and (H) hold.

Using similar arguments to those in [2] we can prove the following lemma.

LEMMA 1. Let F be an admissible mapping. Then F is d-inessential if and only if $d(F^{-1}(B)) = \theta$ or there exists an admissible mapping F' such that

$$F \approx F' \qquad \text{and} \qquad d(F^{-1}(B)) \neq d(F'^{-1}(B)). \tag{2}$$

Proof. The necessity follows by the definition of d-inessential mappings and condition (A).

Suppose now that F' satisfies (2). Let H be a mapping associated with F and F' as in condition (H). Denote

$$Z = \bigcup\{H(t, \cdot)^{-1}(B); t \in [0, 1]\}.$$

If $Z = \varnothing$, then $H(1, \cdot)^{-1}(B) = \varnothing$, i.e. $F^{-1}(B) = \varnothing$ and so $d(F^{-1}(B)) = \theta$, which means that F is d-inessential. Next, assume $Z \neq \varnothing$. According to condition (H), the nonvoid closed subsets A and $\mathrm{cl}(Z)$ of the normal topological space X are disjoint. So, by Urysohn's theorem (see [2, theorem 7.4.1]), there exists a continuous function $\eta\colon X \to [0, 1]$ such that

$$\eta(x) = 1 \text{ for } x \in A \qquad \text{and} \qquad \eta(x) = 0 \text{ for } x \in \mathrm{cl}(Z).$$

The mapping $F^* = H(\eta(\cdot), \cdot)$ is admissible,

$$F|_A = F^*|_A \qquad \text{and} \qquad F'^{-1}(B) = F^{*-1}(B). \tag{3}$$

Therefore,

$$d(F^{-1}(B)) \neq d(F'^{-1}(B)) = d(F^{*-1}(B)). \tag{4}$$

The relations (3) and (4) show that F is d-inessential. The proof of lemma 1 is complete.

Our generalization of the topological transversality theorem is the follow theorem.

THEOREM 1. Let F and F' be two admissible mappings such that

$$F \approx F'.$$

Then F and F' are both d-essential or both d-inessential and in the first case one has

$$d(F^{-1}(B)) = d(F'^{-1}(B)) \neq \theta. \tag{5}$$

Proof. Assume F is d-inessential. If $d(F'^{-1}(B)) = \theta$ then, clearly, F' is d-inessential. Thus, we may assume $d(F'^{-1}(B)) \neq \theta$.

In the case $d(F^{-1}(B)) = \theta$, the d-inessentiality of F' follows from lemma 1 and the symmetry of relation $\approx$. Suppose now that $d(F^{-1}(B)) \neq \theta$ too. Then, lemma 1 implies that there exists an admissible mapping F'' such that

$$F \approx F'' \qquad \text{and} \qquad d(F^{-1}(B)) \neq d(F''^{-1}(B)).$$

Using the symmetry and the transitivity of relation $\approx$, from $F \approx F'$ and $F \approx F''$, we obtain that

$$F' \approx F''.$$

Now, if $d(F'^{-1}(B)) \neq d(F''^{-1}(B))$, then lemma 1 applied to F' and F'' implies that F' is d-inessential, while if $d(F'^{-1}(B)) = d(F''^{-1}(B))$, we deduce that $d(F'^{-1}(B)) \neq d(F^{-1}(B))$ and the d-inessentiality of F' follows once again from lemma 1, this time applied to F' and F. Therefore, F' is also d-inessential.

Now suppose that F and F' are d-essential mappings. Clearly, $d(F^{-1}(B)) \neq \theta$ and $d(F'^{-1}(B)) \neq \theta$. Since $F \approx F'$, we can construct, as in the proof of lemma 1, an admissible mapping F^* such that

$$F|_A = F^*|_A \qquad \text{and} \qquad F'^{-1}(B) = F^{*-1}(B).$$

Consequently,

$$d(F'^{-1}(B)) = d(F^{*-1}(B)). \tag{6}$$

On the other hand, since $F|_A = F^*|_A$ and F is d-essential, we must have

$$d(F^{-1}(B)) = d(F^{*-1}(B)). \tag{7}$$

Now (5) follows from (6) and (7).

2. CONNECTION WITH THE TOPOLOGICAL DEGREE THEORY

In addition to the assumptions made in Section 1 we shall suppose here that

$$\text{if } F|_A = F'|_A \qquad \text{then } d(F^{-1}(B)) = d(F'^{-1}(B)). \tag{8}$$

Then, obviously, an admissible mapping F is d-essential if and only if $d(F^{-1}(B)) \neq \theta$ and it is d-inessential if and only if $d(F^{-1}(B)) = \theta$.

Under assumption (8), theorem 1 says that the equivalence classes contain mappings F having the same "measure" $d(F^{-1}(B))$ for the counter-image $F^{-1}(B)$ of B. Therefore,

$$\text{if } F \approx F' \qquad \text{then } d(F^{-1}(B)) = d(F'^{-1}(B)). \tag{9}$$

Now suppose that no equivalence relation is known on $\mathfrak{A}_A^B(X, Y)$ and introduce the following natural question: if F and F' are admissible mappings, when can we say that $d(F^{-1}(B)) = d(F'^{-1}(B))$? An answer to this question is the following theorem.

THEOREM 2. Assume that d satisfies condition (8) and let F and F' be two admissible mappings. If there exists a mapping H: $[0, 1] \times X \to Y$ such that

(i) $H(0, \cdot) = F'$ and $H(1, \cdot) = F$;

(ii) $H(t_1, \cdot)|_A \neq H(t_2, \cdot)|_A$ for $t_1 \neq t_2$;

(iii) $H(\eta_\lambda(\cdot), \cdot)$ is admissible for each $\eta_\lambda \in C(X; [0, 1])$ with $\eta_\lambda(x) = \lambda$ for all $x \in A$, and every $\lambda \in [0, 1]$;

(iv) $\mathrm{cl}(\bigcup\{H(t,\cdot)^{-1}(B);\ t \in [0,1]\}) \cap A = \varnothing$,
then

$$d(F^{-1}(B)) = d(F'^{-1}(B)).$$

Proof. Consider

$$\underline{\mathfrak{A}}_A^B(X,Y) = \{G\colon X \to Y;\ G(x) = H(\eta_\lambda(x), x) \text{ for } x \in X,\ \lambda \in [0,1],\ \eta_\lambda \in C(X;[0,1]),$$
$$\eta_\lambda(x) = \lambda \text{ for } x \in A\}. \tag{10}$$

By (i) and (iii) we have

$$F, F' \in \underline{\mathfrak{A}}_A^B(X,Y) \subset \mathfrak{A}_A^B(X,Y).$$

Also, for $G = H(\eta_\lambda(\cdot),\cdot)$ and $G' = H(\eta_{\lambda'}(\cdot),\cdot)$, say

$$G \simeq G' \qquad \text{if and only if } \lambda = \lambda' \text{ or } \{\lambda,\lambda'\} = \{0,1\}.$$

Obviously, $\simeq$ is an equivalence relation on the set (10) which, by (ii), satisfies condition (A). Moreover, if for $G \simeq G'$ we set

$$\underline{H}(t,x) = H((1-t)\eta_{\lambda'}(x) + t\eta_\lambda(x), x)$$

then, since

$$\bigcup\{\underline{H}(t,\cdot)^{-1}(B);\ t \in [0,1]\} \subset \bigcup\{H(t,\cdot)^{-1}(B);\ t \in [0,1]\},$$

by (iv), we see that the relation $\simeq$ also satisfies condition (H) with $\underline{H}$ in place of H and with respect to the class (10). Thus, theorem 1 is applicable and since $F \simeq F'$, the conclusion follows by (8).

Remark 1. Theorem 2 can be used in order to derive the homotopy invariance of the topological degree from the axiom of boundary value dependence and the axiom of solution. In fact, condition (8) corresponds to the axiom of boundary value dependence of the topological degree, while the condition $d(\varnothing) = \theta$, or equivalently

$$d(F^{-1}(B)) \neq \theta \qquad \text{implies } F^{-1}(B) \neq \varnothing,$$

corresponds to the axiom of solution. This permits an axiomatic treatment of the topological degree based on the following axioms: (1) axiom of normalization; (2) axiom of additivity; (3) axiom of boundary value dependence; (4) axiom of solution; and (5) axiom of invariance to translations. In this approach, contrary to the papers [3, 4], the homotopy invariance will be a theorem and not an axiom.

Remark 2. In theorem 1 no compatibility between the "measure" d and the class of admissible mappings is assumed. In fact, for that theorem 1 becomes useful, we need that simple d-essential mappings can be identified.

Remark 3. When we only deal with the existence of solutions to (1) and we do not have to "measure" the set of all solutions, it is sufficient to take as d the simplest indicator function

$$d(\varnothing) = 0 \qquad \text{and} \qquad d(C) = 1 \qquad \text{for } C \neq \varnothing \tag{11}$$

taking $\Delta = \{0, 1\}$ and $\theta = 0$. In this case, theorem 1 was proved in our previous paper [9]. When d is given by (11), we concisely speak about essentiality instead of d-essentiality.

In the next sections we shall assume that d is given by (11).

3. CONNECTION WITH FIXED POINT THEORY

Using theorem 1 and a fixed point theorem for mappings leaving invariant a subset of X, we can prove, under some suitable conditions, the existence of fixed points for mappings $f: X \to E$, $X \subset E$, which generally do not leave X invariant.

Let E be a normal topological space, X and A two proper closed subsets of E, $A \subset X$, $A \neq X$. Consider a nonvoid class of mappings

$$\mathcal{C}_A(X, E) \subset \{f: X \to E;\ \mathrm{Fix}(f) \cap A = \emptyset\}$$

where $\mathrm{Fix}(f)$ stands for the set of all fixed points of f. The mappings in $\mathcal{C}_A(X, E)$ are called *admissible*. An admissible mapping f is said to be *essential* if

$$f' \in \mathcal{C}_A(X, E), \qquad f|_A = f'|_A \qquad \text{imply } \mathrm{Fix}(f') \neq \emptyset.$$

Otherwise, f is said to be *inessential*.

Also consider an equivalence relation $\sim$ on $\mathcal{C}_A(X, E)$ and assume that the following conditions are satisfied for f and f' in $\mathcal{C}_A(X, E)$:

(a) if $f|_A = f'|_A$, then $f \sim f'$;

and

(h) if $f \sim f'$, then there is $h: [0, 1] \times X \to E$ such that $h(1, \cdot) = f$, $h(0, \cdot) = f'$, $\mathrm{cl}(\bigcup\{\mathrm{Fix}(h(t, \cdot));\ t \in [0, 1]\}) \cap A = \emptyset$ and $h(\eta(\cdot), \cdot) \in \mathcal{C}_A(X, E)$ for any continuous function $\eta: X \to [0, 1]$ satisfying $\eta(x) = 1$ for all $x \in A$.

It is easy to see that if we put

$$Y = X \times E, \qquad B = \{(x, x);\ x \in X\},$$

$\mathcal{C}_A^B(X, Y) = \{F: X \to Y;\ F(x) = (x, f(x)) \text{ for all } x \in X;\ f \in \mathcal{C}_A(X, E)\}$ and

$$F \approx F' \text{ if and only if } f \sim f'$$

where $F(x) = (x, f(x))$ and $F'(x) = (x, f'(x))$, then F is essential (inessential) in the sense of Section 1 if and only if its corresponding mapping f is essential (inessential) in the sense of this section. Moreover, condition (a) is equivalent to condition (A) and condition (h) on $\sim$ implies condition (H) on $\approx$, where

$$H(t, x) = (x, h(t, x)).$$

Consequently, theorem 1 yields the following proposition.

PROPOSITION 1. If f and f' are in $\mathcal{C}_A(X, E)$ and $f \sim f'$, then f and f' are both essential or both inessential.

The next proposition gives us a scheme to establish the essentiality of some admissible mappings. This is done in terms of fixed point structures.

By a *fixed point structure* on a certain space E we mean a pair (S, M), where S is a class of nonvoid subsets of $E(S \subset P(E))$ and M is a mapping attaching to each set $D \in S$, a family $M(D)$ of mappings from D into D having at least one fixed point each.

PROPOSITION 2. Let (S, M) be a fixed point structure on the normal topological space E and let $f_0 \in \mathfrak{A}_A(X, E)$. If for every $f \in \mathfrak{A}_A(X, E)$ satisfying $f|_A = f_0|_A$, there exist $D_f \in S$ and $\tilde{f} \in M(D_f)$ such that

$$\tilde{f}|_{X \cap D_f} = f|_{X \cap D_f}$$

and

$$\mathrm{Fix}(\tilde{f}) \backslash X = \varnothing,$$

then f_0 is essential.

The proof of proposition 2 is immediate and can be found, together with several examples and applications, in the paper [9] (see also [10]). Another example is described in the next section.

4. MÖNCH TYPE PERTURBATIONS OF LINEAR FREDHOLM MAPPINGS OF INDEX ZERO

Let E be a real Banach space, E_1 a real normed space and let $L: D(L) \subset E \to E_1$ be a linear Fredholm mapping of index zero, i.e.

$$\dim \ker L = \operatorname{codim} R(L) < \infty \qquad \text{and} \qquad R(L) \text{ is closed.}$$

If X is a metric space and $N: X \to E_1$, we say that N is *L-compact* (*L-completely continuous, L-condensing, L-continuous*) on X, if there exists a linear continuous mapping $\Phi: E \to E_0$, $E_1 = R(L) \oplus E_0$, such that $L + \Phi: D(L) \to E_1$ is bijective and $(L + \Phi)^{-1}N: X \to E$ is compact (completely continuous, condensing, continuous, respectively) on X. Also, we say that N is a *L-Mönch* mapping on X if $X \subset E$ and there exists a linear continuous mapping $\Phi: E \to E_0$ such that: $L + \Phi$ is bijective; $(L + \Phi)^{-1}N$ is continuous; if $E_0 \neq \{0\}$, then $(L + \Phi)^{-1}N(X)$ is bounded; and there is $x \in X$ such that

$$C \subset \overline{\operatorname{conv}}(\{x\} \cup (L + \Phi)^{-1}N(C)) + K \qquad \text{implies } \bar{C} \text{ is compact,} \tag{12}$$

whenever $C \subset X$ is countable and $K \subset \ker L$ is compact.

These definitions do not depend on Φ. This easily follows by using the formula

$$(L + \Phi_1)^{-1}N = (L + \Phi_2)^{-1}N + (L + \Phi_1)^{-1}(\Phi_2 - \Phi_1)(L + \Phi_2)^{-1}N \tag{13}$$

and the complete continuity of

$$(L + \Phi_1)^{-1}(\Phi_2 - \Phi_1): E \to \ker L \subset E \qquad \text{(see [6, p. 123]).}$$

Obviously, we have that

L-compact $\Rightarrow$ L-completely continuous $\Rightarrow$ L-condensing $\Rightarrow$ L-continuous and if $X \subset E$ and X is bounded, then L-condensing $\Rightarrow$ L-Mönch $\Rightarrow$ L-continuous.

THEOREM 3. Let $\Omega \subset E$ be an open set and $x_0 \in D(L) \cap \Omega$. Let $\psi: E \to E_0$ be a linear mapping which is L-completely continuous on E and L-compact on $\bar{\Omega}$ and with $\ker(L + \psi) = \{0\}$. Also, consider $N: \bar{\Omega} \to E_1$ an L-Mönch mapping and assume that

$$L(x) + (1 - t)(\psi(x) - z) - tN(x) \neq 0$$

for all $x \in D(L) \cap \partial\Omega$ and $t \in [0, 1]$, where $z = (L + \psi)(x_0)$. Then, the equation

$$L(x) = N(x) \tag{14}$$

has at least one solution in $D(L) \cap \Omega$.

Proof. Recall that, since ψ is linear, $\ker(L + \psi) = \{0\}$ and ψ is L-completely continuous, the mapping $L + \psi$ is bijective and

$$I + (L + \Phi)^{-1}(\psi - \Phi): E \to E$$

is a linear homeomorphism of E, for each linear continuous mapping $\Phi: E \to E_0$ for which $L + \Phi$ is bijective (see [6, p. 124]). Further, using the L-compactness of ψ on $\bar{\Omega}$ and formula (13), we easily see that the mapping

$$(L + \psi)^{-1}\psi$$

is compact on $\bar{\Omega}$. Similarly, $(L + \psi)^{-1}N$ is continuous on $\bar{\Omega}$.

Now, equation (14) is equivalent to

$$x = (L + \psi)^{-1}(N + \psi)(x). \tag{15}$$

We shall first prove the existence of a solution to (15) in the case $\Omega \neq E$, i.e. $\partial\Omega \neq \emptyset$. The proof will be divided into two steps.

Step 1. Application of proposition 1. Consider the class of mappings

$$\begin{aligned} \mathfrak{A}_{\partial\Omega}(\bar{\Omega}, E) = \{f; f(x) = x_0 + \eta_\lambda(x)(L + \psi)^{-1}(N(x) + \psi(x) - z),\ \lambda \in [0, 1], \\ \eta_\lambda \in C(\bar{\Omega}; [0, 1]),\ \eta_\lambda(x) = \lambda \text{ for } x \in \partial\Omega\} \end{aligned} \tag{16}$$

and define an equivalence relation on the set (16):

$$\begin{aligned} f \sim f' \qquad & \text{iff } \lambda = \lambda' \text{ or } \{\lambda, \lambda'\} = \{0, 1\}, \text{ in case } N + \psi \not\equiv z \text{ on } \partial\Omega \\ f|_{\partial\Omega} = f'|_{\partial\Omega}, \qquad & \text{otherwise.} \end{aligned}$$

Here $f = x_0 + \eta_\lambda(L + \psi)^{-1}(N + \psi - z)$ and $f' = x_0 + \eta_{\lambda'}(L + \psi)^{-1}(N + \psi - z)$.

Now if $f|_{\partial\Omega} = f'|_{\partial\Omega}$, then obviously $f \sim f'$ in case $N + \psi \equiv z$ on $\partial\Omega$. In the opposite case, we deduce $\lambda = \lambda'$ and so $f \sim f'$, as well. Thus, condition (a) is satisfied.

On the other hand, if $f \sim f'$ and we set

$$\begin{aligned} h(t, x) &= tf(x) + (1 - t)f'(x) \\ &= x_0 + [t\eta_\lambda(x) + (1 - t)\eta_{\lambda'}(x)](L + \psi)^{-1}(N(x) + \psi(x) - z), \end{aligned}$$

we easily see that condition (h) also holds. Therefore, proposition 1 applies and since

$$(L + \psi)^{-1}(N + \psi) \sim x_0,$$

it remains only to prove that the constant mapping x_0 is essential.

Step 2. Application of proposition 2. We shall use Mönch's fixed point structure (S, M) on the Banach space E, where S is the class of all nonempty closed convex subsets of E and for each

$D \in S$, $M(D)$ is the set of all continuous mappings $\tilde{f}: D \to D$ for which there is some $x \in D$ such that

$$\bar{C} = \overline{\operatorname{conv}}(\{x\} \cup \tilde{f}(C)) \qquad \text{implies } \bar{C} \text{ is compact,} \tag{17}$$

whenever $C \subset D$ is countable (see [8, theorem 2.1]).

In order to show that the constant mapping x_0 is essential in the class (16) we have to prove that any mapping f in (16) satisfying $f|_{\partial\Omega} \equiv x_0$, has a fixed point. Suppose

$$f(x) = x_0 + \eta(x)(L + \psi)^{-1}(N(x) + \psi(x) - z) \qquad \text{for } x \in \bar{\Omega} \tag{18}$$

where $\eta \in C(\bar{\Omega}; [0, 1])$ is constant on $\partial\Omega$. Let

$$D_f = \overline{\operatorname{conv}} f(\bar{\Omega}) \qquad \text{and} \qquad \tilde{f}: D_f \to D_f,$$

$$\tilde{f}(x) = f(x) \text{ if } x \in \bar{\Omega} \qquad \text{and} \qquad \tilde{f}(x) = x_0 \text{ if } x \notin \bar{\Omega}.$$

Clearly, $D_f \in S$ and $\tilde{f}$ is continuous.

Now we show that $\tilde{f} \in M(D_f)$, i.e. $\tilde{f}$ satisfies condition (17). For this, let $C \subset D_f$ be countable so that

$$\bar{C} = \overline{\operatorname{conv}}(\{x_0\} \cup \tilde{f}(C)) = \overline{\operatorname{conv}}(\{x_0\} \cup f(C \cap \bar{\Omega})).$$

Since, by (18)

$$f(x) = (1 - \eta(x))x_0 + \eta(x)(L + \psi)^{-1}(N(x) + \psi(x)),$$

we deduce that

$$C \subset \overline{\operatorname{conv}}(\{x_0\} \cup (L + \psi)^{-1}(N + \psi)(C \cap \bar{\Omega})). \tag{19}$$

On the other hand, the set

$$(L + \psi)^{-1}\psi(C \cap \bar{\Omega}) \subset \ker L$$

is relatively compact. Consequently, from (19) we obtain

$$C \subset \overline{\operatorname{conv}}(\{x_0\} \cup (L + \psi)^{-1}N(C \cap \bar{\Omega})) + K, \tag{20}$$

where $K \subset \ker L$ is a compact set. Since N is an L-Mönch mapping, relation (20) implies that $\bar{C} \cap \bar{\Omega}$ is a compact set. Further, the continuity of $(L + \psi)^{-1}N$ implies that $(L + \psi)^{-1}N(C \cap \bar{\Omega})$ is relatively compact and Mazur's lemma that the second member in (20) is also compact. Hence $\bar{C}$ is compact and consequently, $\tilde{f}$ satisfies condition (17), as claimed.

Finally, since the two conditions on $\tilde{f}$ in proposition 2 are obviously satisfied, the constant mapping x_0 is essential and the proof is complete in the case $\Omega \neq E$.

In the case $\Omega = E$ it is easily seen that the mapping $\tilde{f}: E \to E$,

$$\tilde{f}(x) = (L + \psi)^{-1}(N + \psi)$$

satisfies condition (17) and consequently, it has a fixed point. The proof of theorem 3 is thus complete.

Remark 4. If in theorem 3 the set Ω is bounded, then the L-compactness of ψ on $\bar{\Omega}$ follows from its L-complete continuity on E. For Ω bounded, ψ L-completely continuous on E and N L-condensing on $\bar{\Omega}$, theorem 3 reduces to a result of Volkmann [7].

Remark 5. If we take $E_1 = E$ and $L = I$, the identity of E, then theorem 3 reduces to a fixed point result due to Mönch [8, theorem 2.2].

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