STUDIA UNIV. BABEŞ-BOLYAI, MATHEMATICA, XXXIX, 1, 1994

PERIODIC SOLUTIONS FOR AN INTEGRAL EQUATION FROM BIOMATHEMATICS VIA LERAY-SCHAUDER PRINCIPLE

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Received: May 12, 1994
AMS subject classification: 47H15, 47N60, 34K15

REZUMAT. - Soluţii periodice pentru o ecuaţie integrală din biomatematică via principiul lui Leray-Schauder. Rezultatele stabilite în această lucrare se referă la existenţa, unicitatea şi aproximarea monoton-iterativă a soluţiilor periodice netriviale pentru ecuaţia integrală (1). Demonstraţiile se bazează pe principiul de continuare al lui Leray-Schauder şi pe tehnica iteraţiilor monotone.

Abstract. The main results of this paper concern the existence, the uniqueness and the monotone iterative approximation of periodic nontrivial solutions for the delay integral equation $x(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds$. The proofs are achieved by the Leray-Schauder continuation principle and the monotone iterative technique.

1. Introduction. The delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds \tag{1}$$

is a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $x(t)$ represents the fraction of infectives in the total population at time t, τ is the leangth of time an individual is infective and $f(t, x(t))$ is the proportional of new infectives per unit of time.

In [1-4, 6-11] sufficient conditions were given for the existence of nontrivial ω-periodic continuous solutions to Eq. (1) in case of a ω-periodic contact rate

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$$f(t+\omega,x) = f(t,x), \quad f(t,0) = 0.$$

The tools were Banach's fixed point theorem, topological fixed point principles, fixed point index theory, monotonicity technique.

In [7] we used the Leray-Schauder continuation principle (in Granas' approach) to prove the existence of positive continuous solutions $x(t)$ for Eq. (1) on a given interval $[-\tau,T]$, when it is known the proportion $\phi(t)$ of infectives for $-\tau \le t \le 0$, i.e.

$$x(t) = \phi(t) \quad \text{for } -\tau \le t \le 0. \tag{2}$$

Clearly, we had to assume that ϕ satisfies the following condition

$$\phi_0 = \phi(0) = \int_{-\tau}^{0} f(s,\phi(s))\,ds. \tag{3}$$

Under condition (3) the problem (1)-(2) is equivalent with the initial values problem

$$x'(t) = f(t,x(t)) - f(t-\tau,x(t-\tau)) \quad \text{for } 0 \le t \le T \tag{4}$$

$$x(t) = \phi(t) \quad \text{for } -\tau \le t \le 0.$$

We made use of the following hypotheses:

(i) $f(t,x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $x \ge 0$.

(ii) $\phi(t)$ is continuous, $0 < a \le \phi(t)$ for $-\tau \le t \le 0$ and satisfies (3).

(iii) there exists an integrable function $g(t)$ such that

$$f(t,x) \ge g(t) \quad \text{for } -\tau \le t \le T \text{ and } x \ge a$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \ge a \quad \text{for } 0 \le t \le T.$$

(iv) there exists a positive function $h(x)$ such that $1/h(x)$ is locally integrable on $[a,\infty)$,

$$f(t,x) \le h(x) \quad \text{for } 0 \le t \le T \text{ and } x \ge a$$

and

$$T < \int_{\phi_0}^{\infty} (1/h(x))\,dx.$$

THEOREM A [7]. *Suppose that* (i)-(iv) *are satisfied. Then* Eq. (1) *has at least one continuous solution* $x(t)$, $x(t) \geq a$, *for* $-\tau \leq t \leq T$, *which satisfies* (2).

Approximation schema to solve (1)-(2) under assumptions (i)-(iv), based on the monotone iterative method of Lakshmikantham (see [5]), were described in [8] for the cases where $f(t,x)$ is nondecreasing or nonincreasing with respect to x.

In this paper we shall use a similar technique based upon the Leray-Schauder continuation principle, to establish a new existence result for the periodic solutions of Eq. (1). Finally, the monotone iterative technique is used to prove the uniqueness of the solution and to approximate it, in case $f(t,x)$ is nonincreasing with respect to x.

In [4] (see also [6]) the following conditions are used:

(h_1) $f(t,x)$ is nonnegative and continuous for $-\infty < t < \infty$ and $x \geq 0$.

(h_2) $f(t,0) = 0$ for $-\infty < t < \infty$ and there exists $\omega > 0$ such that $f(t+\omega,x) = f(t,x)$ for $-\infty < t < \infty$ and $x \geq 0$.

(h_3) there exist $0 < a < R$ and a nonnegative locally integrable function $g(t)$ with period ω such that

$$f(t,x) \geq g(t) \quad \text{for } 0 \leq t \leq \omega \text{ and } a \leq x \leq R,$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \geq a \quad \text{for } 0 \leq t \leq \omega.$$

(h_4) $f(t,x) \leq R/\tau$ for $0 \leq t \leq \omega$ and $a \leq x \leq R$.

One of the results in [4] is the following theorem whose proof is based on Schauder's fixed point theorem.

THEOREM B [4]. *If* (h_1)-(h_4) *are satisfied, then* Eq. (1) *has at least one positive and continuous solution* $x(t)$ *with period* ω *and*

$$a \leq \inf_{-\infty<t<\infty} x(t) \leq \sup_{-\infty<t<\infty} x(t) \leq R.$$

Let us remark that for a given function $f(t,x)$ satisfying (h_1) and (h_2) could exist several intervals $[a,R]$ such that (h_3) and (h_4) hold. If these intervals are disjoint, then Theorem B ensures that the corresponding solutions are distinct. For example, if $f(t,x) = x/\tau$ for $-\infty < t < \infty$ and $x \geq 0$, we may take arbitrary $\omega > 0$, $a > 0$ and $R > a$. Clearly, in this case, any nonnegative constant is a solution.

Assumption (h_4) is essential for the domain invariance in Schauder's fixed point theorem. We shall replace (h_4) by another condition which guarantes that the Leray-Schauder boundary condition is satisfied.

We shall see that there are cases where our main existence result, Theorem 1, applies and Theorem B does not, and conversely.

2. Main existence result. We use instead of (h_4) the following hypothesis:

(h_5) there exists a positive function $h(x)$ such that $1/h(x)$ is integrable for $a \leq x \leq R$, and a number b such that $a < b < R$,

$$f(t,x) \leq h(x) \quad \text{for } 0 \leq t \leq \omega \text{ and } a \leq x \leq R, \tag{5}$$

$$\int_b^R (1/h(x))\,dx \geq \omega \tag{6}$$

and

$$f(t,x) < b/\tau \quad \text{for } 0 \leq t \leq \omega \text{ and } b \leq x \leq R. \tag{7}$$

THEOREM 1. *Suppose that* (h_1)-(h_3) *and* (h_5) *are satisfied. Then,* Eq. (1) *has at least one continuous solution* $x(t)$ *with period* ω *and*

$$a \leq \inf_{-\infty<t<\infty} x(t) < b \text{ and } \sup_{-\infty<t<\infty} x(t) < R. \tag{8}$$

Proof. Let E be the real Banach space of all continuous and ω-periodic functions $x(t)$, with norm

$$\|x\| = \sup_{-\infty<t<\infty} |x(t)| = \max_{0\le t\le\omega} |x(t)|.$$

Let $K = \{x \in E\,;\ a \le x(t) \text{ for } 0 \le t \le \omega\}$ and

$$U = \{\, x \in K;\ \min_{0\le t\le\omega} x(t) < b \ \text{ and } \ \|x\| < R \,\}.$$

Obviously, K is a closed convex subset of E and U is bounded and open in K. Consider the homotopy

$$H : [0,1] \times \bar{U} \to K,$$

$$H(\lambda, x)(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s, x(s))\,ds$$

for $0 \le \lambda \le 1$, $x \in \bar{U}$ and $0 \le t \le \omega$. It is easy to show that, by (h_1)-(h_2), H is well-defined and completely continuous. We claim now that H satisfies the Leray-Schauder condition on the boundary ∂U of U with respect to K, i.e. $H(\lambda, \cdot)$ is fixed point free on ∂U for each $\lambda \in [0,1]$. Assume, by contradiction, that there would exist $\lambda \in (0,1]$ and $x \in \partial U$ such that $H(\lambda,x)=x$, that is

$$x(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s, x(s))\,ds \quad \text{for } -\infty < t < \infty. \tag{9}$$

Since x is on ∂U, we have either

$$\|x\| = R \ \text{ and } \ \min_{0\le t\le\omega} x(t) < b, \tag{10}$$

or

$$\|x\| \le R \ \text{ and } \ \min_{0\le t\le\omega} x(t) = b. \tag{11}$$

First, suppose (10). Then, by differentiating (9), we obtain

$$x'(t) = \lambda f(t, x(t)) - \lambda f(t-\tau, x(t-\tau)).$$

Hence, by (h_1) and (5), we have

$$x'(t) \le \lambda f(t, x(t)) \le \lambda h(x(t)) \le h(x(t)).$$

Let $0 \le t_0 \le \omega$ be such that $x(t_0) = \min_{0\le t\le\omega} x(t)$. Integration from t_0 to t yields

$$\int_{t_0}^{t}(x'(s)/h(x(s)))\,ds \le t - t_0 \le \omega \quad \text{for} \quad t_0 \le t \le t_0 + \omega .$$

Thus,

$$\int_{x(t_0)}^{x(t)}(1/h(u))\,du \le \omega \quad \text{for} \quad t_0 \le t \le t_0 + \omega .$$

Since $x(t_0) < b$, by (6), we deduce that $x(t) < R$ for all $t_0 \le t \le t_0 + \omega$, equivalently for all $-\infty < t < \infty$. Therefore, $\|x\| < R$, a contradiction. Next, suppose (11). Let $0 \le t_0 \le \omega$ be such that $x(t_0) = \min_{0 \le t \le \omega} x(t) = b$. Then, by (9) and (7), we obtain

$$\begin{aligned} b = x(t_0) = (1-\lambda)a + \lambda \int_{-\tau}^{t_0} f(s, x(s))\,ds < \\ < (1-\lambda)b + \lambda b = b, \end{aligned} \tag{12}$$

again a contradiction. Thus, H is an admissible homotopy on $\bar{U}$. On the other hand, the mapping $H(0,\cdot)$ is essential (its fixed point index $i(H(0,\cdot), U, K)$ equals 1) because $H(0,\cdot)=a$ and the constant function a belongs to U. Consequently, by the Leray-Schauder principle, $H(1,\cdot)$ is essential too. Therefore, there exists at least one fixed point of $H(1,\cdot)$ in U, that is a continuous solution with period ω for Eq. (1) satisfying (8). Thus, Theorem 1 is proved.

Remark 1. Theorem 1 remains true if instead of (7) we only assume that there exists a locally integrable function $\ell(t)$ with period ω such that

$$f(t, x) \le \ell(t) \quad \text{for} \quad 0 \le t \le \omega \quad \text{and } b \le x \le R$$

and

$$\int_{-\tau}^{t} \ell(s)\,ds < b \quad \text{for} \quad 0 \le t \le \omega .$$

Indeed, under this more general assumption, the strict inequality (12) also holds.

Remark 2. Here is an example for which Theorem 1 applies but Theorem B does not. Let $\tau = \omega = 1$ and let $f(t, x) = h_1(x)$ $(-\infty < t < \infty)$, where

$$\begin{aligned} h_1(x) &= 5x && \text{for } 0 \le x \le 1 \\ &= -4x + 9 && \text{for } 1 \le x \le 2 \\ &= 1 && \text{for } 2 \le x \le 3 \\ &= 3x - 8 && \text{for } 3 \le x \le 5 \\ &= x + 2 && \text{for } 5 \le x . \end{aligned}$$

Conditions (h_1)-(h_3) and (h_5) are fulfilled with $a = 1$, $b = 2$, $R = 3$, $g(t) = 1$ and $h(x) = h_1(x)$, but for any $R > 0$ there is no $a < R$ such that (h_3)-(h_4) be satisfied.

Remark 3. For a given function $f(t,x)$ satisfying (h_1)-(h_2) there could exist several intervals $[a, R]$ such that (h_3) and (h_5) hold. If these intervals are disjoint, then the corresponding solutions by Theorem 1 are distinct. Here is an example: Let τ and $h_1(x)$ be as in Remark 2 and let

$$f(t,x) = g_1(t)h_n(x) \text{ for } -\infty < t < \infty,\ 4(n-1) \le x \le 4n,\ n = 1, 2, \ldots,$$

where $h_n(x) = 4(n-1) + h_1(x - 4(n-1))$ and $g_1(t)$ is any continuous nonnegative function with a period $\omega > 0$ such that

$$\int_{t-1}^{t} g_1(s)\,ds \ge 1 \text{ for } 0 \le t \le \omega.$$

It is easy to see that if $\omega \cdot \max_{0 \le t \le \omega} g_1(t) \le (4(n-1)+1)^{-1}$, all the assumptions of Theorem 1 are fulfilled for $a = 4(n-1)+1$, $b = 4(n-1)+2$, $R = 4(n-1)+3$, $g(t) = (4(n-1)+1)g_1(t)$ and $h(x) = \max_{0 \le t \le \omega} g_1(t)\,h_n(x)$. Therefore, for each nonnull natural number n so that $4(n-1)+1 \le (\omega \max_{0 \le t \le \omega} g_1(t))^{-1}$, Eq. (1) has at least one continuous solution $x_n(t)$ with period ω, such that

$$4(n-1) + 1 \le \inf_{-\infty < t < \infty} x_n(t) < 4(n-1) + 2$$

and

$$\sup_{-\infty < t < \infty} x_n(t) < 4(n-1) + 3.$$

For example, in case $g_1(t) = 1$, such solutions are the following constant ones

$$x_n(t) = 4(n-1) + 9/5,\ n = 1, 2, \ldots.$$

Notice that none of these constant solutions can be obtained by means of Theorem B.

Example 1. Let us give another function which satisfies the assumptions of Theorem 1:

$$f(t,x) = g_0(t)h(x),\ -\infty < t < \infty,\ x \ge 0,$$

where $h(x) = (x - 1/2)(x - 2)(x - 3) + 3$ and $g_0(t)$ is a continuous function with period

$$\omega \leq \int_{2.6}^{2.7} (1/h(x))\,dx$$

and satisfies the following conditions

$$0 \leq g_0(t) \leq 1 \text{ for } 0 \leq t \leq \omega,$$

$$\int_{t-1}^{t} g_0(s)\,ds \geq 1/h((11+\sqrt{19})/6) \text{ for } 0 \leq t \leq \omega.$$

For this function we take $\tau = 1$, $a = 1$, $b = 2.6$, $R = 2.7$ and $g(t) = h((11+\sqrt{19})/6)\,g_0(t)$, where $h((11+\sqrt{19})/6) > 1$ is the minimum of $h(x)$ for $1 \leq x \leq 2.7$.

3. Uniqueness and monotone iterative approximation schema. Under the assumptions of Theorem 1, denote by A the completely continuous operator from $P = \{x \in E;\ 0 \leq x(t) \text{ for } 0 \leq t \leq \omega\}$ into P,

$$Ax(t) = \int_{t-\tau}^{t} f(s, x(s))\,ds, \quad -\infty < t < \infty,\ x \in P.$$

Also define the following sequence of functions in P :

$$v_0(t) = R,\ v_n(t) = Av_{n-1}(t),\ n = 1, 2, \ldots.$$

THEOREM 2. *Let* (h_1)-(h_3) *and* (h_5) *hold and suppose that* $f(t, x)$ *is nonicreasing in* x *for* $a \leq x \leq R$ *and there exists* $\alpha \in (-1, 0)$ *such that*

$$f(t, \gamma x) \leq \gamma^{\alpha} f(t, x) \tag{13}$$

for all $t \in [0, \omega]$, $\gamma \in (0, 1)$ *and* $x \in [a, R]$ *with* $\gamma x \in [a, R]$. *If*

$$A^2(R)(t) \leq R \quad \text{for } 0 \leq t \leq \omega, \tag{14}$$

then Eq. (1) *has a unique continuous solution* $x^*(t)$ *of period* ω *such that* $a \leq x^*(t) \leq R$ *for* $0 \leq t \leq \omega$. *Moreover,*

$$a \leq v_1(t) \leq v_3(t) \leq \ldots \leq v_{2n+1}(t) \leq \ldots \leq x^*(t) \leq \ldots$$
$$\leq v_{2n}(t) \leq \ldots \leq v_4(t) \leq v_2(t) \leq v_0(t) = R \text{ for } t \in [0, \omega], \tag{15}$$

$v_n(t) \to x^*(t)$ *uniformly for* $0 \le t \le \omega$ *as* $n \to \infty$.

Proof. By Theorem 1, there exists at least one continuous solution $x(t)$ of period ω for Eq. (1), such that $a \le x(t) \le R$ for $0 \le t \le \omega$. Now let $x(t)$ be any solution of this type for Eq. (1). Since $f(t,x)$ is nonicreasing in x for $a \le x \le R$, from

$$a \le x(t) \le R = v_0(t) \text{ for } 0 \le t \le \omega,$$

we get

$$a \le A(R)(t) \le x(t) \text{ for } 0 \le t \le \omega.$$

It follows that

$$a \le A(R)(t) \le x(t) \le A^2(R)(t) \text{ for } 0 \le t \le \omega.$$

By (14), this yields

$$a \le A(R)(t) \le A^3(R)(t) \le x(t) \le A^2(R)(t) \le R \text{ for } 0 \le t \le \omega.$$

Finally, we obtain

$$\begin{aligned} a &\le v_1(t) \le v_3(t) \le \dots \le v_{2n+1}(t) \le \dots \le x(t) \le \dots \\ &\le v_{2n}(t) \le \dots \le v_2(t) \le v_0(t) = R \text{ for } 0 \le t \le \omega. \end{aligned} \tag{16}$$

By the complete continuity of A^2, the sequence $(v_{2n+1}(t))_{n\ge 0}$ contains a subsequence uniformly convergent to some $x_*(t)$ in K and, similarly, $(v_{2n}(t))_{n\ge 0}$ contains a subsequence convergeing uniformly to some $x^*(t)$ in K. From (16), we see that the entier sequences $(v_{2n+1}(t))_{n\ge 0}$ and $(v_{2n}(t))_{n\ge 0}$ converge uniformly to $x_*(t)$ and $x^*(t)$, respectively, for $0 \le t \le \omega$, and that

$$a \le x_*(t) \le x(t) \le x^*(t) \le R \quad \text{for} \quad 0 \le t \le \omega. \tag{17}$$

Obviously, we have

$$x_*(t) = A\,x^*(t) \text{ and } x^*(t) = Ax_*(t).$$

Now we prove that under assumption (13), we have indeed $x_*(t) = x^*(t)$ for all $0 \le t \le \omega$.

To this end, let

$$\gamma_0 = \min_{0 \le t \le \omega} (x_*(t)/x^*(t)).$$

From (17), we see that $0 < a/R \le \gamma_0 \le 1$. We show that $\gamma_0 = 1$. Suppose $\gamma_0 < 1$. Since $x_*(t) \ge \max\{a, \gamma_0 x^*(t)\} = \gamma_0 \max\{a/\gamma_0, x^*(t)\} \ge a$ for $0 \le t \le \omega$, by (13), we get that

$$x^*(t) = Ax_*(t) \le A(\gamma_0 \max\{a/\gamma_0, x^*\})(t) \le$$

$$\le \gamma_0^{\alpha} A(\max\{a/\gamma_0, x^*\})(t).$$

One has $x^*(t) \le \max\{a/\gamma_0, x^*(t)\} \le R$ and so,

$$A(\max\{a/\gamma_0, x^*\})(t) \le Ax^*(t) = x_*(t).$$

It follows $x^*(t) \le \gamma_0^{\alpha} x_*(t)$ for $0 \le t \le \omega$. Hence $\gamma_0^{-\alpha} \le \gamma_0$ or, equivalently, $\alpha \le -1$, a contradiction. Thus, $\gamma_0 = 1$ as claimed. Consequently, $x_*(t) = x(t) = x^*(t)$ for $0 \le t \le \omega$ and the proof is complete.

Remark 4. A sufficient condition for (14) is that

$$A(a)(t) \le R \quad \text{for} \quad 0 \le t \le \omega. \tag{18}$$

Indeed, from $a \le A(R)(t) \le A(a)(t)$, we get that

$$A^2(a)(t) \le A^2(R)(t) \le A(a)(t) \le R,$$

whence (14).

If in Theorem 2 we use (18) instead of (14), then we have in addition that for any continuous function $x_0(t)$ with period ω satisfying $a \le x_0(t) \le R$, one has

$$x_n(t) \to x^*(t) \text{ uniformly for } 0 \le t \le \omega \tag{19}$$

as $n \to \infty$, where $x_n = Ax_{n-1}$, $n = 1, 2, \ldots$. Indeed, in this case, from $a \le x_0(t) \le R$, we obtain

$$a \le v_1(t) \le x_1(t) \le A(a) \le R = v_0(t)$$

whence,

$$a \le v_1(t) \le x_2(t) \le v_2(t) \le A(a) \le R = v_0(t)$$

and, in general,

$$a \le v_1(t) \le v_3(t) \le \ldots \le v_{2[(n-1)/2]+1}(t) \le$$

$$\le x_n(t) \le v_{2[n/2]}(t) \le \ldots \le v_2(t) \le v_0(t) = R,$$

$n = 1,2, \ldots$. Since $v_n(t) \to x^*(t)$, it follows that $x_n(t) \to x^*(t)$, as claimed.

Remark 5. If $f(t,x)$ satisfies (h_4) then (18) holds. Indeed, from $f(t,a) \le R/\tau$ for any t, by integrating, we obtain (18).

Conversely, if $f(t,x)$ is constant in $t (f(t,x) = h(x))$ and satisfies all assumptions of Theorem 2 and (18), then (h_4) is fulfilled. Indeed, for any $x_0 \in [a, R]$, we have $A(R) \le \le A(x_0) \le A(a) \le R$. Hence, $h(x_0) = \tau^{-1}A(x_0) \le R/\tau$.

The next theorem completes the results in [4].

THEOREM 3. *Let* (h_1)-(h_4) *hold and suppose that* $f(t,x)$ *is nonincreasing in* x *for* $a \le x \le R$ *and there exists* $\alpha \in (-1, 0)$ *such that* (13) *is satisfied. Then* Eq. (1) *has a unique continuous solution* $x^*(t)$ *of period* ω *such that* $a \le x^*(t) \le R$ *for* $0 \le t \le \omega$. *Moreover*, (19) *holds*.

The proof is similar with that of Theorem 2, so we omit the details.

Example 2. Suppose that $f(t,x)$ satisfies (h_1)-(h_3) and

$$f(t,x) = g(t)(3/x)^{1/2} \text{ for } -\infty < t < \infty \text{ and } 1 \le x \le 3,$$

where $g(t)$ is a continuous function with period ω such that

$$0 \le g(t) \le \sqrt{3} \quad \text{for } 0 \le t \le \omega$$

and

$$\int_{t-1}^{t} g(s)\,ds \ge 1 \quad \text{for } 0 \le t \le \omega.$$

The assumptions of Theorem 3 are fulfilled with $\tau = 1$, $a = 1$, $R = 3$ and $\alpha = -1/2$.

We conclude with a simple example of functions which satisfy all assumptions of

Theorem 2, but not (h_4).

Example 4. Suppose that $f(t,x)$ satisfies (h_1)-(h_2) and

$$f(t,x) = M(3/x)^{1/2} \text{ for } -\infty < t < \infty \text{ and } 1 \le x \le 3,$$

where M is any constant such $\sqrt{3} < M < 5\sqrt{30}/12$. All assumptions of Theorem 2 are satisfied with $\tau = 1$, $a = 1$, $b = 2.5$, $R = 3$, provided that $M\omega \le 2 - 5\sqrt{30}/18$, while (h_4) does not hold. Therefore, there are cases where Theorem 3 fails and Theorem 2 applies.

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