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CONTINUATION THEOREMS FOR MAPPINGS OF CARISTY TYPE

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Dedicated to Professor Ioan A. Rus on his 60th anniversary

Abstract. In this paper we prove some continuation theorems for mappings of Caristi type. Our results generalize Caristi's fixed point theorem and the continuation theorem for contractions.

1. Preliminaries

We first recall Ekeland's variational principle.

Theorem 1. *Let M be a complete metric space and let $\Phi : M \to]-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon > 0$ be given and $x \in M$ be such that*

$$\Phi(x) \leq \inf_M \Phi + \varepsilon.$$

Then there exists $y \in M$ such that

$$\Phi(y) \leq \Phi(x), \quad d(x,y) \leq 1$$

and, for each $z \neq y$ in M,

$$\Phi(z) \quad > \quad \Phi(y) - \varepsilon d(y,z).$$

For the proof see, for example, [4]. The following famous fixed point theorem due to Caristi [1] (see also [4], [7]), is a simple consequence of Ekeland's principle.

Theorem 2. *Let M be a complete metric space, $\varphi : M \to R_+$ a lower semicontinuous function and $T : M \to M$ a mapping such that*

$$d(x, T(x)) \leq \varphi(x) - \varphi(T(x)) \tag{1}$$

for each $x \in M$. Then T has at least one fixed point.

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For the proof, apply Theorem 1 with $\varepsilon = 1/2$ to get $y \in M$ such that

$$(1/2)d(y,T(y)) \geq \varphi(y) - \varphi(T(y)).$$

This, by (1), yields $(1/2)d(y,T(y)) \geq d(y,T(y))$ whence $T(y) = y$.

Remark 1. *If $T : M \to M$ is a contraction, that is*

$$d(T(x),T(y)) \leq \alpha d(x,y)$$

for some $\alpha \in [0,1[$ and all $x,y \in M$, then T satisfies (1) with $\varphi(x) = (1-\alpha)^{-1}d(x,T(x))$. Thus, Caristi's theorem is a generalization of Banach's fixed point theorem. Nevertheless, a mapping satisfying (1) can be not continuous. For an example, take $M = R_+$, $\varphi(x) = x$, $T(x) = x$ for $0 \leq x < 1$ and $T(x) = x - 1$ for $1 \leq x < +\infty$.

Recently, Granas [3] and Frigon-Granas [2] proved some continuation theorems of Leray-Schauder for contractions in complete metric spaces. Also, in [5], we have obtained some improvements of the continuation principle for nonexpansive mappings, while in [6], we have presented a very general continuation principle. Motivated by these results, in this paper we shall state and prove continuation theorems for mappings of Caristi type.

2. Main results

Theorem 3. *Let M be a complete metric space, $X \subset M$ a closed nonempty set, $\psi : X \times [0,1] \to R_+$ a lower semicontinuous function and $N : X \times [0,1] \to M$ a mapping. Let X_λ be the biggest subset invariated by $N_\lambda = N(.,\lambda)$, i.e.*

$$X_\lambda = \cap\{(N_\lambda^k)^{-1}(X);\ k = 1,2,...\}.$$

Suppose that

(i) *$d(x,N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X_\lambda$ and $\lambda \in [0,1]$, where $\psi_\lambda = \psi(.,\lambda)$;*

(ii) *there is a closed nonempty set $S \subset \{(x,\lambda) \in X \times [0,1],\ x \in X_\lambda\}$ such that:*

if $(x_0,\lambda_0) \in S$ and $\lambda_0 < 1$, then there exists $(x,\lambda) \in S$ such that

$$\lambda_0 < \lambda,\quad d(x_0,x) \leq \psi_{\lambda_0}(x_0) - \psi_\lambda(x), \tag{2}$$

$$(N_1(x_0),1) \in S \text{ whenever } (x_0,1) \in S. \tag{3}$$

Then, if N_0 has a fixed point x with $(x,0) \in S$, N_1 also has a fixed point.

Proof. We define an order relation on S, namely

$$(x,\lambda) \preceq (y,\eta) \text{ if } \lambda \leq \eta \text{ and } d(x,y) \leq \psi_\lambda(x) - \psi_\eta(y).$$

Let us show that Zorn's lemma is applicable. Suppose $S_0 \subset S$ is a totally ordered set and denote

$$\psi^* = \inf\{\psi_\lambda(x);\ (x,\lambda) \in S_0\}.$$

Consider a sequence $(x_n, \lambda_n) \in S_0$ such that $\psi_{\lambda_n}(x_n)$ decreases to ψ^* as $n \to \infty$. Then, since S_0 is totally ordered, we have

$$(x_1,\lambda_1) \preceq (x_2,\lambda_2) \preceq \ldots \preceq (x_n,\lambda_n) \preceq \ldots .$$

From

$$d(x_n, x_{n+p}) \leq \psi_{\lambda_n}(x_n) - \psi_{\lambda_{n+p}}(x_{n+p}) \to 0 \text{ as } n \to \infty,$$

uniformly with respect to p, it follows that there exists $x^* \in X$ such that $x_n \to x^*$. Denote $\lambda_* = \lim \lambda_n$. Since ψ is lower semicontinuous, we then have $\psi_{\lambda_*}(x^*) = \psi^*$. S being closed, $(x^*, \lambda_*) \in S$. In addition, $(x_n, \lambda_n) \preceq (x^*, \lambda_*)$. Two cases are possible:

Case 1. There is no $(x,\lambda) \in S_0$ with $(x^*,\lambda_*) \prec (x,\lambda)$. Then (x^*,λ_*) is a upper bound for S_0. Indeed, let (x,λ) be any element of S_0.

a) if $(x,\lambda) \preceq (x_n,\lambda_n)$ for some n, then, since $(x_n,\lambda_n) \preceq (x^*,\lambda_*)$, it clearly follows $(x,\lambda) \preceq (x^*,\lambda_*)$.

b) if $(x_n,\lambda_n) \prec (x,\lambda)$ for some n, then obviously, $\psi_\lambda(x) = \psi^*$ and $\lambda_* \leq \lambda$. If we would have $\lambda_* < \lambda$, then $(x^*,\lambda_*) \prec (x,\lambda)$, which has been excluded by the beginning. Hence $\lambda_* = \lambda$. On the other hand, $d(x_n, x) \leq \psi_{\lambda_n}(x_n) - \psi_\lambda(x) \to 0$, and so $x^* = x$. Therefore $(x,\lambda) = (x^*,\lambda_*)$.

Case 2. There is $(x,\lambda) \in S_0$ with $(x^*,\lambda_*) \prec (x,\lambda)$. Then, $x = x^*$. Let

$$\lambda^* = \sup\{\lambda;\ (x^*,\lambda) \in S_0,\ (x^*,\lambda_*) \preceq (x^*,\lambda)\}.$$

We have $\lambda_* < \lambda^* \leq 1$. Let us consider a sequence $(x^*, \lambda'_n) \in S_0$ such that λ'_n increases to λ^* and $(x^*,\lambda_*) \preceq (x^*,\lambda'_n)$. The element (x^*,λ^*) is an upper bound for S_0. Indeed, let $(x,\lambda) \in S_0$.

a) if $(x,\lambda) \preceq (x^*,\lambda'_n)$ for some n, then clearly, $(x,\lambda) \preceq (x^*,\lambda^*)$.

b) if $(x^*,\lambda'_n) \prec (x,\lambda)$ for every n, then $x = x^*$ and $\lambda > \lambda'_n$, whence $\lambda \geq \lambda^*$. Consequently, $\lambda = \lambda^*$.

Therefore we can apply Zorn's lemma and obtain a maximal element $(x_0, \lambda_0) \in S$. According to (ii), $\lambda_0 = 1$ and $x_0 \in X_1$. Now, using (i) and (3), we get $(x_0, 1) \preceq (N_1(x_0), 1)$ whence, due to the maximality of $(x_0, 1)$, $x_0 = N_1(x_0)$. □

Theorem 3 together with Theorem 2 immediately yield the following result for continuous mappings N.

Corollary 1. *Let M be a complete metric space, $X \subset M$ a closed set, $\psi : X \times [0,1] \to R_+$ a lower semicontinuous function and $N : X \times [0,1] \to M$ a continuous mapping. Suppose*

1) *$d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ for all $x \in X_\lambda$ and $\lambda \in [0,1]$;*

2) *if $N_{\lambda_0}(x_0) = x_0$ and $\lambda_0 < 1$, there exists $\lambda \in]\lambda_0, 1[$ such that $x_0 \in X_\lambda$ and $\psi_\lambda(x_0) \leq \psi_{\lambda_0}(x_0)$.*

Then, if $X_0 \neq \emptyset$, each mapping N_λ, $\lambda \in [0,1]$, has at least one fixed point.

Proof. In order to apply Theorem 3, take

$$S = \{(x, \lambda) \in X \times [0,1];\ N_\lambda(x) = x\}.$$

Since N is continuous, the sets S and X_λ are closed. Hence X_0 is a closed nonempty subset of M. In addition, $N_0(X_0) \subset X_0$. Consequently, by Theorem 2, there exists x with $N_0(x) = x$. It remains to show (2). For this, suppose $(x_0, \lambda_0) \in S$ and $\lambda_0 < 1$. By 2), $x_0 \in X_\lambda$ for some $\lambda \in]\lambda_0, 1[$. Further, by 1), the sequence $(N_\lambda^k(x_0))$ is fundamental and so convergent to some x. Clearly, $(x, \lambda) \in S$ and

$$d(x_0, x) \leq \psi_\lambda(x_0) - \psi_\lambda(x) \leq \psi_{\lambda_0}(x_0) - \psi_\lambda(x).$$

□

Remark 2. *For $X = M$, $N_\lambda = T$ continuous and $\psi_\lambda = \varphi$ for all $\lambda \in [0,1]$, Corollary 4 reduces to Caristi's theorem for continuous mappings.*

A simple consequence of Corollary 4 is the following result by Granas.

Corollary 2. ([3]) *Let M be a complete metric space, $U \subset M$ an open set, and $N : \overline{U} \times [0,1] \to M$ a mapping such that the following conditions hold:*

(h1) *$N(x, \lambda) \neq x$ for all $x \in \partial U$ and $\lambda \in [0,1]$;*

(h2) *there is $\alpha \in [0,1[$ such that*

$$d(N(x,\lambda), N(y,\lambda)) \leq \alpha d(x,y)$$

for all $x, y \in \overline{U}$ *and* $\lambda \in [0,1]$;

(h3) *there is a nondecreasing lower semicontinuous function* $\omega : [0,1] \to R$ *such that*

$$d(N(x,\lambda), N(x,\eta)) \leq | \omega(\lambda) - \omega(\eta) |$$

for all $\lambda, \eta \in [0,1]$ *and* $x \in \overline{U}$.

Then N_1 *has a fixed point if and only if* N_0 *has one.*

Proof. Apply Corollary 4 to $X = \overline{U}$ and

$$\psi_\lambda(x) = (1-\alpha)^{-1}\left[d(x, N_\lambda(x)) + \omega(1) - \omega(\lambda)\right].$$

□

We finish with a continuation theorem for not necessarily continuous mappings of Caristi type.

Theorem 4. *Let* M *be a complete metric space,* $X \subset M$ *a closed set,* $\psi : M \times [0,1] \to R_+$ *a lower semicontinuous function, and* $N : X \times [0,1] \to M$ *a mapping. Suppose that the following conditions hold:*

(i) X_λ *is closed for every* $\lambda \in [0,1]$;

(ii) $d(x, N_\lambda(x)) \leq \psi_\lambda(x) - \psi_\lambda(N_\lambda(x))$ *for all* $x \in X$ *and* $\lambda \in [0,1]$;

(iii) $\psi_\lambda(x) \leq d(x, \partial X)$ *for all* $\lambda \in [0,1]$ *and whenever* $N_\eta(x) = x$ *for some* $\eta \in [0,1]$.

Then, if $X_0 \neq \emptyset$*, each mapping* N_λ*,* $\lambda \in [0,1]$*, has at least one fixed point.*

Proof. Since X_0 is a closed nonempty set and $N_0(X_0) \subset X_0$, by Theorem 2, there exists $x_0 \in X$ such that $N_0(x_0) = x_0$. Further, by (ii) and (iii),

$$d(x_0, N_\lambda^k(x_0)) \leq \psi_\lambda(x_0) - \psi_\lambda(N_\lambda^k(x_0)) \leq \psi_\lambda(x_0) \leq d(x_0, \partial X),$$

whence $N_\lambda^k(x_0) \in X$ for all $k \in$ N. Consequently, $x_0 \in X_\lambda$ for every $\lambda \in [0,1]$. Hence, for each $\lambda \in [0,1]$, $X_\lambda \neq \emptyset$ and we can apply Theorem 2. □

Remark 3. *In particular, if* $X = M$, $N_\lambda = T$ *and* $\psi_\lambda = \varphi$ *for all* $\lambda \in [0,1]$*, Theorem 6 reduces to Theorem 2. Indeed, in this case, we have* $X_\lambda = M$, $\partial X = \emptyset$ *and* $d(x, \partial X) = +\infty$.

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