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MONOTONE ITERATIONS FOR DECREASING MAPS IN ORDERED BANACH SPACES

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Abstract. In this paper we propose a version of the monotone iterations method for decreasing maps in ordered Banach spaces. In some particular cases, this principle has been already applied in [3] and [4], to solve a nonlinear integral equation from biomathematics. Our theorem is new and complements the existing results for increasing maps (see [2, Chapter 6]).

Key words: ordered Banach space, increasing or decreasing map, compact map, fixed point.

1 Introduction

Let X be a real Banach space. By a cone $K \subset X$ we understand a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

Given a cone $K \subset X$, one defines a partial ordering $\leq$ with respect to K by $x \leq y$ iff $y - x \in K$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$. It is easily seen that: $x \leq y$ and $\lambda \geq 0$ imply $\lambda x \leq \lambda y$; if $x_i \leq y_i$, $i = 1,2$, then $x_1 + x_2 \leq y_1 + y_2$; if $0 \leq x_n$ and $x_n \to x$ as $n \to \infty$, then $0 \leq x$. Conversely, if for some partial ordering $\leq$ on X the above three properties hold, then the set $K = \{x \in X;\ 0 \leq x\}$ is a cone and relation $\leq$ is exactly the partial ordering with respect to K.

We use the standard terminology concerning concepts connected with $\leq$. Thus, for example, if $x_0 \leq y_0$, we denote by $[x_0, y_0]$ the order interval, i.e. the set $\{x \in X;\ x_0 \leq x \leq y_0\}$. Obviously, $[x_0, y_0]$ is a closed convex subset of X.

Let X be a Banach space, $K \subset X$ a cone and $\leq$ the partial ordering with respect to K. We say that the norm on X is monotone if $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, and semimonotone if $\|x\| \leq \gamma\|y\|$ for some $\gamma \geq 1$ and all x,y such that $0 \leq x \leq y$. It is known (see, for example, (1, Propozitia II 1.1.2)) that the norm is semimonotone iff K is normal, i.e. $0 \leq x_n \leq y_n$ and $y_n \to 0$ as $n \to \infty$ imply $x_n \to 0$ as $n \to \infty$. There is no difficulty in verifying that every order interval is bounded (with respect the norm) iff K is normal.

A map $T: D \subset X \to X$ is said to be increasing (decreasing) if $T(x) \leq T(y)$ $(T(y) \leq T(x))$ whenever $x,y \in D$ and $x \leq y$.

The following theorem is known as the monotone iterations principle for increasing maps in ordered Banach spaces with normal cone (see, for example, (1, Propozitia V 2.1.1) or (2, Theorem 6.19.1)):

Theorem 1. Let X be a Banach space partially ordered by the normal cone K. Let (x_0,y_0) be an order interval and $T: (x_0,y_0) \to (x_0,y_0)$ be increasing and compact (i.e. continuous and with $T((x_0,y_0))$ relatively compact). Then the sequences $(T^n(x_0))$ and $(T^n(y_0))$ are increasing, respectively decreasing, and converge to the fixed points of T, x^* and respectively y^*. In addition, $x_0 \leq x^* \leq y^* \leq y_0$, x^* is the minimal fixed point of T in (x_0,y_0) while y^* is the maximal fixed point of T in (x_0,y_0).

The goal of this paper is to obtain a similar result in case that T is decreasing.

2 Main result

Theorem 2. Let X be a Banach space partially ordered by the normal cone K. Let (x_0,y_0) be an order interval, $0 < x_0 \leq y_0$, such that

(1) for every x,y with $x_0 \leq x \leq y \leq y_0$ there is $\mu \in (0,1)$ such that $\mu y \leq x$.

Suppose $T: (0,y_0) \to X$ is decreasing on $(0,y_0)$, continuous on (x_0,y_0) with $T((x_0,y_0)) \subset (x_0,y_0)$ and $T((x_0,y_0))$ relatively compact. Also suppose that

(2) there is $\alpha \in (-1,0)$ with $T(\mu x) \leq \mu^\alpha T(x)$ for all $\mu \in (0,1)$ and $x \in (x_0,y_0)$.

Then T has a unique fixed point x^* in (x_0,y_0),

(3) $x_0 \leq T(y_0) \leq T^2(x_0) \leq \dots \leq T^{2n}(x_0) \leq T^{2n+1}(y_0) \leq \dots \leq x^* \leq \dots$
$\leq T^{2n+1}(x_0) \leq T^{2n}(y_0) \leq \dots \leq T^2(y_0) \leq T(x_0) \leq y_0$

and $T^n(x)$ converges to x^* for any $x \in (x_0,y_0)$.

Proof. Since (x_0,y_0) is a bounded closed convex subset of X, $T((x_0,y_0)) \subset (x_0,y_0)$ and T is compact on (x_0,y_0), by the Schauder fixed point theorem, there exists at least one fixed point of T in (x_0,y_0).

Now, let $x_1 \in (x_0,y_0)$ be any fixed point of T. Then, since T is decreasing and $T((x_0,y_0)) \subset (x_0,y_0)$, we have

$$x_0 \leq T(y_0) \leq x_1 \leq T(x_0) \leq y_0.$$

Next

$$x_0 \leq T(y_0) \leq T^2(x_0) \leq x_1 \leq T^2(y_0) \leq T(x_0) \leq y_0.$$

Finally

(4) $x_0 \leq T(y_0) \leq T^2(x_0) \leq \dots \leq T^{2n}(x_0) \leq T^{2n+1}(y_0) \leq \dots \leq x_1$
$\dots \leq T^{2n+1}(x_0) \leq T^{2n}(y_0) \leq \dots \leq T^2(y_0) \leq T(x_0) \leq y_0.$

T being compact on (x_0,y_0), the sequence $T^{2n}(x_0) = T(T^{2n-1}(x_0))$ contains a subsequence convergent to some $x^* \in X$. Similarly, the sequence $T^{2n}(y_0) = T(T^{2n-1}(y_0))$ contains a subsequence convergent to some $y^* \in X$. Obviously, $x_0 \leq x^* \leq x_1 \leq y^* \leq y_0$ and $T^{2n}(x_0) \leq x^*$, $y^* \leq T^{2n}(y_0)$ for all n. Further, by (4), we can show that

(5) $T^{2n}(x_0) \to x^*$, $T^{2n+1}(y_0) \to x^*$,
$T^{2n}(y_0) \to y^*$, $T^{2n+1}(x_0) \to y^*$.

Let us prove, for example, that $T^{2n}(x_0) \to x^*$. Suppose x^* is the limit of the subsequence $T^{2k(n)}(x_0)$. Then, for each $\varepsilon > 0$ there is $n_0 \in N$ with

$$\| x^* - T^{2k(n_0)}(x_0) \| \leq \varepsilon .$$

For $m \geq k(n_0)$ we have $0 \leq x^* - T^{2m}(x_0) \leq x^* - T^{2k(n_0)}(x_0)$, and since the norm is semimonotone, we deduce

$$\| x^* - T^{2m}(x_0) \| \leq \gamma \| x^* - T^{2k(n_0)}(x_0) \| \leq \gamma\varepsilon$$

for all $m \geq k(n_0)$. Thus the enter sequence $T^{2n}(x_0)$ converges to x^*. From (5) we obtain

$$x^* = T(y^*), \quad y^* = T(x^*).$$

Now we prove that under assumptions (1) and (2) we have indeed

$x^* = y^*$. According to (1), let $\mu_0 = \sup\{\mu \in [0,1];\ \mu y^* \le x^*\}$. Clearly, $\mu_0 y^* \le x^*$. We have to prove that $\mu_0 = 1$. Suppose, by contradiction, $\mu_0 < 1$. Then, by (2), $y^* = T(x^*) \le T(\mu_0 y^*) \le \mu_0^{\alpha} T(y^*) = \mu_0^{\alpha} x^*$. Consequently, $\mu_0^{-\alpha} \le \mu_0$, that is $-\alpha \ge 1$, a contradiction. Thus, $x^* = y^* = x_1$.

Finally, let x be any element of $[x_0, y_0]$. Then

$T^{2n}(x_0) \le T^{2n}(x) \le T^{2n}(y_0)$ and $T^{2n+1}(y_0) \le T^{2n+1}(x) \le T^{2n+1}(x_0)$.

By using (5), these imply that $T^n(x) \to x^*$.

Example. Consider the nonlinear integral equation

$$u(t) = \int_0^1 f(t,s,u(s))ds, \quad 0 \le t \le 1, \tag{6}$$

with $f: [0,1] \times [0,1] \times [0,b] \to R$ continuous and $f(t,s,.)$ decreasing on $[0,b]$ $(b > 0)$ for every $t,s \in [0,1]$. Suppose $0 < a < b$,

$$a \le \int_0^1 f(t,s,b)ds \quad \text{and} \quad \int_0^1 f(t,s,a)ds \le b$$

for $t \in [0,1]$. Also assume that there is $\alpha \in (-1,0)$ such that $f(t,s,\mu u) \le \mu^{\alpha} f(t,s,u)$ for all $t,s \in [0,1]$, $u \in [a,b]$ and $\mu \in (0,1]$. Then (6) has a unique solution $u \in C[0,1]$ with $a \le u(t) \le b$ for all $t \in [0,1]$. In particular, the equation

$$u(t) = \int_0^1 g(t,s)u^{\alpha}(s)ds, \quad 0 \le t \le 1, \tag{7}$$

has a unique solution $u \in C[0,1]$ with $a \le u(t) \le b$, where $0 < a < b$, provided that g is continuous and nonnegative on $[0,1]^2$, $\alpha \in (-1,0)$ and $ab^{-\alpha} \le \int_0^1 g(t,s)ds \le a^{-\alpha} b$. For other examples see [3] and [4].

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