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ON THE CONTINUATION PRINCIPLE FOR NONEXPANSIVE MAPS

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Abstract. In this note the continuation principle (nonlinear alternative) for nonexpansive maps on Hilbert spaces (see [5]) is extended in two directions: 1) to the case of uniformly convex Banach spaces; 2) for nonexpansive maps on a not necessarly convex set of a Hilbert space. In the proofs we use the Leray-Schauder continuation principle for condensing maps [7], [9] (we can also use Granas' continuation principle for contractions on complete metric spaces [6]).

In [5], the following nonlinear alternative for nonexpansive maps was proved by means of the Banach fixed point theorem.

Theorem A [5]. *Let H be a Hilbert space and C the closed ball $\{x \in H; | x | \leq c\}$. Then each nonexpansive map $T: C \to H$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial C$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

In what follows we shall prove the following two generalizations of Theorem A:

Theorem 1. *Let E be an uniformly convex Banach space and U a bounded open convex set of E with $0 \in U$. Then each nonexpansive map $T : \bar{U} \to E$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial U$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

Theorem 2. *Let H be a Hilbert space and U a bounded open set of H (not necessarily convex) with $0 \in U$. Then each nonexpansive map $T : \bar{U} \to H$ has at least one of the following properties:*

(a) *T has a fixed point.*

(b) *There is $x \in \partial U$ and $\lambda \in]0,1[$ such that $x = \lambda T(x)$.*

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Recall that a Banach space E is said to be *uniformly convex* provided that for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\| x + y \| \leq 2(1 - \delta)$ for every $x, y \in E$ satisfying $\| x \| = \| y \| = 1$ and $\| x - y \| \geq \varepsilon$.

Each uniformly convex Banach space is reflexive (see, for example, [4]), and each Hilbert space is uniformly convex as follows from the parallelogram equation $| x - y |^2 + | x + y |^2 = 2(| x |^2 + | y |^2)$. For example, the spaces $L^p(\Omega)$ with $\Omega \subset \mathbf{R}^N$ measurable are uniformly convex for $1 < p < \infty$ (see [4]).

For the proofs we need some lemmas essentially due to Browder.

Lemma 1. *Let E be an uniformly convex Banach space, D a bounded convex set of E and $T : D \to E$ a nonexpansive map. Then, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0, x_1 \in D$, $\| x_0 - T(x_0) \| \leq \delta$ and $\| x_1 - T(x_1) \| \leq \delta$, it follows $\| x - T(x) \| \leq \delta$ for any x of the form $x = (1 - \lambda)x_0 + \lambda x_1$ with $\lambda \in]0, 1[$.*

For the proof see [2] or [8, Teorema 1.4.2].

Lemma 2. *Let E be an uniformly convex Banach space, D a bounded closed convex set of E and $T : D \to E$ a nonexpansive map. If $(x_n) \subset D, x_n \to x_0$ weakly and $x_n - T(x_n) \to y_0$ in norm, then $x_0 - T(x_0) = y_0$.*

For the proof see the proof of Teorema 1.4.3 a) in [8].

Proof of Theorem 1. Suppose (b) does not hold. Then, $x \neq \lambda T(x)$ for all $x \in \partial U$ and $\lambda \in [0, 1[$. For each fixed $\lambda \in]0, 1[$, the map λT is a contraction and so, it is condensing. Then, by the Leray-Schauder continuation principle for condensing maps (see [7], [9]), there exists $x_\lambda \in U$ such that $x_\lambda - \lambda T(x_\lambda) = 0$. Let us denote by x_n such an element x_λ for $\lambda = 1 - 1/n$, $n \in \mathbf{N}^*$. Then, passing if necessarily to a subsequence, we may suppose that (x_n) converges weakly to some x_0. On the other hand, from $x_n - (1 - 1/n)T(x_n) = 0$, it follows that $x_n - T(x_n) \to 0$ in norm. Then, from Lemma 2, we get $x_0 - T(x_0) = 0$. Thus, (a) holds and the proof is complete.

Remark. If in particular, $T(\bar{U}) \subset \bar{U}$, then (b) in Theorem 1, clearly, does not hold. In this case, conclusion (a) follows directly by the following theorem of Browder-Kirk: *If E is an uniformly convex Banach space, D is a bounded closed convex set of E and $T : D \to D$ is nonexpansive, then there exists $x \in D$ with $T(x) = x$.*

In the case of Hilbert spaces, we may renounce at the assumption that U is convex and also give a much simpler proof:

Proof of Theorem 2. Also suppose (b) does not hold. The sequence (x_n) obtained in the proof of Theorem 1 satisfies:

$$\langle (n-1)^{-1}x_n - (m-1)^{-1}x_m, x_n - x_m\rangle = \langle T(x_n) - T(x_m), x_n - x_m\rangle - | x_n - x_m |^2 \leq 0$$

for all $n, m > 1$. Denote $r_n = (n-1)^{-1}$ and use the equality

$$2\langle r_n x_n - r_m x_m, x_n - x_m\rangle = (r_n + r_m) | x_n - x_m |^2 + (r_n - r_m)(| x_n |^2 - | x_m |^2).$$

Then, we obtain

$$0 \leq (r_n + r_m) | x_n - x_m |^2 \leq (r_n - r_m)(| x_m |^2 - | x_n |^2).$$

Since (r_n) is a decreasing sequence, we get that $(| x_n |)$ is an increasing sequence. In addition, U being bounded, $(| x_n |)$ is also bounded and thus, convergent. Next, from

$$| x_n - x_m |^2 \leq (| x_m |^2 - | x_n |^2)(r_n - r_m)/(r_n + r_m),$$

it follows that (x_n) is convergent. It is clear that its limit is a fixed point of T and the proof is complete.

Example. Let H be a Hilbert space and let us consider the boundary value problem

$$\begin{cases} u'' = f(t, u, u') & \text{for } 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $f : [0,1] \times H^2 \to H$ satisfies

(i) $f(\cdot, u, v)$ is measurable for any fixed $u, v \in H$; there exist $1 < p < \infty$ and $h \in L^\infty(0,1)$ such that $f(\cdot, 0, 0) \in L^p(0,1;H)$ and

$$| f(t, u_1, v_1) - f(t, u_2, v_2) | \leq h(t)(| u_1 - u_2 | + | v_1 - v_2 |)$$

for all $u_1, u_2, v_1, v_2 \in H$ and a.e. $t \in [0,1]$.

We look for a weak solution $u \in W_0^{1,p}(0,1;H) \cap W^{2,p}(0,1;H)$ to problem (1).

Let $G(t,s)$ be the Green function, i.e. $G(t,s) = (1-t)s$ for $s \leq t$ and $G(t,s) = (1-s)t$ for $s > t$. Also, denote by C the smallest constant in the Wirtinger-Poincaré inequality:

$$\int_0^1 | u |^p \, dt \leq C^p \int_0^1 | u' |^p \, dt, \qquad u \in W_0^{1,p}(0,1;H)$$

(see [1]).

Theorem 3. *Let* (i) *holds. Also assume*

(ii) *there is* $r > 0$ *such that*

$$\langle u, f(t,u,v)\rangle + | v |^2 > 0 \quad \textit{for} \quad a.e. \qquad t \in [0,1]$$

and whenever $| u | \geq r$ *and* $\langle u, v\rangle = 0$;

(iii) $(C+1)^p \int_0^1 \left\{ \int_0^1 (| G_t(t,s) | \cdot h(s))^q \, ds \right\}^{p/q} dt \leq 1.$

Then (1) *has at least one solution.*

Proof. Problem (1) is equivalent with

$$u(t) = -\int_0^1 G(t,s) f(s, u(s), u'(s)) \, ds, \qquad 0 < t < 1.$$

We shall apply Theorem 1 to $E = W_0^{1,p}(0,1;H)$ and $T : E \to E$,

$$T(u)(t) = -\int_0^1 G(t,s) f(s, u(s), u'(s)) \, ds.$$

By the uniform convexity of $L^p(0,1;H)$, it easily follows that $W_0^{1,p}(0,1;H)$ endowed with norm

$$\| u \| = \left(\int_0^1 | u' |^p \, dt \right)^{1/p},$$

is also uniformly convex.

Next we have

$$\| T(u) - T(v) \|^p = \int_0^1 \left| \int_0^1 G_t(t,s) \left(f(s,u(s),u'(s)) - f(s,v(s),v'(s)) \right) ds \right|^p dt$$

$$\leq \int_0^1 \left\{ \int_0^1 | G_t(t,s) | \cdot h(s) (| u(s) - v(s) | + | u'(s) - v'(s) |) \, ds \right\}^p dt \leq$$

$$\leq \int_0^1 (| u(s) - v(s) | + | u'(s) - v'(s) |)^p \, ds \cdot \int_0^1 \left\{ \int_0^1 (| G_t | \cdot h(s))^q \, ds \right\}^{p/q} dt$$

where $1/p + 1/q = 1$, by Hölder's inequality.

Further, since

$$\int_0^1 (| u(s) - v(s) | + | u'(s) - v'(s) |)^p \, ds \leq$$

$$\leq \left\{ \left(\int_0^1 | u(s) - v(s) |^p \, ds \right)^{1/p} + \left(\int_0^1 | u'(s) - v'(s) |^p \, ds \right)^{1/p} \right\}^p \leq$$

$$\leq (C+1)^p \, \| u - v \|^p,$$

we obtain

$$\| T(u) - T(v) \| \leq (C+1)B \| u - v \|$$

where $B = \left[\int_0^1 \left\{\int_0^1 (\mid G_t(t,s) \mid h(s))^q \, ds\right\}^{p/q} dt\right]^{1/p}$.

Thus, by (iii), T is nonexpansive.

Finally, by a standard reasoniement, from (ii), we get a number $R > 0$ such that $\| u \| < R$ for each $u \in W_0^{1,p}(0,1;H)$ solution to $u = \lambda T(u)$ for some $\lambda \in]0,1[$. Therefore, (b) does not hold and so T has a fixed point.

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