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ANALYSIS OF A NONLINEAR INTEGRAL EQUATION MODELLING INFECTION DISEASES

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Abstract

To describe the spread of virus diseases with contact rate that varies seasonally, the following delay integral equation has been proposed by K.L. Cooke and J.L. Kaplan

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds.$$

This model can also be interpreted as an evolution equation of a single species population. The purpose of this paper is to describe and improve recent results on this equation, obtained by the authors in the last decade. Our analysis is concerned with the existence, uniqueness, approximation and continuous dependence on data of the positive solutions of the initial-value problem, and of the periodic solutions. We use topological methods (fixed point theorems, continuation principle) and monotone iterative techniques.

Keywords: nonlinear integral equation, positive solutions, periodic solutions, fixed point, continuation principle, monotone iterations, continuous dependence, population dynamics.
AMS subject classification: 45G10, 45M15, 47H15.

1 Introduction

In this paper we are concerned with the following nonlinear delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds. \tag{1}$$

This equation and similar others appear when investigating the spread of virus diseases or, more generally, the growth of single species populations. Delay equations also arise from the study of materials with thermal- or shape-memory (see [28]).

Several results regarding various mathematical aspects of Eq.(1), or of equations of type (1), have been obtained by K.L. Cooke and J.L. Kaplan [1], H.L. Smith [2],[8], R.D. Nussbaum [3], J. Kaplan, M. Sorg and J. Yorke [4], S. Busenberg and K. Cooke [5], R.W. Leggett and L.R. Williams [7],[9], A. Cañada [11], D. Guo and V. Lakshmikantham [12], I.A. Rus [13], S.G. Hristova and D.D. Bainov [14], N.G. Kazakova and D.D. Bainov [15], A.M. Fink and J.A. Gatica [16], R. Precup [17],[20],[24], R. Torrejòn [19], E. Kirr [21],[25], A. Cañada and A. Zertiti [22],[23], Ait Dads, K. Ezzinbi and O. Arino [26]. Eq.(1) also appears in the monographs [6],[10, Example 20.1] and [18].

Let us first describe the meaning of Eq.(1) in terms of epidemics. In this case, it is assumed that the total number of population members is constant; $x(t)$ represents the proportion of infectives in population at time t, regarded as a continuous quantity; τ is the length of time an individual remains infectious (duration of infectivity); $f(t, x(t))$ means the proportion of new infectives per unit time (instantaneous contact rate). Then, $f(t, x(t))\, dt$ represents the proportion of individuals infected within the period t, $t + dt$. In consequence, the number of infectious individuals at time t equals the sum of all individuals infected between $t - \tau$ and t.

Let us now interpret Eq.(1) as a growth equation of a single species population when the birth rate varies seasonally. In this case, $x(t)$ is the number of individuals of a single species population at time t, $f(t, x(t))$ is the number of new births per unit time, and τ is the lifetime. It is assumed that each individual lives to the age τ exactly and then dies.

In this paper we report on two distinct problems on Eq.(1). In both cases, because of the biological interpretation, we shall be interested in positive solutions.

(I) *The initial-values problem (IVP)*
We look for positive continuous solutions $x(t)$ of Eq.(1), for $-\tau \leq t \leq T$, when it is known the proportion $\varphi(t)$ of infectives for $-\tau \leq t \leq 0$, i.e.,

$$x(t) = \varphi(t) \quad \text{for } -\tau \leq t \leq 0. \tag{2}$$

Obviously, we have to assume that $\varphi(t)$ is a positive continuous function on $[-\tau, 0]$ and satisfies

$$\varphi(0) = \int_{-\tau}^{0} f(s, \varphi(s))\, ds. \tag{3}$$

It is easy to see that, under assumption (3), problem (1)-(2) is equivalent with the following initial-values problem

$$\begin{cases} x'(t) = f(t, x(t)) - f(t-\tau, x(t-\tau)) & \text{for } 0 \le t \le T \\ x(t) = \varphi(t) & \text{for } -\tau \le t \le 0. \end{cases} \tag{4}$$

(II) *The periodic problem (PP)*

Because of seasonal factors, the rate $f(t, x)$ may be a ω-periodic function of t and, in such situations, one is interested in ω-periodic solutions of Eq.(1).

I. THE INITIAL-VALUES PROBLEM

2 Existence results

A. *Positive solutions in space.*

We are looking for solutions of (1)-(2) in the space C of all continuous functions $x(t)$ satisfying $x(t) \ge a$ for $-\tau \le t \le T$, where $a \ge 0$ is a given number.

Let us list our assumptions:

(a1) $f(t, x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $x \ge a$.

(a2) $\varphi(t)$ is continuous, satisfies (3) and $\varphi(t) \ge a$ for $-\tau \le t \le 0$.

(a3) There exists a continuous function $g(t)$ such that

$$f(t, x) \ge g(t) \quad \text{for } -\tau \le t \le T \text{ and } x \ge a$$

and

$$\int_{t-\tau}^{t} g(s)\, ds \ge a \quad \text{for } 0 \le t \le T.$$

(a4) There exists a positive continuous function $h(x)$ on $[a, \infty)$ such that

$$f(t, x) \le h(x) \quad \text{for } 0 \le t \le T \text{ and } x \ge a$$

and

$$T < \int_{\varphi(0)}^{\infty} (1/h(x))\, dx.$$

Denote $b = \varphi(0)$ and let R_0 be given by

$$T = \int_{b}^{R_0} (1/h(x))\, dx. \tag{5}$$

Theorem 2.1 ([17]). *Suppose* (a1)-(a4) *are satisfied. Then the problem* (1)-(2) *has at least one solution* $x(t) \in C$. *Moreover, any solution in* C *satisfies*

$$x(t) \le R_0 \text{ for } 0 \le t \le T. \tag{6}$$

Proof. Let E be the Banach space of all continuous functions $x(t)$ defined on $[0,T]$, endowed with the uniform norm. Consider the closed convex set of E, $K = \{x \in E;\ x(t) \geq a \text{ for } 0 \leq t \leq T\}$, and let

$$X = \{x \in E;\ x(0) = b \text{ and } x(t) \geq a \text{ for } 0 \leq t \leq T\}.$$

Also consider the homotopy

$$H : K \times [0,1] \to X,$$

$$H(x,\lambda)(t) = (1-\lambda)\, b + \lambda \textstyle\int_{t-\tau}^{t} f(s, \tilde{x}(s))\, ds,$$

where $\tilde{x}(t) = x(t)$ for $0 < t \leq T$ and $\tilde{x}(t) = \varphi(t)$ for $-\tau \leq t \leq 0$. By (a3) and $b \geq a$, H is well-defined, i.e., $H(K \times [0,1]) \subset X$, while by means of Ascoli-Arzelà theorem, it is completely continuous.

Next we establish the *a priori* boundedness of the set of all solutions of equations $H(x,\lambda) = x,\ \lambda \in [0,1]$. Let x be such a solution. Then, for each $t \in [0,T]$, we have

$$x'(t) = \lambda f(t, x(t)) - \lambda f(t-\tau, x(t-\tau)).$$

Since f is nonnegative, we get

$$x'(t) \leq \lambda f(t, x(t)).$$

Further, by (a4),

$$x'(t) \leq \lambda h(x(t)).$$

It follows that

$$\int_0^t (x'(s) / h(x(s)))\, ds \leq \lambda t \leq \lambda T \leq T,$$

for all $t \in [0,T]$. Hence

$$\int_b^{x(t)} (1/h(u))\, du \leq T \quad \text{for } 0 \leq t \leq T$$

whence, by (5), we see that x satisfies (6).

Therefore, if we choose any $R > R_0$, we have that H is an admissible (fixed point free on boundary) homotopy on the closure of the open bounded set of X,

$$U = \{x \in X;\ x(t) < R \ \text{ for } 0 \leq t \leq T\}.$$

On the other hand, the constant map $H(.,0) = b$ is essential (see [27, Theorem 2.2]). Consequently, by the topological transversality theorem ([27, Theorem 2.5]), the map $A = H(.,1)$ is essential too. It follows that A has at least one fixed point $x \in U$. Clearly, x is a solution of (1)-(2). □

Remark 2.1. Let us assume that instead of (a4) one has

(a4') $\lim_{x\to\infty} \sup f(t,x)/x = \mu(t)$ uniformly in $t \in [0,T]$ and $\mu = \sup_{0\le t\le T} \mu(t) < \infty$.

Then, choosing $\alpha > \mu$, we get $\beta \geq 0$ such that

$$f(t,x) \leq \alpha x + \beta \quad \text{for } 0 \leq t \leq T \text{ and } x \geq a. \tag{7}$$

Hence (a4) is fulfilled by $h(x) = \alpha x + \beta$, and

$$\int_b^\infty (1/h(u))\, du = \infty.$$

If in addition, in (a4'), we suppose $\mu < 1/\tau$ (this is assumption (H5) in [12]), then taking $\mu < \alpha < 1/\tau$ we can choose $R > b$ such that

$$\alpha R + \beta \leq R/\tau, \tag{8}$$

in order that A maps $\overline{U}$ into itself and so, in this situation, Theorem 2.1 follows directly by Schauder's fixed point theorem. Next we show that this is also true for an arbitrary value of μ.

Indeed, let us use an equivalent norm on E, namely

$$\|x\|_\theta = \max_{0\le t\le T}(|x(t)| \exp(-\theta t))$$

with a suitable positive number θ. By (8), we get

$$A(x)(t) \leq \tau\gamma + \textstyle\int_0^t (\alpha x(s) + \beta)\, ds =$$

$$\tau\gamma + \beta t + \alpha \textstyle\int_0^t x(s) \exp(-\theta s) \exp(\theta s)\, ds \leq \tau\gamma + \beta T + \alpha \|x\|_\theta \int_0^t \exp(\theta s)\, ds \leq$$

$$\tau\gamma + \beta T + (\alpha/\theta) \|x\|_\theta \exp(\theta t),$$

where $\gamma = \max_{-\tau\le t\le 0} f(t,\varphi(t))$. Thus

$$A(x)(t) \exp(-\theta t) \leq (\alpha/\theta) \|x\|_\theta + \tau\gamma + \beta T.$$

Now, if we choose $\theta > \alpha$ and $R > b$ such that

$$(\alpha/\theta) R + \tau\gamma + \beta T \leq R,$$

we see that A maps $\{x \in X;\ \|x\|_\theta \leq R\}$ into itself and so Schauder's fixed point theorem applies.

Let us now consider instead of (a4) a more restrictive condition than (a4'), namely

(a4") There exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y| \tag{9}$$

for all $t \in [-\tau, T]$ and $x, y \in [a, \infty)$.

Theorem 2.2 ([17]). *Suppose* (a1)-(a3) *and* (a4″) *are satisfied. Then the problem* (1)-(2) *has a unique solution* $x(t) \in C$. *Moreover,*

$$x_n(t) \to x(t) \quad \text{as } n \to \infty, \text{uniformly in } t \in [0, T],$$

where $x_0(t) = b$ *and* $x_n(t) = A(x_{n-1})(t)$ *for* $n = 1, 2, \ldots$.

Proof. Similar arguments as in Remark 2.1 yield to the conclusion that the map $A : X \to X$ is a contraction with respect to a suitable norm $\|.\|_\theta$. Thus, Banach's fixed point theorem is applicable. □

B. *Positive solutions in a ball.*
Suppose we are interested in solutions $x(t) \in C$ of (1)-(2), in a given ball of E, say of ray R. Obviously, in this situation, the contact rate $f(t, x)$ may be known only for $a \le x \le R$.

Let us list the hypotheses corresponding to this case.

(h1) $f(t, x)$ is nonnegative and continuous for $-\tau \le t \le T$ and $a \le x \le R$.

(h2) $\varphi(t)$ is continuous, satisfies (3) and $a \le \varphi(t) \le R$ for $-\tau \le t \le 0$.

(h3) There exists a continuous function $g(t)$ such that

$$f(t, x) \ge g(t) \quad \text{for } -\tau \le t \le T \text{ and } a \le x \le R$$

and

$$\int_{t-\tau}^{t} g(s)\, ds \ge a \quad \text{for } 0 \le t \le T. \tag{10}$$

(h4) There exists a positive continuous function $h(x)$ on $[a, R]$ such that

$$f(t, x) \le h(x) \quad \text{for } 0 \le t \le T \text{ and } a \le x \le R \tag{11}$$

and

$$T < \int_{\varphi(0)}^{R} (1/h(x))\, dx. \tag{12}$$

Theorem 2.3. *Suppose* (h1)-(h4) *are satisfied. Then the problem* (1)-(2) *has at least one continuous solution* $x(t)$ *such that* $a \le x(t) \le R$ *for* $-\tau \le t \le T$. *In addition, any such solution satisfies* (6).

Proof. The proof is the same as for Theorem 2.1. There is only one difference, the fact that the homotopy H can be defined only on $\overline{U} \times [0, 1]$. □

Let us now suppose that instead of (h4) the following condition is satisfied:

(h4*) There exists $L > 0$ such that (9) holds for all $t \in [-\tau, T]$ and $x, y \in [a, R]$.

Theorem 2.4. *Suppose* (h1)-(h3) *and* (h4*) *are satisfied. Then there exists* T_0, $0 < T_0 \le T$, *such that* (1)-(2) *has a unique continuous solution* $x(t)$ *on* $[-\tau, T_0]$ *satisfying* $a \le x(t) \le R$ *for* $-\tau \le t \le T_0$.

Proof. By (9), we obtain

$$f(t,x) \le L(x-a) + \max_{0\le t\le T} f(t,a) =: h(x) \quad \text{for } a \le x \le R.$$

So (11) holds. Now we choose $T_0 \le T$ such that

$$T_0 < \int_{\varphi(0)}^{R} (1/h(x))\, dx,$$

and we apply Theorem 2.3 with T_0 instead of T. Thus the existence of solutions is proved. To show the uniqueness, suppose $x_1(t)$ and $x_2(t)$ are two solutions on $[-\tau, T_0]$. Then, by (9), we have

$$|x_1(t) - x_2(t)| \le \int_{t-\tau}^{t} |f(s,\tilde{x}_1(s)) - f(s,\tilde{x}_2(s))|\, ds \le$$

$$L\int_{t-\tau}^{t} |\tilde{x}_1(s) - \tilde{x}_2(s)|\, ds \le L\int_0^t |x_1(s) - x_2(s)|\, ds,$$

for $0 \le t \le T_0$. This, by Gronwall's inequality, implies $x_1(t) = x_2(t)$. □

3 Continuous dependence on data

Suppose the data f, φ and τ satisfy (h1)-(h4) and that the corresponding IVP, (1)-(2), has a unique continuous solution $x(t)$ satisfying $a \le x(t) \le R$ for $-\tau \le t \le T$.

Let (τ_n) be a nonincreasing sequence of positive numbers and let (φ_n) and (f_n) be two sequences of nonnegative continuous functions defined on $[-\tau_n, 0]$ and $[-\tau_n, T] \times [a, R]$, respectively. We suppose that

$$a \le \varphi_n \le R, \quad \varphi_n(0) = \int_{-\tau_n}^{0} f_n(s,\varphi_n(s))\, ds, \quad \tau_n \to \tau,$$
$$\varphi_n \to \varphi \ \text{ and } f_n \to f \text{ uniformly,} \tag{13}$$

i.e., for each $\varepsilon > 0$ there is $n_\varepsilon \ge 1$ such that, for every $n \ge n_\varepsilon$, one has

$$|\tau_n - \tau| < \varepsilon, \quad |\varphi_n(t) - \varphi(t)| < \varepsilon \ \text{ for } -\tau_n \le t \le 0$$

and

$$|f_n(t,x) - f(t,x)| < \varepsilon \ \text{ for } -\tau_n \le t \le T \text{ and } a \le x \le R.$$

Finally, let us consider the IVP corresponding to f_n, φ_n and τ_n :

$$x_n(t) = \int_{t-\tau_n}^{t} f_n(s, x_n(s))\, ds \quad \text{for } 0 \le t \le T,$$

$$x_n(t) = \varphi_n(t) \quad \text{for } -\tau_n \le t \le 0,$$

denoted by (1n)-(2n).

If inequality in (10) is strict, by Theorem 2.3, it follows that for sufficiently large n, say $n \geq n_0$, (1n)-(2n) has at least one continuous solution $x_n(t)$ satisfying

$$a \leq x_n(t) \leq R \text{ for } -\tau_n \leq t \leq T. \tag{14}$$

The question is: if for each n we choose an arbitrary continuous solution $x_n(t)$ of (1n)-(2n) satisfying (14), does the sequence $(x_n(t))$ converge to $x(t)$ uniformly in $t \in [0,T]$? The answer is positive as shows the following theorem essentially established in [25].

Theorem 3.1. *Suppose* (h1)-(h4) *are satisfied and that* (1)-(2) *has a unique continuous solution $x(t)$ such that $a \leq x(t) \leq R$ on $[-\tau, T]$. If the sequences (τ_n), (φ_n) and (f_n) satisfy* (13), *and $(x_n(t))$ is any sequence of continuous solutions of* (1n)-(2n) *satisfying* (14), *then*

$$x_n(t) \to x(t) \text{ as } n \to \infty, \text{ uniformly in } t \in [0,T].$$

Proof. From (14) we have that $(x_n(t))$ is bounded in $C[0,T]$. On the other hand, by (13) and

$$x_n'(t) = f_n(t, x_n(t)) - f_n(t-\tau, x_n(t-\tau)) \quad \text{for } 0 \leq t \leq T,$$

we easily see that the sequence $(x_n'(t))$ is also bounded in $C[0,T]$. Thus, the sequence $(x_n(t))$ is equibounded and equicontinuous on $[0,T]$. By Ascoli-Arzelà theorem, there is a convergent subsequence $(x_{k_n}(t))$ of $(x_n(t))$. Suppose $x_{k_n}(t) \to \overline{x}(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$. Now taking the limit as $n \to \infty$ in $(1k_n)-(2k_n)$, we obtain that $\overline{x}(t)$ is solution of (1)-(2). Finally, the uniqueness of the solution implies $\overline{x}(t) = x(t)$ and that the entire sequence $(x_n(t))$ converges uniformly to $x(t)$. □

4 Minimal and maximal solutions

Theorem 4.1 ([24]). *Suppose* (a1)-(a4) *are satisfied. In addition assume that $f(t,x)$ is nondecreasing in x for $a \leq x \leq R_0$. Denote*

$$u_0(t) = a, \quad u_n(t) = A(u_{n-1})(t) \quad \text{for } 0 \leq t \leq T, \ n = 1, 2, \ldots.$$

Then, $u_n(t) \to x_*(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$, $x_*(t)$ is the minimal solution of (1)-(2) *in C, and*

$$a \leq u_1(t) \leq \ldots \leq u_n(t) \leq \ldots \leq x_*(t) \leq R_0 \quad \text{for } 0 \leq t \leq T.$$

Proof. By Theorem 2.1, there exists in C at least one solution of (1)-(2). Moreover, any such solution $x(t)$ satisfies $a \leq x(t) \leq R_0$ for $0 \leq t \leq T$. Let $x_1(t) \in C$ be an arbitrary solution. Then, $a = u_0(t) \leq x_1(t) \leq R_0$ for $0 \leq t \leq T$. Since $f(t,x)$ is nondecreasing in x for $a \leq x \leq R_0$, it follows that the map A is nondecreasing on the interval $[a, R_0]$ of E. Thus, $u_1(t) = A(u_0)(t) \leq A(x_1)(t) = x_1(t)$. On the other hand, since $A(K) \subset X$, we have $u_1(t) = A(u_0)(t) \geq a = u_0(t)$. Hence $u_0(t) \leq u_1(t) \leq x_1(t)$ for $0 \leq t \leq T$. Further, we inductively find

$$a \leq u_1(t) \leq u_2(t) \leq ... \leq u_n(t) \leq ... \leq x_1(t) \quad \text{for } 0 \leq t \leq T.$$

Since A is completely continuous, the sequence $(u_n)_{n\geq 1} = A\left((u_n)_{n\geq 0}\right)$ must contain a subsequence, say (u_{k_n}), convergent to some $x_* \in X$. Now taking into account the monotonicity of $(u_n(t))$, we easily see that the entire sequence (u_n) converges to x_*, uniformly on $[0,T]$, and

$$u_n(t) \leq x_*(t) \leq x_1(t) \quad \text{for } 0 \leq t \leq T,\ n = 0, 1, ...$$

Letting $n \to \infty$ in $A(u_n)(t) = u_{n+1}(t)$, we get $A(x_*)(t) = x_*(t)$, i.e., $x_*(t)$ is a solution of (1)-(2). Finally, since inequality $x_*(t) \leq x_1(t)$ holds for any solution $x_1(t) \in C$, we see that $x_*(t)$ is the minimal solution in C of (1)-(2). □

The next result deals with the existence and approximation of the maximal solution in C of (1)-(2).

Theorem 4.2 ([24]). *Suppose* (a1)-(a4) *are satisfied. In addition assume that there is $R \geq R_0$ such that*

$$f\left(t, \tilde{R}\right) \leq R/\tau \quad \textit{for } -\tau \leq t \leq T \tag{15}$$

(i.e., $f(t, \varphi(t)) \leq R/\tau$ for $-\tau \leq t \leq 0$ and $f(t,R) \leq R/\tau$ for $0 < t \leq T$), and $f(t,x)$ is nondecreasing in x for $a \leq x \leq R$. Denote

$$v_0(t) = R, \quad v_n(t) = A(v_{n-1})(t) \quad \textit{for } 0 \leq t \leq T,\ n = 1, 2, ...$$

Then, $v_n(t) \to x^*(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$, $x^*(t)$ is the maximal solution in C of (1)-(2), *and*

$$a \leq x^*(t) \leq ... \leq v_n(t) \leq ... \leq v_1(t) \leq R \quad \textit{for } 0 \leq t \leq T.$$

Proof. By (15), we have $v_1(t) \leq v_0(t) = R$ for $0 \leq t \leq T$. Further the proof is analog with that of Theorem 4.1. □

Theorem 4.3. *Suppose the assumptions of Theorem4.1 are satisfied. In addition assume $a > 0$ and that there is a function $\chi : [a/R_0, 1) \to \mathbf{R}$ such that for all $\rho \in [a/R_0, 1)$, $t \in [0,T]$ and $x \in [a, R_0]$ with $\rho x \geq a$, one has*

$$1 \geq \chi(\rho) > \rho \quad \textit{and } f(t, \rho x) \geq \chi(\rho) f(t,x). \tag{16}$$

Then (1)-(2) *has a unique solution in C.*

Proof. Let $x_1(t) \in C$ be any solution of (1)-(2). We will show that $x_1(t) = x_*(t)$. Let $\rho_0 = \min_{0 \le t \le T} (x_*(t)/x_1(t))$. Since $a \le x_*(t) \le x_1(t) \le R_0$, we then have $a/R_0 \le \rho_0 \le 1$. Now we show that $\rho_0 = 1$. Suppose $\rho_0 < 1$. Since $x_*(t) \ge \max\{a, \rho_0 x_1(t)\} = \rho_0 \max\{a/\rho_0, x_1(t)\} \ge a$ for $0 \le t \le T$, by (16), we get

$$x_*(t) = A(x_*)(t) \ge A(\rho_0 \max\{a/\rho_0, x_1(t)\}) \ge$$

$$\chi(\rho_0) A(\max\{a/\rho_0, x_1(t)\}) \ge \chi(\rho_0) A(x_1)(t) = \chi(\rho_0) x_1(t).$$

It follows $\rho_0 \ge \chi(\rho_0)$, a contradiction. Therefore $\rho_0 = 1$ and so $x_*(t) = x_1(t)$. □

Remark 4.1. For $\chi(\rho) = \rho^\alpha$, $\alpha \in (0,1)$, Theorem 4.3 becomes Theorem 4 in [24]. An other example of function χ satisfying (16), is $\chi(\rho) = \log(1 + a\rho) / \log(1 + a)$, for $f(t,x)$ of the form $q(t)\log(1+x)$ (see [26, Example 17]).

Corollary 4.1. *Suppose the assumptions of Theorems 4.2 and 4.3 are satisfied. Then,* (1)-(2) *has a unique solution $x_*(t)$ in C, and for any $x_0(t) \in E$ with $a \le x_0(t) \le R$ for $0 \le t \le T$, one has $x_n(t) \to x_*(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$, where $x_n(t) = A(x_{n-1})(t)$, $n = 1, 2, \ldots$.*

Proof. From $a = u_0(t) \le x_0(t) \le v_0(t) = R$, one gets $u_n(t) \le x_n(t) \le v_n(t)$ for $n = 1, 2, \ldots$. On the other hand, Theorems 4.2 and 4.3 imply that $u_n(t) \to x_*(t)$ and $v_n(t) \to x_*(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$. □

The last result of this section refers to functions $f(t,x)$ which are nonincreasing in x.

Theorem 4.5. *Suppose* (a1)-(a4) *are satisfied. Denote*

$$R = \max\left\{R_0, \max_{0 \le t \le T} |u_1(t)|\right\}$$

and suppose $f(t,x)$ is nonincreasing in x for $0 < a \le x \le R$. Also suppose that there is a function $\chi : [a/R, 1) \to \mathbf{R}$ such that for all $\rho \in [a/R, 1)$, $t \in [0,T]$ and $x \in [a, R]$ with $\rho x \ge a$, one has

$$1 \le \chi(\rho) < 1/\rho \quad \text{and } f(t, \rho x) \le \chi(\rho) f(t,x). \tag{17}$$

Then (1)-(2) *has a unique solution $x_*(t)$ in C,*

$$a = u_0(t) \le v_1(t) \le \ldots \le u_{2n}(t) \le v_{2n+1}(t) \le \ldots \le x_*(t) \le \ldots$$

$$\le u_{2n+1}(t) \le v_{2n}(t) \le \ldots \le u_1(t) \le v_0(t) = R \quad \text{for } 0 \le t \le T,$$

and $u_n(t) \to x_*(t)$, $v_n(t) \to x_*(t)$ as $n \to \infty$, uniformly in $t \in [0,T]$.

Proof. By Theorem 4.1, there is in C at least one solution $x_1(t)$ of (1)-(2), and $a \le x_1(t) \le R_0$ for $0 \le t \le T$. We have $a = u_0(t) \le x_1(t) \le v_0(t) = R$, whence $v_1(t) \le x_1(t) \le u_1(t)$. By (a3), $a \le v_1(t)$. Also $u_1(t) \le \max_{0 \le t \le T} |u_1(t)| \le R$. Hence

$$u_0(t) \le v_1(t) \le x_1(t) \le u_1(t) \le v_0(t).$$

We then have successively,

$$\begin{aligned} a = u_0(t) \le v_1(t) \le \dots \le u_{2n}(t) \le v_{2n+1}(t) \le \dots \le x_1(t) \le \dots \\ \le u_{2n+1}(t) \le v_{2n}(t) \le \dots \le u_1(t) \le v_0(t) = R. \end{aligned} \tag{18}$$

Since A is completely continuous, there are two subsequences of $(A(u_{2n-1}))$ and $(A(v_{2n-1}))$, convergent to some $y_*(t) \in X$ and $y^*(t) \in X$, respectively. Then, by (18), it follows that

$$\begin{aligned} u_{2n}(t) \to y_*(t), \quad v_{2n+1}(t) \to y_*(t), \\ u_{2n+1}(t) \to y^*(t), \quad v_{2n}(t) \to y^*(t), \end{aligned} \tag{19}$$

uniformly in $t \in [0, T]$, and

$$y_*(t) \le x_1(t) \le y^*(t).$$

By (19), we obtain

$$y^*(t) = A(y_*)(t) \quad \text{and } y_*(t) = A(y^*)(t).$$

Next we show that (17) implies $y_*(t) = y^*(t)$. To do this, let

$$\rho_0 = \min_{0 \le t \le T} (y_*(t) / y^*(t)).$$

Obviously, $a/R \le \rho_0 \le 1$. We will show that $\rho_0 = 1$. Suppose $\rho_0 < 1$. Then (17) yields

$$y^* = A(y_*) \le A(\rho_0 \max\{a/\rho_0, y^*\}) \le$$

$$\chi(\rho_0) A(\max\{a/\rho_0, y^*\}) \le \chi(\rho_0) A(y^*) = \chi(\rho_0) y_*.$$

Thus, $\chi(\rho_0) \ge 1/\rho_0$, a contradiction. Therefore, $\rho_0 = 1$ as claimed. □

Remark 4.2. For $\chi(\rho) = \rho^\alpha$, $\alpha \in (-1, 0)$, Theorem 4.5 becomes Theorem 6 in [24].

II. THE PERIODIC PROBLEM

5 Existence of periodic solutions

We are interested in periodic continuous solutions $x(t)$ of Eq.(1), such that $0 \leq a \leq x(t) \leq R$ for all $t \in \mathbf{R}$. Our hypotheses are as follows:

(H1) $f(t,x)$ is nonnegative and continuous for $t \in \mathbf{R}$ and $a \leq x \leq R$.

(H2) There is $\omega > 0$ such that $f(t+\omega, x) = f(t,x)$ for $t \in \mathbf{R}$ and $a \leq x \leq R$.

(H3) There exists a continuous function $g(t)$ with period ω such that

$$f(t,x) \geq g(t) \quad \text{for } 0 \leq t \leq \omega \text{ and } a \leq x \leq R,$$

and

$$\int_{t-\tau}^{t} g(s)\,ds \geq a \quad \text{for } 0 \leq t \leq \omega.$$

(H4) There is a positive continuous function $h(x)$ for $a \leq x \leq R$, and a number b such that $a < b < R$,

$$f(t,x) \leq h(x) \quad \text{for } 0 \leq t \leq \omega \text{ and } a \leq x \leq R,$$

$$\textstyle\int_b^R (1/h(x))\,dx \geq \omega$$

and

$$f(t,x) < b/\tau \quad \text{for } 0 \leq t \leq \omega \text{ and } b \leq x \leq R. \tag{20}$$

Theorem 5.1 ([20]). *Suppose* (H1)-(H4) *are satisfied. Then* (1) *has at least one continuous solution* $x(t)$ *with period* ω *satisfying*

$$a \leq \min_{0 \leq t \leq \omega} x(t) < b \quad \textit{and} \quad \max_{0 \leq t \leq \omega} x(t) < R.$$

Proof. Let E be the Banach space of all continuous ω-periodic functions $x(t)$ on $\mathbf{R}$, endowed with the uniform norm $\|x\| = \max_{0 \leq t \leq \omega} |x(t)|$. Let

$$K = \{x \in E;\ a \leq x(t) \quad \text{for } 0 \leq t \leq \omega\}$$

and

$$U = \left\{x \in K;\ \min_{0 \leq t \leq \omega} x(t) < b \text{ and } \|x\| < R\right\}.$$

Obviously, K is a closed convex set of E, and U is bounded and open in K. We consider the homotopy

$$H : \overline{U} \times [0,1] \to K, \quad H(x,\lambda)(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s, x(s))\, ds.$$

By (H1)-(H3), H is well-defined and completely continuous. We claim that, for each λ, $H(.,\lambda)$ is fixed point free on the boundary ∂U of U with respect to K. Assume, by contradiction, that there would exist $\lambda \in (0,1]$ and $x \in \partial U$ such that $H(x,\lambda) = x$, that is

$$x(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s, x(s))\, ds \quad \text{for } t \in \mathbf{R}. \tag{21}$$

Since x is on ∂U, we have either

$$\|x\| = R \quad \text{and} \quad \min_{0 \le t \le \omega} x(t) < b, \tag{22}$$

or

$$\|x\| \le R \quad \text{and} \quad \min_{0 \le t \le \omega} x(t) = b. \tag{23}$$

First, suppose (22). Then, by differentiating (21), we get

$$x'(t) = \lambda f(t, x(t)) - \lambda f(t-\tau, x(t-\tau)).$$

It follows

$$x'(t) \le \lambda f(t, x(t)) \le \lambda h(x(t)) \le h(x(t)).$$

Let $t_0 \in [0,\omega]$ be such that $x(t_0) = \min_{0 \le t \le \omega} x(t)$. Integration from t_0 to t yields

$$\int_{t_0}^{t} (x'(s)/h(x(s)))\, ds \le t - t_0 \le \omega \quad \text{for } t_0 \le t \le t_0 + \omega.$$

Thus,

$$\int_{x(t_0)}^{x(t)} (1/h(u))\, du \le \omega \quad \text{for } t_0 \le t \le t_0 + \omega.$$

Since $x(t_0) < b$, by (H4), we deduce that $x(t) < R$ for $t_0 \le t \le t_0 + \omega$, equivalently for all $t \in \mathbf{R}$. Therefore, $\|x\| < R$, a contradiction. Next, suppose (23). Let $0 \le t_0 \le \omega$ be such that $x(t_0) = \min_{0 \le t \le \omega} x(t) = b$. Then, by (21) and (20), we obtain

$$b = x(t_0) = (1-\lambda)a + \lambda \int_{t_0-\tau}^{t_0} f(s, x(s))\, ds <$$

$$(1-\lambda)b + \lambda b = b,$$

again a contradiction. Thus, H is an admissible homotopy on $\overline{U}$. On the other hand, the constant map $H(.,0) \equiv a$ is essential because $a \in U$. Consequently, by the topological transversality theorem, $H(.,1)$ is essential too. □

6 Monotone iterative approximation

Under the assumptions of Theorem 5.1, denote by A the completely continuous map from $P = \{x \in E; a \leq x(t) \leq R \text{ for } 0 \leq t \leq \omega\}$ into K,

$$A(x)(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \quad t \in \mathbf{R},\ x \in P.$$

Theorem 6.1. *Suppose* (H1)-(H4) *are satisfied. In addition suppose that* $a > 0$, $f(t,x)$ *is nonincreasing in* x *for* $a \leq x \leq R$ *and there exists a function* $\chi : [a/R, 1) \to \mathbf{R}$ *satisfying* (17) *for all* $t \in [0,\omega]$, $\rho \in [a/R, 1)$ *and* $x \in [a, R]$ *with* $\rho x \geq a$. *If*

$$A^2(R)(t) \leq R \quad \text{for } 0 \leq t \leq \omega, \tag{24}$$

then (1) *has a unique solution* $x^*(t) \in P$. *Moreover, the sequence* $v_0(t) = R$, $v_n(t) = A(v_{n-1}(t))$, $n = 1, 2, \ldots$, *converges to* $x^*(t)$, *uniformly in* $t \in [0,\omega]$, *and*

$$a \leq v_1(t) \leq v_3(t) \leq \ldots \leq v_{2n+1}(t) \leq \ldots \leq x^*(t) \leq \ldots$$

$$\leq v_{2n}(t) \leq \ldots \leq v_4(t) \leq v_2(t) \leq v_0(t) = R.$$

Proof. By Theorem 5.1, there exists at least one solution in P. Let $x(t) \in P$ be any solution of (1). Since $f(t,x)$ is nonincreasing in x for $a \leq x \leq R$, from $a \leq x(t) \leq R = v_0(t)$, we get $a \leq A(R)(t) \leq A(x)(t) = x(t)$. Then,

$$a \leq A(R)(t) \leq x(t) \leq A^2(R)(t).$$

This, by (24), yields

$$a \leq A(R)(t) \leq A^3(R)(t) \leq x(t) \leq A^2(R)(t) \leq R.$$

We successively obtain

$$\begin{aligned} a \leq v_1(t) \leq v_3(t) \leq \ldots \leq v_{2n+1}(t) \leq \ldots \leq x(t) \leq \ldots \\ \leq v_{2n}(t) \leq \ldots \leq v_4(t) \leq v_2(t) \leq v_0(t) = R. \end{aligned} \tag{25}$$

Since A is completely continuous, there are two subsequences of (v_{2n+1}) and (v_{2n}) uniformly convergent to some $x_* \in P$ and $x^* \in P$, respectively. By (25) we see that the entire sequences (v_{2n+1}) and (v_{2n}) converge uniformly to x_* and x^*, respectively, and

$$a \leq x_*(t) \leq x(t) \leq x^*(t) \leq R.$$

Obviously,

$$x_*(t) = A(x^*)(t) \quad \text{and } x^*(t) = A(x_*)(t).$$

Now we prove that (17) implies $x_*(t) = x^*(t)$ for all $t \in \mathbf{R}$. To this end, let $\rho_0 = \min_{0\le t\le \omega} (x_*(t)/x^*(t))$. Clearly, $0 < a/R \le \rho_0 \le 1$. We have to show that $\rho_0 = 1$. Suppose $\rho_0 < 1$. Since $x_*(t) \ge \max\{a, \rho_0 x^*(t)\} = \rho_0 \max\{a/\rho_0, x^*(t)\} \ge a$, by (17), we get

$$x^* = A(x_*) \le A(\rho_0 \max\{a/\rho_0, x^*\}) \le \chi(\rho_0) A(\max\{a/\rho_0, x^*\}) \le$$

$$\chi(\rho_0) A(x^*) = \chi(\rho_0) x_*.$$

It follows that $\chi(\rho_0) \ge 1/\rho_0$, a contradiction. Thus $\rho_0 = 1$ as claimed. Consequently, $x_*(t) = x(t) = x^*(t)$ and the proof is complete. □

Corollary 6.1. *Suppose the assumptions of Theorem 6.1 hold with*

$$A(a)(t) \le R \quad \textit{for } 0 \le t \le \omega \tag{26}$$

instead of (24). *Then* (1) *has a unique solution* $x^*(t) \in P$ *and* $A^n(x_0)(t) \to x^*(t)$ *as* $n \to \infty$, *uniformly in* $t \in [0, \omega]$, *for any* $x_0(t) \in P$.

Proof. Let us remark that (26) implies (24). Indeed, from $a \le A(R)(t) \le A(a)(t)$, we get

$$A^2(a)(t) \le A^2(R)(t) \le A(a)(t) \le R,$$

whence (24). Thus, Theorem 6.1 applies.

Further, if $x_0(t)$ is any function in P, then from $a \le x_0(t) \le R$, we obtain

$$a \le v_1(t) \le A(x_0)(t) \le A(a)(t) \le R = v_0(t).$$

This yields

$$a \le v_1(t) \le A^2(x_0)(t) \le v_2(t) \le A(a)(t) \le R,$$

and, in general,

$$a \le v_1(t) \le v_3(t) \le ... \le v_{2[(n-1)/2]+1}(t) \le$$

$$A^n(x_0)(t) \le v_{2[n/2]}(t) \le ... \le v_2(t) \le v_0(t) = R,$$

for $n = 1, 2, ...$ Since $v_n(t) \to x^*(t)$, it follows that $A^n(x_0)(t) \to x^*(t)$, as claimed. □

Remark 6.1. A sufficient condition for (26) is that $f(t, a) \le R/\tau$ for all $t \in \mathbf{R}$.

For the next results, let us replace (H4) by the following assumption used in [12]:

(H4') $f(t, x) \le R/\tau$ for $0 \le t \le \omega$ and $a \le x \le R$.

The following theorems complement the results in [12].

Theorem 6.2. *Suppose* (H1)-(H3) *and* (H4') *are satisfied. In addition suppose* $a > 0$, $f(t,x)$ *is nonincreasing in* x *for* $a \leq x \leq R$, *and there is a function* $\chi : [a/R, 1) \to \mathbf{R}$ *satisfying* (17) *for all* $t \in [0,\omega]$, $\rho \in [a/R, 1)$ *and* $x \in [a, R]$ *with* $\rho x \geq a$. *Then* (1) *has a unique solution* $x^*(t) \in P$ *and* $A^n(x_0)(t) \to x^*(t)$ *as* $n \to \infty$, *uniformly in* $t \in [0,\omega]$, *for any* $x_0(t) \in P$.

Theorem 6.3. *Suppose* (H1)-(H3) *and* (H4') *are satisfied. In addition suppose* $a > 0$, $f(t,x)$ *is nondecreasing in* x *for* $a \leq x \leq R$, *and there is a function* $\chi : [a/R, 1) \to \mathbf{R}$ *satisfying* (16) *for all* $t \in [0,\omega]$, $\rho \in [a/R, 1)$ *and* $x \in [a, R]$ *with* $\rho x \geq a$. *Then* (1) *has a unique solution* $x^*(t) \in P$ *and* $A^n(x_0)(t) \to x^*(t)$ *as* $n \to \infty$, *uniformly in* $t \in [0,\omega]$, *for any* $x_0(t) \in P$.

The proofs of Theorems 6.2 and 6.3 are similar with that of Theorem 6.1, so we omit the details.

Remark 6.2. For $\chi(\rho) = \rho^\alpha$, $\alpha \in (-1, 0)$, Theorems 6.1 and 6.2 have been established in [20]. Also, in [20], several examples can be found.

The monotone iterative approximation of periodic solutions of Eq.(1), for the case when $f(t,x)$ is nondecreasing in x, was discussed in [12].

Finally, for similar results by means of more subtle conditions than (a4) and (H4), we send to [21] and [25].

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