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## MONOTONE APPROXIMATION FOR AN INTEGRAL EQUATION MODELING INFECTIOUS DISEASE

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#### Abstract

Monotone technique is used to approximate the periodic solutions of a delay integral equation modeling epidemics and population growth. Both cases of nondecreasing and nonincreasing contact rate are considered.

Keywords: population dynamics, nonlinear integral equation, periodic solution, fixed point, continuation principle, monotone iterations.

In this paper we are concerned with the following nonlinear delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s)) ds.$$
 (1)

This equation and similar others appear when investigating the spread of virus diseases or, more generally, the growth of single species populations [2].

Let us first describe the meaning of Eq.(1) in terms of epidemics. In this case, it is assumed that the total number of population members is constant; x(t) represents the proportion of infectives in population at time t, regarded as a continuous quantity;  $\tau$  is the length of time an individual remains infectious (duration of infectivity); f(t, x(t)) means the proportion of new infectives per unit time (instantaneous contact rate). Then, f(t, x(t)) dt represents the proportion of individuals infected within the period t, t + dt. In consequence, the number of infectious individuals at time t equals the sum of all individuals infected between  $t - \tau$  and t.

Let us now interpret Eq.(1) as a growth equation of a single species population when the birth rate varies seasonally. In this case, x(t) is the number of individuals of a single species population at time t, f(t, x(t)) is the number of new births per unit time, and  $\tau$  is the lifetime. It is assumed that each individual lives to the age  $\tau$  exactly and then dies.

We are interested in periodic continuous solutions x(t) of Eq.(1), such that  $0 \le a \le x(t) \le R$  for all  $t \in \mathbf{R}$ . Our hypotheses are as follows:

- **(H1)** f(t,x) is nonnegative and continuous for  $t \in \mathbf{R}$  and  $a \leq x \leq R$ .
- **(H2)** There is  $\omega > 0$  such that  $f(t + \omega, x) = f(t, x)$  for  $t \in \mathbf{R}$  and  $a \le x \le R$ .
- (H3) There exists a continuous function g(t) with period  $\omega$  such that

$$f(t,x) \ge g(t)$$
 for  $0 \le t \le \omega$  and  $a \le x \le R$ ,

and

$$\int_{t-\tau}^{t} g(s) ds \ge a \text{ for } 0 \le t \le \omega.$$

(H4) There is a positive continuous function h(x) for  $a \le x \le R$ , and a number b such that a < b < R,  $f(t, x) < h(x) \quad \text{for } 0 \le t \le \omega \text{ and } a \le x \le R,$ 

$$\int_{h}^{R} (1/h(x)) dx \ge \omega$$

and

$$f(t,x) < b/\tau \text{ for } 0 \le t \le \omega \text{ and } b \le x \le R.$$
 (2)

**Theorem 1** ([6]). Suppose (H1)-(H4) are satisfied. Then Eq.(1) has at least one continuous solution x(t) with period  $\omega$  satisfying

$$a \le \min_{0 \le t \le \omega} x(t) < b \text{ and } \max_{0 \le t \le \omega} x(t) < R.$$

**Proof.** Let E be the Banach space of all continuous  $\omega$ -periodic functions x(t) on R, endowed with the uniform norm  $||x|| = \max_{0 \le t \le \omega} |x(t)|$ . Let

$$K = \{x \in E; \ a \le x (t) \text{ for } 0 \le t \le \omega\}$$

and

$$U = \left\{ x \in K; \min_{0 \le t \le \omega} x(t) < b \text{ and } ||x|| < R \right\}.$$

Obviously, K is a closed convex set of E, and U is bounded and open in K. We consider the homotopy

$$H: \overline{U} \times [0,1] \to K, \quad H(x,\lambda)(t) = (1-\lambda)a + \lambda \int_{t-\tau}^{t} f(s,x(s)) ds.$$

By (H1)-(H3), H is well-defined and completely continuous. We claim that, for each  $\mathcal{H}(.,\lambda)$  is fixed point free on the boundary  $\partial U$  of U with respect to K. Assume, by contradiction, that there would exist  $\lambda \in (0,1]$  and  $x \in \partial U$  such that  $H(x,\lambda) = x$ , that is

$$x(t) = (1 - \lambda) a + \lambda \int_{t-\tau}^{t} f(s, x(s)) ds \text{ for } t \in \mathbf{R}.$$

Since x is on  $\partial U$ , we have either

$$||x|| = R$$
 and  $\min_{0 \le t \le \omega} x(t) < b$ ,

$$||x|| \le R$$
 and  $\min_{0 \le t \le \omega} x(t) = b.$  (5)

First, suppose (4). Then, by differentiating (3), we get

$$x'(t) = \lambda f(t, x(t)) - \lambda f(t - \tau, x(t - \tau)).$$

It follows

$$x'(t) \le \lambda f(t, x(t)) \le \lambda h(x(t)) \le h(x(t))$$
.

Let  $t_{0} \in [0, \omega]$  be such that  $x(t_{0}) = \min_{0 \leq t \leq \omega} x(t)$ . Integration from  $t_{0}$  to t yields

$$\int_{t_0}^{t} (x'(s)/h(x(s))) ds \le t - t_0 \le \omega \text{ for } t_0 \le t \le t_0 + \omega.$$

Thus,

$$\int_{x(t_0)}^{x(t)} (1/h(u)) du \le \omega \text{ for } t_0 \le t \le t_0 + \omega.$$

Since  $x(t_0) < b$ , by (H4), we deduce that x(t) < R for  $t_0 \le t \le t_0 + \omega$ , equivalently for all  $t \in \mathbf{R}$ . Therefore, ||x|| < R, a contradiction. Next, suppose (5). Let  $0 \le t_0 \le \omega$  be such that  $x(t_0) = \min_{0 \le t \le \omega} x(t) = b$ . Then, by (3) and (2), we obtain

$$b = x(t_0) = (1 - \lambda) a + \lambda \int_{t_0 - \tau}^{t_0} f(s, x(s)) ds < (1 - \lambda) b + \lambda b = b,$$

again a contradiction. Thus, H is an admissible homotopy on  $\overline{U}$ . On the other hand, the constant map  $H(.,0) \equiv a$  is essential because  $a \in U$ . Consequently, by the topological transversality theorem, H(.,1) is essential too.

Denote

$$P = \{x \in E; \ a \le x(t) \le R \text{ for } 0 \le t \le \omega\}.$$

Under the above assumptions, consider the completely continuous map from P into K,

$$A(x)(t) = \int_{t-\tau}^{t} f(s, x(s)) ds, \quad t \in \mathbf{R}, \ x \in P.$$

**Theorem 2.** Suppose (H1)-(H4) are satisfied. In addition suppose that a > 0, f(t,x) is nonincreasing in x for  $a \le x \le R$  and there exists a function  $\chi : [a/R, 1) \to \mathbf{R}$  such that

$$1 \le \chi(\rho) < 1/\rho \text{ and } f(t, \rho x) \le \chi(\rho) f(t, x)$$
 (6)

for all  $t \in [0, \omega]$ ,  $\rho \in [a/R, 1)$  and  $x \in [a, R]$  with  $\rho x \ge a$ . If

$$A^{2}(R)(t) \leq R \text{ for } 0 \leq t \leq \omega, \tag{7}$$

then Eq.(1) has a unique solution  $x^* \in P$ . Moreover, the sequence  $v_0 \equiv R$ ,  $v_n = A(v_{n-1})$ , n = 1, 2, ..., converges to  $x^*$ , uniformly on  $[0, \omega]$ , and

$$a \le v_1 \le v_3 \le \dots \le v_{2n+1} \le \dots \le x^* \le \dots$$

$$\leq v_{2n} \leq \ldots \leq v_4 \leq v_2 \leq v_0 = R.$$

**Proof.** By Theorem 1, there exists at least one solution in P. Let  $x \in P$  be any solution of Eq.(1). Since f(t,x) is nonincreasing in x for  $a \le x \le R$ , from  $a \le x(t) \le R = v_0(t)$ , we get  $a \le A(R)(t) \le A(x)(t) = x(t)$ . Then,

$$a \le A(R)(t) \le x(t) \le A^2(R)(t).$$

This, by (7), yields

$$a \le A(R)(t) \le A^3(R)(t) \le x(t) \le A^2(R)(t) \le R.$$

We successively obtain

$$a \le v_1(t) \le v_3(t) \le \dots \le v_{2n+1}(t) \le \dots \le x(t) \le \dots$$

$$\le v_{2n}(t) \le \dots \le v_4(t) \le v_2(t) \le v_0(t) = R.$$
(8)

Since A is completely continuous, there are two subsequences of  $(v_{2n+1})$  and  $(v_{2n})$  uniformly convergent to some  $x_* \in P$  and  $x^* \in P$ , respectively. By (8) we see that the entire sequences  $(v_{2n+1})$  and  $(v_{2n})$  converge uniformly to  $x_*$  and  $x^*$ , respectively, and

$$a \le x_*(t) \le x(t) \le x^*(t) \le R.$$

Obviously,

$$x_*(t) = A(x^*)(t)$$
 and  $x^*(t) = A(x_*)(t)$ .

Now we prove that (6) implies  $x_*(t) = x^*(t)$  for all  $t \in \mathbf{R}$ . To this end, let  $\rho_0 = \min_{0 \le t \le \omega} (x_*(t)/x^*(t))$ . Clearly,  $0 < a/R \le \rho_0 \le 1$ . We have to show that  $\rho_0 = 1$ . Suppose  $\rho_0 < 1$ . Since  $x_*(t) \ge \max\{a, \rho_0 x^*(t)\} = \rho_0 \max\{a/\rho_0, x^*(t)\} \ge a$ , by (6), we get

$$x^* = A(x_*) \le A(\rho_0 \max\{a/\rho_0, x^*\}) \le \chi(\rho_0) A(\max\{a/\rho_0, x^*\}) \le \chi(\rho_0) A(\max\{a/\rho_0, x^*\})$$

$$\chi\left(\rho_{0}\right)A\left(x^{*}\right)=\chi\left(\rho_{0}\right)x_{*}.$$

It follows that  $\chi(\rho_0) \geq 1/\rho_0$ , a contradiction. Thus  $\rho_0 = 1$  as claimed. Consequently,  $x_*(t) = x(t) = x^*(t)$  and the proof is complete.

Corollary 3. Suppose the assumptions of Theorem 2 hold with

$$A(a)(t) \le R \text{ for } 0 \le t \le \omega$$
 (9)

instead of (7). Then Eq.(1) has a unique solution  $x^* \in P$  and  $A^n(x_0)(t) \to x^*(t)$  as  $n \to \infty$ , uniformly in  $t \in [0, \omega]$ , for any  $x_0 \in P$ .

**Proof.** Let us remark that (9) implies (7). Indeed, from  $a \leq A(R)(t) \leq A(a)(t)$ , we get

$$A^{2}(a)(t) \leq A^{2}(R)(t) \leq A(a)(t) \leq R,$$

whence (7). Thus, Theorem 2 applies.

Further, if  $x_0$  is any function in P, then from  $a \le x_0(t) \le R$ , we obtain

$$a \le v_1(t) \le A(x_0)(t) \le A(a)(t) \le R = v_0(t)$$
.

This yields

$$a \le v_1(t) \le A^2(x_0)(t) \le v_2(t) \le A(a)(t) \le R$$

and, in general,

$$a \le v_1(t) \le v_3(t) \le \dots \le v_{2[(n-1)/2]+1}(t) \le$$

$$A^{n}(x_{0})(t) \leq v_{2[n/2]}(t) \leq ... \leq v_{2}(t) \leq v_{0}(t) = R,$$

for n=1,2,... Since  $v_{n}\left(t\right)\to x^{*}\left(t\right)$ , it follows that  $A^{n}\left(x_{0}\right)\left(t\right)\to x^{*}\left(t\right)$ , as claimed.

**Remark.** A sufficient condition for (9) is that  $f(t, a) \leq R/\tau$  for all  $t \in \mathbb{R}$ .

For the next result, let us replace (H4) by the following assumption used in [4]:

**(H4')** 
$$f(t,x) \leq R/\tau$$
 for  $0 \leq t \leq \omega$  and  $a \leq x \leq R$ .

The next theorems complement the results in [4].

**Theorem 4.** Suppose (H1)-(H3) and (H4') are satisfied. In addition suppose a > 0, f(t, x) is nonincreasing in x for  $a \le x \le R$ , and there is a function  $\chi : [a/R, 1) \to \mathbf{R}$  satisfying (6) for all  $t \in [0, \omega]$ ,  $\rho \in [a/R, 1)$  and  $x \in [a, R]$  with  $\rho x \ge a$ . Then Eq.(1) has a unique solution  $x^* \in P$  and  $A^n(x_0)(t) \to x^*(t)$  as  $n \to \infty$ , uniformly in  $t \in [0, \omega]$ , for any  $x_0 \in P$ .

**Theorem 5.** Suppose (H1)-(H3) and (H4') are satisfied. In addition suppose a > 0, f(t, x) is nondecreasing in x for  $a \le x \le R$ , and there is a function  $\chi: [a/R, 1] \to \mathbf{R}$  such that

$$1 \ge \chi(\rho) > \rho$$
 and  $f(t, \rho x) \ge \chi(\rho) f(t, x)$ 

for all  $t \in [0, \omega]$ ,  $\rho \in [a/R, 1)$  and  $x \in [a, R]$  with  $\rho x \ge a$ . Then Eq.(1) has a unique solution  $x^*$  in P and  $A^n(x_0)(t) \to x^*(t)$  as  $n \to \infty$ , uniformly in  $t \in [0, \omega]$ , for any  $x_0 \in P$ .

**Proof.** The proofs are similar with that of Theorem 2, so we omit the details.

**Remark.** For  $\chi(\rho) = \rho^{\alpha}$ ,  $\alpha \in (-1,0)$ , Theorem 2 and Theorem 4 have been established in [6]. Also, in [6], several examples can be found.

The monotone iterative approximation of periodic solutions of Eq.(1), for the case when f(t,x) is nondecreasing in x, was discussed in [4]. Related topics can be found in [1], [3], [5] and [7]

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