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Fixed Point Theorems for Set-Valued Maps and Existence Principles for Integral Inclusions

Donal O'Regan

Department of Mathematics, National University of Ireland, Galway, Ireland

and

Radu Precup

Department of Applied Mathematics, "Babeş-Bolyai" University, Cluj, Romania

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New fixed point theorems of Mönch type are presented for set-valued maps. These theorems are then used to establish general existence principles for Hammerstein integral inclusions in Banach spaces. © 2000 Academic Press

1. INTRODUCTION

In [13] Mönch obtained the following common generalization of the fixed point theorems of Schauder, Krasnoselskii, Darbo, and Sadovskii:

THEOREM 1.1. [13]. *Let D be a closed, convex subset of a Banach space X and $N\colon D \to D$ continuous with the further property that for some $x_0 \in D$ one has*

$$C \subset D \text{ countable}, \qquad \overline{C} = \overline{\operatorname{conv}}\big(\{x_0\} \cup N(C)\big) \qquad \Longrightarrow \qquad \overline{C} \text{ compact}. \tag{1.1}$$

Then N has a fixed point.

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The continuation theorem of Leray–Schauder type accompanying Theorem 1.1 is also due to Mönch [13] (for both results see alternatively [7]):

THEOREM 1.2. [13]. *Let K be a closed, convex subset of a Banach space X, U a relatively open subset of K, and $N: \overline{U} \to K$ continuous with the further property that for some $x_0 \in U$ one has*

$$C \subset \overline{U} \text{ countable}, \qquad C \subset \overline{\text{conv}}\big(\{x_0\} \cup N(C)\big) \qquad \Longrightarrow \qquad \overline{C} \textit{ compact}. \tag{1.2}$$

In addition, assume

$$x \neq (1-\lambda)x_0 + \lambda N(x) \qquad \textit{for all } x \in \overline{U} \setminus U \quad \textit{and} \quad \lambda \in (0,1).$$

Then N has a fixed point in $\overline{U}$.

Applications of Theorems 1.1 and 1.2 to differential and integral equations in abstract spaces can be found in Mönch [13], Mönch and von Harten [14], Deimling [7], Guo et al. [10], Agarwal and O'Regan [1], and O'Regan and Precup [17, 18].

To check (1.1) or (1.2) in applications we make use of the following result of Heinz [11] (see also [17] for an easier proof).

THEOREM 1.3. [11]. *Let E be a Banach space and $C \subset L^1([a,b]; E)$ countable with $|u(t)| \leq h(t)$ for a.e. $t \in [a,b]$ and every $u \in C$, where $h \in L^1[a,b]$. Then the function $\varphi(t) = \alpha(C(t))$ belongs to $L^1[a,b]$ and satisfies*

$$\alpha\left(\left\{\int_a^b u(t)\,dt : u \in C\right\}\right) \leq 2\int_a^b \alpha\big(C(t)\big)\,dt$$

(*here α is the Kuratowski measure of noncompactness*).

We note that the factor 2 in the above inequality (and in all inequalities based on it) can be dropped if instead of α we use β, the ball measure of noncompactness.

In our recent paper [17] we discussed integral equations of the form

$$u(t) = \int_0^t k(t,s)g(s,u(s))\,ds, \qquad t \in [0,T]$$

and

$$u(t) = \int_0^T k(t,s)g(s,u(s))\,ds, \qquad t \in [0,T]$$

in a ball of a Banach space. There we assumed a condition of the form

$$\alpha\,(g(s,M)) \leq \omega\,(s, \alpha(M))$$

for a.e. $s \in [0,T]$ and any bounded set M, and we established some existence criteria using Theorems 1.1–1.3.

The goal of this paper is to extend our results from [17] to Hammerstein and Volterra integral inclusions. Such inclusions arise naturally in mathematical physics, control theory, and critical point theory for nonsmooth energy functionals (see [2, 4]).

For recent existence results for integral inclusions in abstract spaces we refer the reader to Agarwal and O'Regan [1], Appel et al. [2], Cardinali and Papageorgiou [4], Frigon [9], O'Regan [15, 16], and the references therein. Most of the results proved in the above papers are based on the multivalued analogs of the Banach, Schauder, Sadovskii fixed point theorems and of the corresponding theorems of Leray–Schauder type.

In this paper we first extend the Mönch theorems to set-valued maps. The extension is possible if we replace (1.1), (1.2) by some more general conditions which are expressed in terms of a pair (M, C) instead of a single set C. Next, the fixed point theorems are applied to establish existence principles for Hammmerstein integral inclusions in general Banach spaces. The results extend and complement the existing literature.

2. PRELIMINARIES

For any nonempty set X we let 2^X be the collection of all subsets of X and $P(X) = 2^X \setminus \{\emptyset\}$. In the case that X is a Hausdorff topological space, we let

$$P_f(X) = \{A \subset X : \ A \text{ is nonempty, closed}\},$$
$$P_k(X) = \{A \subset X : \ A \text{ is nonempty, compact}\}.$$

If X is a closed, convex subset of a normed space $(E, |\cdot|)$, then we define

$$P_c(X) = \{A \subset X : A \text{ is nonempty, convex}\},$$
$$P_{fc}(X) = \{A \subset X : A \text{ is nonempty, closed, convex}\},$$
$$P_{kc}(X) = \{A \subset X : A \text{ is nonempty, compact, convex}\},$$

and for any $A \in P(E)$ we let $|A| = \sup\{|x| : x \in A\}$ and by $\operatorname{conv}(A)$ we mean the convex hull of A.

Let X, Y be two sets, $N\colon X \to 2^Y$ a set-valued map, and $A \subset Y$. We define

$$\operatorname{graph}(N) = \{(x, y) : x \in X, y \in N(x)\} \qquad (\text{the } \textit{graph} \text{ of } N),$$
$$N(A) = \cup\{N(x) : x \in A\} \qquad (\text{the } \textit{image} \text{ of } A),$$
$$N^-(A) = \{x \in X : N(x) \cap A \neq \emptyset\} \qquad (\text{the } \textit{weak inverse} \text{ of } A).$$

Suppose now that X, Y are Hausdorff topological spaces. A map $N : X \to P(Y)$ is said to be *upper semicontinuous* if $N^{-}(A)$ is closed for every closed set $A \in P(Y)$. The map N is said to be *lower semicontinuous* if $N^{-}(A)$ is open for every open set $A \in P(Y)$.

We now state some well-known results of set-valued analysis (for proof see [3, 8, 12]).

(P1) Let X, Y be Hausdorff topological spaces and N: $X \to P_f(Y)$. If N is upper semicontinuous, then graph(N) is closed in $X \times Y$. Conversely, if graph(N) is closed and $\overline{N(X)}$ is compact, then N is upper semicontinuous.

(P2) Let X, Y be Hausdorff topological spaces and N: $X \to P_k(Y)$ upper semicontinuous. Then $N(A)$ is compact for each compact $A \subset X$.

(P3) Bohnenblust–Karlin's fixed point theorem states if X is a Banach space, $K \in P_{kc}(X)$, and N: $K \to P_{fc}(K)$ is upper semicontinuous, then N has a fixed point (i.e., there exists $x \in K$ with $x \in N(x)$).

(P4) Michael's selection theorem states if X is a metric space, Y a Banach space, and N: $X \to P_{fc}(Y)$ is lower semicontinuous, then there exists a continuous map N_0: $X \to Y$ with $N_0(x) \in N(x)$ for all $x \in X$.

Throughout this paper E will be a real Banach space with norm $|\cdot|$. We denote by $C([a, b]; E)$ the space of continuous functions u: $[a, b] \to E$ and by $|\cdot|_\infty$ its max-norm $|u|_\infty = \max_{t\in[a, b]} |u(t)|$. For any subset $M \subset E$, we denote by $C([a, b]; M)$ the set of all functions in $C([a, b]; E)$ which take values in M.

A function u: $[a, b] \to E$ is said to be *strongly measurable* on $[a, b]$ if there exists a sequence of finitely valued functions u_n with

$$u_n(t) \to u(t) \qquad \text{as } n \to \infty, \qquad \text{a.e. } t \in [a, b].$$

By $\int_a^b u(t)\, dt$ we mean the Bochner integral of u, assuming it exists. Recall that a strongly measurable function u is Bochner integrable if and only if $|u|$ is Lebesgue integrable.

For any real $p \in [1, \infty]$, we consider the space $L^p([a, b]; E)$ of all strongly measurable functions $u : [a, b] \to E$ such that $|u|^p$ is Lebesgue integrable on $[a, b]$. $L^p([a, b]; E)$ is a Banach space under the norm

$$|u|_p = \left(\int_a^b |u(s)|^p\, ds \right)^{1/p}$$

for $p < \infty$ and

$$|u|_\infty = \operatorname*{ess\,sup}_{t\in[a, b]} |u(t)| = \inf \{c \geq 0 : |u(t)| \leq c \text{ a.e. } t \in [a, b]\}.$$

In particular, $L^1([a, b]; E)$ is the space of Bochner integrable functions on $[a, b]$. When $E = \mathbf{R}$, the space $L^p([a, b]; \mathbf{R})$ is simply denoted by $L^p[a, b]$.

We say that a map ϕ: $[a, b] \times D \to P(E)$, $D \subset E$, is *L^p-Carathéodory* $(1 \le p \le \infty)$ if

(C1) $\phi(\cdot, x)$ has a strongly measurable selection for each $x \in D$;

(C2) $\phi(t, \cdot)$ is upper semicontinuous for a.e. $t \in [a, b]$;

(C3) for each $r > 0$ there exists $h_r \in L^p[a, b]$ with $|\phi(t, x)| \le h_r(t)$ for all $x \in D$ satisfying $|x| \le r$ and a.e. $t \in [a, b]$.

Finally we recall the definition of the *Kuratowski measure of noncompactness.* Let $M \subset E$ be bounded. Then

$$\alpha(M) = \inf\left\{\varepsilon > 0 : M \subset \bigcup_{j=1}^{m} M_j \text{ and diam } (M_j) \le \varepsilon\right\}.$$

3. FIXED POINT THEOREMS FOR SET-VALUED MAPS

Our first result is the set-valued analog of the Mönch's fixed point theorem for a self-map of a closed, convex subset of a Banach space (Theorem 1.1).

THEOREM 3.1. *Let D be a closed, convex subset of a Banach space X and N: $D \to P_c(D)$. Assume graph(N) is closed, N maps compact sets into relatively compact sets, and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{l} M \subset D,\ M = \operatorname{conv}\big(\{x_0\} \cup N(M)\big) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \quad \Longrightarrow \quad \overline{M} \text{ compact}. \tag{3.1}$$

Then there exists $x \in D$ with $x \in N(x)$.

Remark. (a) Under the assumptions of Theorem 3.1, N: $D \to P_{kc}(D)$.

(b) According to (P1) and (P2), all the assumptions on N in Theorem 3.1 are fulfilled if N: $D \to P_{kc}(D)$ is upper semicontinuous and satisfies (3.1).

Proof. (a) Define

$$M_0 = \{x_0\}, \qquad M_n = \operatorname{conv}\big(\{x_0\} \cup N(M_{n-1})\big) \quad \text{for } n \ge 1.$$

It is clear that $M_n \subset D$ and M_n is convex. Also, it is immediately seen by induction that $M_{n-1} \subset M_n$. Consequently, the set $M = \cup\{M_n : n \ge 0\}$ is convex and $M \subset D$. In addition, it is easy to show that $M = \operatorname{conv}(\{x_0\} \cup N(M))$.

(b) M_n is relatively compact. This follows by induction, using the property of N of sending compact sets into relatively compact sets and Mazur's lemma. Consequently, there exists a countable set $C_n \subset M_n$ with $\overline{C_n} = \overline{M_n}$. Let $C = \cup\{C_n : n \geq 0\}$. It is clear that $C \subset M$ is countable and $\overline{M} = \overline{C}$. Now (3.1) guarantees that the set $K = \overline{M}$ is compact (nonempty and convex, obviously). Hence $K \in P_{kc}(X)$.

(c) We also have $K \subset N^{-}(K)$. Indeed, if $x \in K$, then $x_n \to x$ as $n \to \infty$, for some $x_n \in M$. Now take any $y_n \in N(x_n)$. Since $N(M) \subset M$, we have $y_n \in K$. Due to the compactness of K, we may suppose that $y_n \to y$ as $n \to \infty$, for some $y \in K$. Since $(x_n, y_n) \in \text{graph}(N)$ and $\text{graph}(N)$ is closed, we also have $(x, y) \in \text{graph}(N)$. Thus, $y \in N(x) \cap K$, that is, $x \in N^{-}(K)$, as claimed.

(d) Define the map $\widehat{N}: K \to P_{fc}(K)$ by

$$\widehat{N}(x) = N(x) \cap K.$$

Notice $\widehat{N}(x) \neq \emptyset$ because of $K \subset N^{-}(K)$.

It is clear from the definition that $\text{graph}(\widehat{N})$ is closed like $\text{graph}(N)$. Then, from (P1), we find that $\widehat{N}$ is upper semicontinuous. The Bohnenblust–Karlin fixed point theorem for $\widehat{N}$ guarantees the existence of an $x \in K$ with $x \in \widehat{N}(x)$. Since $K \subset D$ and $\widehat{N}(x) \subset N(x)$, we also have $x \in D$ and $x \in N(x)$. ∎

Remark. In case that N also satisfies

$$N(\overline{M}) \subset \overline{N(M)} \qquad \text{for every } M \subset D \tag{3.2}$$

(which always holds if N is lower semicontinuous), a sufficient condition for (3.1) is the Mönch's original condition,

$$C \subset D \text{ countable}, \quad \overline{C} = \overline{\text{conv}}\big(\{x_0\} \cup N(C)\big) \quad \Longrightarrow \quad \overline{C} \text{ compact}. \tag{3.3}$$

Indeed, if $M \subset D$ satisfies $M = \text{conv}(\{x_0\} \cup N(M))$ and $\overline{M} = \overline{C}$ for some countable set $C \subset M$, then from (3.2), we obtain

$$N(M) \subset N(\overline{M}) = N(\overline{C}) \subset \overline{N(C)} \subset \overline{\text{conv}}\big(\{x_0\} \cup N(C)\big).$$

This implies

$$\overline{\text{conv}}\big(\{x_0\} \cup N(M)\big) \subset \overline{\text{conv}}\big(\{x_0\} \cup N(C)\big).$$

The converse inclusion being obvious, we find that

$$\overline{\text{conv}}\big(\{x_0\} \cup N(M)\big) = \overline{\text{conv}}\big(\{x_0\} \cup N(C)\big).$$

Consequently, $\overline{C} = \overline{\text{conv}}(\{x_0\} \cup N(C))$. Now (3.3) guarantees that $\overline{C} = \overline{M}$ is compact.

The next result is the continuation principle accompanying Theorem 3.1.

THEOREM 3.2. *Let K be a closed, convex subset of a Banach space X, U a relatively open subset of K, and $N: \overline{U} \to P_c(K)$. Assume graph$(N)$ is closed, N maps compact sets into relatively compact sets, and that for some $x_0 \in U$, the following two conditions are satisfied*:

$$\left.\begin{array}{l} M \subset \overline{U}, M \subset \operatorname{conv}(\{x_0\} \cup N(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \quad \Longrightarrow \quad \overline{M} \text{ compact}, \tag{3.4}$$

$$x \notin (1-\lambda)x_0 + \lambda N(x) \qquad \text{for all } x \in \overline{U} \setminus U, \lambda \in (0,1). \tag{3.5}$$

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

Proof. We may assume that (3.5) holds for all $\lambda \in [0,1]$ (the conclusion being trivial if (3.5) does not hold for $\lambda = 1$). Also, we may suppose that $U \neq K$ (otherwise the conclusion follows immediately from Theorem 3.1), so $\overline{U} \setminus U \neq \emptyset$.

Define $H: \overline{U} \times [0,1] \to P(K)$ by

$$H(x, \lambda) = (1-\lambda)x_0 + \lambda N(x).$$

Let

$$\Sigma = \big\{x \in \overline{U} : x \in H(x, \lambda) \text{ for some } \lambda \in [0,1]\big\}.$$

The sets Σ and $\overline{U} \setminus U$ are nonempty ($x_0 \in \Sigma$) and disjoint by (3.5) extended to all $\lambda \in [0,1]$. We claim that Σ is closed like is $\overline{U} \setminus U$. Indeed, consider $x_n \in \Sigma$ with $x_n \to x$ as $n \to \infty$. Then there exist $\lambda_n \in [0,1]$ and $y_n \in N(x_n)$ with

$$x_n - (1-\lambda_n)\, x_0 = \lambda_n y_n. \tag{3.6}$$

Clearly, we may suppose that $\lambda_n \to \lambda$ as $n \to \infty$ for some $\lambda \in [0,1]$. Two cases are possible:

(a) $\lambda \neq 0$. Then $y_n = \lambda_n^{-1}(x_n - (1-\lambda_n)x_0) \to \lambda^{-1}(x - (1-\lambda)x_0)$ and, since graph(N) is closed, we deduce that

$$\lambda^{-1}\,(x - (1-\lambda)\,x_0) \in N\,(x)\,,$$

i.e., $x \in (1-\lambda)x_0 + \lambda N(x)$. This shows that $x \in \Sigma$.

(b) $\lambda = 0$. Since $\overline{\{x_n : n \geq 1\}}$ is compact, its image by N is relatively compact and, in consequence, bounded. It follows that $\{y_n : n \geq 1\}$ is bounded. Now letting $n \to \infty$ in (3.6), we obtain $x - x_0 = 0$. Thus $x = x_0 \in \Sigma$.

Therefore Σ is closed as claimed. Urysohn's lemma now guarantees the existence of a continuous function $\theta : \overline{U} \to [0,1]$ with $\theta(x) = 0$ on $\overline{U} \setminus U$ and $\theta(x) = 1$ on Σ.

Now we define

$$D = \overline{\mathrm{conv}}\big(\{x_0\} \cup N(\overline{U})\big)$$

and $\widetilde{N}$: $D \to P_c(D)$ by

$$\widetilde{N}(x) = \begin{cases} (1-\theta(x))\, x_0 + \theta(x) N(x) & \text{for } x \in D \cap U \\ x_0 & \text{for } x \in D \setminus U. \end{cases}$$

It is not difficult to prove that graph($\widetilde{N}$) is closed and $\widetilde{N}$ maps compact sets into relatively compact sets. Next we check (3.1) for $\widetilde{N}$. Let $M \subset D$ be such that $M = \mathrm{conv}(\{x_0\} \cup \widetilde{N}(M))$ and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Since

$$M \cap U \subset M \subset \mathrm{conv}\left(\{x_0\} \cup \widetilde{N}(M)\right) = \mathrm{conv}\left(\{x_0\} \cup N(M \cap U)\right),$$

$$\overline{M \cap U} = \overline{C \cap U} \qquad \text{and} \qquad C \cap U \subset M \cap U \text{ is countable,} \tag{3.7}$$

from (3.4) we deduce that $M \cap U$ is relatively compact. Then $N(M \cap U)$ is relatively compact and Mazur's lemma and (3.7) imply that $\overline{M}$ is compact. Thus (3.1) holds for $\widetilde{N}$.

Now we apply Theorem 3.1 to deduce that there exists an $x \in D \subset \overline{U}$ with $x \in \widetilde{N}(x)$. The case $x \notin U$ is impossible because, otherwise, $x \in \widetilde{N}(x) = \{x_0\}$, that is, $x = x_0$, which contradicts $x_0 \in U$. Hence $x \in U$ and, consequently,

$$x \in (1-\theta(x))\, x_0 + \theta(x) N(x). \tag{3.8}$$

Thus $x \in \Sigma$ and so $\theta(x) = 1$. Then (3.8) shows that $x \in N(x)$. ∎

The analogs of Theorems 3.1 and 3.2 for lower semicontinuous maps are as follows:

THEOREM 3.3. *Let D be a closed, convex subset of a Banach space X and N: $D \to P_{fc}(D)$. Assume N is lower semicontinuous and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{l} M \subset D, M \subset \mathrm{conv}\left(\{x_0\} \cup N(M)\right) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \implies \overline{M} \text{ compact.} \tag{3.9}$$

Then there exists $x \in D$ with $x \in N(x)$.

Proof. Michael's selection theorem guarantees the existence of a continuous selection N_0: $D \to D$ of N. Now (3.9) implies that N_0 satisfies (3.1). Thus, Theorem 3.1 applies to N_0. ∎

THEOREM 3.4. *Let K be a closed, convex subset of a Banach space X, U a relatively open subset of K, and $N\colon \overline{U} \to P_{fc}(K)$. Assume N is lower semicontinuous and that for some $x_0 \in U$,* (3.4) *and* (3.5) *hold. Then there exists $x \in \overline{U}$ with $x \in N(x)$.*

Proof. Use a continuous selection of N and apply Theorem 3.2. ∎

Remark (the use of two norms on X). Let $|\cdot|$ be the norm of the Banach space X and let $|\cdot|'$ be another norm on X not necessarily complete. Assume that there is a constant $c > 0$ such that

$$|x|' \le c|x| \qquad \text{for all } x \in X.$$

Then the results in Theorems 3.1–3.4 remain true if the conclusion in the implications (3.1), (3.4), and (3.9) is that M is relatively compact with respect to the topology induced by $|\cdot|'$, provided that all the topological properties of $N\colon D(N) \to P(X)$ are understood with respect to the $|\cdot|'$-topology on $D(N)$ and the $|\cdot|$-topology on X. Indeed, if the graph of N is $(|\cdot|', |\cdot|)$-closed, then it is also $(|\cdot|, |\cdot|)$-closed; if N maps $|\cdot|'$-compact sets into $|\cdot|$-relatively compact maps, then it also maps $|\cdot|$-compact sets into $|\cdot|$-relatively compact sets; if N is $(|\cdot|', |\cdot|)$-upper (lower) semicontinuous, then N is also $(|\cdot|, |\cdot|)$-upper (respectively, lower) semicontinuous.

4. HAMMERSTEIN INTEGRAL INCLUSIONS

In this section we study the Hammerstein integral inclusion

$$u(t) \in \int_0^T k(t,s) g\big(s, u(s)\big)\, ds, \qquad t \in [0,T] \tag{4.1}$$

in a ball of a Banach space $(E, |\cdot|)$; here k is a real single-valued function and g is a set-valued map with compact, convex values in E. In particular, if

$$k(t,s) = 0 \qquad \text{for } 0 \le t < s \le T$$

(the *Volterra case*), (4.1) becomes the Volterra integral inclusion

$$u(t) \in \int_0^t k(t,s) g\big(s, u(s)\big)\, ds, \qquad t \in [0,T]. \tag{4.2}$$

Thus, the existence theorems for (4.1) yield automatically existence results for (4.2). However, some of the assumptions made for (4.1) can be stated in a different way or improved for (4.2).

Let $0 < T < \infty$, $0 < R < \infty$, and let B be the closed ball $\{x \in E : |x| \le R\}$. Suppose that for some $p \in (1, \infty]$ we have

$$k\colon [0,T]^2 \to \mathbf{R}, \quad k(t,\cdot) \in L^p[0,T] \qquad \text{for each } t \in [0,T] \tag{4.3}$$

and let g: $[0, T] \times B \to P(E)$. By a solution of (4.1) we mean

$$u \in C\big([0, T]; B\big) \quad \text{with } u(t) = \int_0^T k(t, s)w(s)\,ds \qquad \text{for } t \in [0, T],$$

where

$$w \in L^q\big([0, T]; E\big) \quad \text{and } w(s) \in g\big(s, u(s)\big) \qquad \text{for a.e. } s \in [0, T].$$

Throughout this paper it is assumed that $1/p + 1/q = 1$.

Let $U = \{u \in C([0, T]; E) : |u|_\infty < R\}$. Clearly, $\overline{U} = C([0, T]; B)$. Assign to g and $q \in [1, \infty)$, a set-valued operator (the *Nemitsky operator*)

$$G_q\colon \overline{U} \to 2^{L^q([0,T];E)}$$

by letting

$$G_q(u) = \big\{w \in L^q\big([0, T]; E\big) : w(s) \in g(s, u(s)) \qquad \text{for a.e. } s \in [0, T]\big\}.$$

If, in addition to (4.3), we assume that

$$\big|k(t, \cdot) - k(t', \cdot)\big|_p \to 0 \qquad \text{as } t' \to t, \tag{4.4}$$

then we may define the linear operator

$$J_q\colon L^q([0, T]; E) \to C([0, T]; E),$$

$$J_q(w)(t) = \int_0^T k(t, s)w(s)\,ds, \qquad t \in [0, T].$$

Now, the solutions of (4.1) appear as fixed points of the set-valued operator

$$N\colon \overline{U} \to 2^{C([0,T];E)}, \qquad N = J_q \circ G_q.$$

Theorem 3.2 yields the following very general existence principle for Hammerstein integral inclusions of upper semicontinuous type.

THEOREM 4.1. *Assume that* (4.3), (4.4) *hold. In addition suppose*

(H1) $N\colon \overline{U} \to P_c(C([0, T]; E))$ *has closed graph and maps compact sets into relatively compact sets.*

(H2) $\left.\begin{array}{l} M \subset \overline{U}, M \subset \operatorname{conv}(\{0\} \cup N(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \implies \overline{M}$ *compact.*

(H3) $u \notin \lambda N(u)$ *for all* $u \in \overline{U}$ *with* $|u|_\infty = R$ *and all* $\lambda \in (0, 1)$.

Then (4.1) *has a solution in* $C([0, T]; B)$.

Proof. Apply Theorem 3.2 with $K = X = C([0, T]; E)$, X endowed with the norm $|\cdot|_\infty$, and $x_0 = 0$. ∎

Remark. If $N(\overline{U}) \subset \overline{U}$, then (H3) trivially holds and the conclusion of Theorem 4.1 follows directly from Theorem 3.1.

To obtain applicable existence criteria we have to find sufficient conditions for (H1)–(H3).

LEMMA 4.2. *Assume that* (4.3) *and* (4.4) *hold. If* $g\colon [0,T]\times B \to P_{kc}(E)$ *is* L^q*-Carathéodory, then* N *satisfies* (H1).

Proof. (a) $N(u) \neq \emptyset$ for each $u \in \overline{U}$. Indeed, since g takes nonempty, compact values and satisfies (C1) and (C2), for each $u \in C([0,T];B)$ there exists a strongly measurable selection w of $g(\cdot, u(\cdot))$ (see [7, Proposition 3.5(a)]). Next, (C3) guarantees $w \in L^q([0,T];E)$. Then $v = J_q(w) \in N(u)$.

(b) $N(u)$ is convex. This follows immediately from the convexity of the values of g.

(c) graph(N) is closed. To show this, let $(u_n, v_n) \in \text{graph}(N)$, $n \geq 1$, with $|u_n - u|_\infty$, $|v_n - v|_\infty \to 0$ as $n \to \infty$. Let

$$\begin{aligned} &v_n = J_q(w_n), \\ &w_n \in L^q([0,T];E), \qquad w_n(s) \in g(s, u_n(s)) \qquad \text{a.e. } s \in [0,T]. \end{aligned} \tag{4.5}$$

The sequence (w_n) has an L^q-weakly convergent subsequence; see [7, Proposition 9.4 and Problem 9.6]. Let w be its limit. As in [8, Proof of Lemma 2.3], we can show that $w(s) \in g(s, u(s))$ for a.e. $s \in [0,T]$. Also, from (4.5), we deduce $v = J_q(w)$. Therefore, $(u, v) \in \text{graph}(N)$.

(d) $N(M)$ is relatively compact for each compact $M \subset \overline{U}$. To prove this, let $M \subset \overline{U}$ be a compact set and let (v_n) be any sequence of elements of $N(M)$. We show that (v_n) has a convergent subsequence by using the Arzèla–Ascoli criterion of compactness in $C([0,T];E)$ (see [6, Proposition 7.3(b)]). Since $v_n \in N(M)$, there exist $u_n \in M$ and $w_n \in L^q([0,T];E)$ with

$$v_n = J_q(w_n) \quad \text{and} \quad w_n(s) \in g(s, u_n(s)) \qquad \text{for a.e. } s \in [0,T].$$

Using Theorem 1.3, we obtain that

$$\alpha\left(\{v_n(t) : n \geq 1\}\right) \leq 2\int_0^T \alpha\left(\{k(t,s)w_n(s) : n \geq 1\}\right) ds. \tag{4.6}$$

On the other hand, since $g(s, \cdot)$ is upper semicontinuous with compact values and $M(s)$ is compact in E, the set $g(s, M(s))$ is compact. Consequently, $\alpha(\{w_n(s) : n \geq 1\}) = 0$ for a.e. $s \in [0,T]$. Furthermore,

$$\alpha\left(\{k(t,s)w_n(s) : n \geq 1\}\right) = |k(t,s)|\alpha\left(\{w_n(s) : n \geq 1\}\right) = 0$$

for a.e. $s \in [0,T]$. Now (4.6) implies that $\{v_n(t) : n \geq 1\}$ is relatively compact in E, for each $t \in [0,T]$. In addition, we have

$$|v_n(t) - v_n(t')| \leq |k(t,\cdot) - k(t',\cdot)|_p \, |w_n|_q \leq |k(t,\cdot) - k(t',\cdot)|_p \, |h|_q, \tag{4.7}$$

where $h \in L^q[0, T]$ is such that $|g(s, x)| \leq h(s)$ for a.e. $s \in [0, T]$ and all $x \in B$. This, and (4.4), shows that $\{v_n : n \geq 1\}$ is equicontinuous. Consequently, $\{v_n : n \geq 1\}$ is relatively compact in $C([0, T]; E)$.

Thus, N satisfies (H1). ∎

Remark. Another way to assure (H1) is to put conditions so that the set-valued Nemitsky operator G_q is upper semicontinuous from the L^p-topology to the L^q-topology and then use the remark after Theorem 3.4; here $|\cdot|_p \leq c|\cdot|_\infty$. For a result concerning the upper semicontinuity of the Nemitsky set-valued operator from L^p to L^q, we refer the reader to Cellina *et al.* [5]. See also Couchouron and Kamenski [6] for a direct result about the upper semicontinuity of N.

LEMMA 4.3. *Assume that* (4.3), (4.4) *hold and that* g: $[0, T] \times B \to P(E)$ *satisfies*

$$\left|g(s, x)\right| \leq h(s) \tag{4.8}$$

for a.e. $s \in [0, T]$, *all* $x \in B$, *and some* $h \in L^q[0, T]$. *In addition, suppose that there exists an* L^q*-Carathéodory function* ω: $[0, T] \times [0, 2R] \to \mathbf{R}_+$ *such that*

$$\alpha\left(g(s, M)\right) \leq \omega\left(s, \alpha(M)\right) \tag{4.9}$$

for a.e. $s \in [0, T]$, *every* $M \subset B$, *and that the unique solution* $\varphi \in C([0, T]; [0, 2R])$ *of the inequality*

$$\varphi(t) \leq 2\int_0^T \left|k(t, s)\right| \omega\left(s, \varphi(s)\right) ds, \qquad t \in [0, T] \tag{4.10}$$

is $\varphi \equiv 0$. *Then* N *satisfies* (H2).

Proof. Suppose $M \subset \overline{U}$, $M \subset \operatorname{conv}(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using (4.8) and an estimation of type (4.7), we see that $N(M)$ is equicontinuous. Then, from $M \subset \operatorname{conv}(\{0\} \cup N(M))$, we deduce that M is equicontinuous too. In order to apply the Arzèla–Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in [0, T]$. Since

$$C \subset M \subset \operatorname{conv}\left(\{0\} \cup N(M)\right) \qquad \text{and} \qquad C \text{ is countable,}$$

we can find a countable set $V = \{v_n\colon n \geq 1\} \subset N(M)$ with

$$C \subset \operatorname{conv}\left(\{0\} \cup V\right).$$

Then, there exist $u_n \in M$ and $w_n \in L^q([0, T]; E)$ with

$$v_n = J_q(w_n) \quad \text{and} \quad w_n(s) \in g\left(s, u_n(s)\right) \qquad \text{for a.e. } s \in [0, T].$$

From $M \subset \overline{C} \subset \overline{\mathrm{conv}}(\{0\} \cup V)$, according to Theorem 1.3, we have

$$\alpha(M(t)) \le \alpha\left(\overline{C}(t)\right) \le \alpha(V(t)) = \alpha\left(\left\{J_q(w_n)(t) : n \ge 1\right\}\right)$$
$$\le 2\int_0^T \alpha\left(\left\{k(t,s)w_n(s) : n \ge 1\right\}\right) ds.$$

Now, since $w_n(s) \in g(s, u_n(s))$ and $u_n \in M$, (4.9) guarantees

$$\alpha\left(\left\{k(t,s)w_n(s) \colon n \ge 1\right\}\right) \le \left|k(t,s)\right| \alpha\big(g(s, M(s))\big)$$
$$\le \left|k(t,s)\right| \omega\left(s, \alpha(M(s))\right).$$

It follows that

$$\alpha\left(M(t)\right) \le 2\int_0^T \left|k(t,s)\right| \omega\left(s, \alpha(M(s))\right) ds, \qquad t \in [0, T].$$

Also, the function φ given by $\varphi(t) = \alpha(M(t))$ belongs to $C([0, T]; [0, 2R])$. Consequently, $\varphi \equiv 0$, that is, $\alpha(M(t)) = 0$ for all $t \in [0, T]$.

Now, by the Arzèla–Ascoli theorem, M is relatively compact in $C([0, T]; E)$. ∎

Remark. In the Volterra case, the inequality (4.10) is

$$\varphi(t) \le 2\int_0^t \left|k(t,s)\right| \omega\left(s, \varphi(s)\right) ds, \qquad t \in [0, T].$$

LEMMA 4.4. *Assume* (4.3), (4.4). *In addition, suppose that* $|k(t, \cdot)|_p \le 1$ *for all* $t \in [0, T]$, g: $[0, T] \times B \to P(E)$, *and that there exist* $\delta \in L^q[0, T]$ *and* ψ: $(0, R] \to (0, \infty)$ *continuous and nondecreasing such that*

$$\left|g(s, x)\right| \le \delta(s)\psi(|x|)$$

for a.e. $s \in [0, T]$ *and all* $x \in B \setminus \{0\}$.

(a) *If*

$$|\delta|_q \le R/\psi(R), \tag{4.11}$$

then N *satisfies* (H3) *with* $N(\overline{U}) \subset \overline{U}$.

(b) *In the Volterra case, if*

$$|\delta|_q^q \le q\int_0^R \left(r^{q-1}/\psi(r)^q\right) dr, \tag{4.12}$$

then N *satisfies* (H3).

Proof. (a) Let $v \in N(u)$ with $u \in \overline{U}$. Then $v = J_q(w)$ for some $w \in L^p([0,T];E)$ with $w(s) \in g(s, u(s))$ a.e. on $[0,T]$. Since $|u(s)| \le R$, ψ is nondecreasing, and $|k(t,\cdot)|_p \le 1$, we have

$$|v(t)| \le \int_0^T |k(t,s)|\,|w(s)|\,ds \le \int_0^T |k(t,s)|\delta(s)\psi(|u(s)|)\,ds$$
$$\le \psi(R)\int_0^T \left|k(t,s)\right|\delta(s)\,ds \le \psi(R)\left|k(t,\cdot)\right|_p |\delta|_q \le \psi(R)|\delta|_q$$

(we put $\psi(0) = \lim_{t\to 0+}\psi(t)$). This together with (4.11) implies $|v(t)| \le R$ for all $t \in [0,T]$. Hence $v \in \overline{U}$.

(b) Assume the Volterra case. Let $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. Then

$$\left|u(t)\right| \le \lambda \int_0^t \left|k(t,s)\right|\delta(s)\psi\left(\left|u(s)\right|\right)\,ds$$
$$\le \lambda\left(\int_0^t \left[\delta(s)\psi\left(|u(s)|\right)\right]^q\,ds\right)^{1/q} \tag{4.13}$$

for all $t \in [0,T]$. Let

$$c(t) = \min\left\{R, \lambda\left(\int_0^t \left[\delta(s)\psi\left(|u(s)|\right)\right]^q\,ds\right)^{1/q}\right\}.$$

Clearly c is nondecreasing. We claim that $c(T) < R$. Suppose the contrary. Then, since $c(0) = 0$, there exists a subinterval $[a,b] \subset [0,T]$ with

$$c(a) = 0, \qquad c(b) = R \qquad \text{and} \quad 0 < c(t) < R \qquad \text{on } (a,b).$$

Since by (4.13), $|u(t)| \le c(t) \le R$ on $[a,b]$ and ψ is nondecreasing, we have

$$\left(c(s)^q\right)' = \lambda^q\left[\delta(s)\psi\left(|u(s)|\right)\right]^q \le \lambda^q\delta(s)^q\psi(c(s))^q \qquad \text{a.e. } s \in [a,b].$$

This yields

$$\int_a^b \frac{(c(s)^q)'}{\psi(c(s))^q}\,ds = q\int_0^R \frac{r^{q-1}}{\psi(r)^q}\,dr \le \lambda^q \int_a^b \delta(s)^q\,ds$$
$$\le \lambda^q|\delta|_q^q < |\delta|_q^q,$$

which contradicts (4.12). Notice we may assume $|\delta|_q > 0$ for the last inequality, since otherwise we have nothing to prove. ∎

Remark. Condition (4.12) is less restrictive than (4.11). Indeed, since ψ is nondecreasing, one has $\psi(r) \le \psi(R)$ for all $r \in [0,R]$. In consequence,

$$q\int_0^R \left(r^{q-1}/\psi(r)^q\right)dr \ge q\left(1/\psi(R)\right)^q \int_0^R r^{q-1}\,dr = \left(R/\psi(R)\right)^q,$$

which proves our claim.

Now, according to Theorem 4.1 and Lemmas 4.2–4.4, we can state the main existence result for (4.1).

THEOREM 4.5. *Let* k: $[0, T]^2 \to \mathbf{R}$ *and* g: $[0, T] \times B \to P_{kc}(E)$ *be* L^q-*Carathéodory for some* $q \in [1, \infty)$. *Suppose*

(A) $k(t, \cdot) \in L^p[0, T]$, $|k(t, \cdot)|_p \le 1$ *for each* $t \in [0, T]$ *and*

$$\left|k(t, \cdot) - k(t', \cdot)\right|_p \to 0 \qquad as\ t' \to t(1/p + 1/q = 1.$$

(B) *There exists an* L^q *-Carathéodory function* ω: $[0, T] \times [0, 2R] \to \mathbf{R}_+$ *such that*

$$\alpha(g(s, M)) \le \omega(s, \alpha(M)) \tag{4.14}$$

for a.e. $s \in [0, T]$, *any* $M \subset B$, *and that the unique solution in* $C([0, T]; [0, 2R])$ *of the inequality*

$$\varphi(t) \le 2 \int_0^T |k(t, s)| \omega(s, \varphi(s))\, ds, \qquad t \in [0, T], \tag{4.15}$$

is $\varphi \equiv 0$.

(C) *There exists* $\delta \in L^q[0, T]$ *and* ψ: $(0, R] \to (0, \infty)$ *continuous and nondecreasing such that*

$$|g(s, x)| \le \delta(s)\psi(|x|) \tag{4.16}$$

for a.e. $s \in [0, T]$, $x \in B \setminus \{0\}$, *and*

$$|\delta|_q \le R/\psi(R). \tag{4.17}$$

Then (4.1) *has a solution in* $C([0, T]; B)$.

For the Volterra case, we have:

THEOREM 4.6. *Assume the Volterra case. Suppose that all the assumptions of Theorem* 4.5 *are satisfied except* (4.17), *which is relaxed to*

$$|\delta|_q^q \le q \int_0^R \left(r^{q-1}/\psi(r)^q\right) dr. \tag{4.18}$$

Then (4.2) *has a solution in* $C([0, T]; B)$.

Much more applicable results can be derived from Theorems 4.5 and 4.6.

COROLLARY 4.7. *Let* k: $[0, T]^2 \to \mathbf{R}$ *and* $g : [0, T] \times B \to P_{fc}(E)$ *with* $g = g_1 + g_2$, *where* g_1 *is single-valued and* $g_1(\cdot, 0) = 0$ *and the values of* g_2

are subsets of a compact set of E. Assume (A) *holds. In addition suppose*

(B*) *There exists* $\delta \in L^q[0, T]$ *and* $\psi_1: (0, 2R] \to (0, \infty)$ *continuous and nondecreasing with*

$$2|\delta|_q < \inf_{r \in (0, 2R]} \big(r/\psi_1(r)\big) \tag{4.19}$$

and

$$\big|g_1(s, x) - g_1(s, y)\big| \le \delta(s)\psi_1(|x - y|) \tag{4.20}$$

for a.e. $s \in [0, T]$ *and all* $x, y \in B$, $x \neq y$.

(C*) *There exists* $\psi_2: [0, R] \to \mathbf{R}_+$ *continuous and nondecreasing such that*

$$\big|g_2(s, x)\big| \le \delta(s)\psi_2(|x|) \tag{4.21}$$

for a.e. $s \in [0, T]$, *all* $x \in B$, *and*

$$|\delta|_q \le R/\psi(R) \tag{4.22}$$

where $\psi = \psi_1 + \psi_2$.

Then (4.1) *has a solution in* $C([0, T]; B)$.

Proof. We show that (B) holds. Indeed, by (4.20), (4.14) holds with $\omega(s, r) = \delta(s)\psi_1(r)$. Next, suppose that $\varphi \in C([0, T]; [0, 2R])$ solves (4.15) and $\varphi \not\equiv 0$. Then $r_0 = \max_{t \in [0, T]} \varphi(t) \in (0, 2R]$. Let $t_0 \in [0, T]$ be such that $\varphi(t_0) = r_0$. From (4.15), using Hölder's inequality, $|k(t, \cdot)|_p \le 1$, and ψ_1 nondecreasing, we obtain

$$r_0 = \varphi(t_0) \le 2\left(\int_0^T \big[\delta(s)\psi_1(\varphi(s))\big]^q \, ds\right)^{1/q} \le 2\psi_1(r_0)|\delta|_q.$$

Hence $r_0/\psi_1(r_0) \le 2|\delta|_q$, which contradicts (4.19). Thus $r_0 = 0$.

Next we check (C). Indeed, from (4.20) and (4.21), since $g_1(\cdot, 0) = 0$, we have

$$\begin{aligned} \big|g(s, x)\big| &= \big|(g_1 + g_2)(s, x)\big| \le \big|g_1(s, x) - g_1(s, 0)\big| + |g_2(s, x)| \\ &\le \delta(s)\psi_1(|x|) + \delta(s)\psi_2(|x|) = \delta(s)\psi(|x|). \end{aligned}$$

Thus (4.16) holds. Now (4.17) is just (4.22). ∎

In [19] it is shown that a condition like (4.19) can be relaxed in some particular cases.

The above result can be improved for the Volterra case as follows.

COROLLARY 4.8. *Assume the Volterra case. Suppose that all the assumptions of Corollary* 4.7 *are satisfied except* (4.19) *and* (4.22) *which are relaxed to*

$$\int_0^{2R} \left(r^{q-1}/\psi_1(r)^q\right) dr = \infty \tag{4.23}$$

and (4.18), *respectively. Then* (4.2) *has a solution in* $C([0,T];B)$.

Proof. We check the conditions of Theorem 4.6. First we check (B). From (4.20) we see that (4.14) holds with $\omega(s,r) = \delta(s)\psi_1(r)$. Now, let $\varphi \in C([0,T];[0,2R])$ be any solution of (4.15). By Hölder's inequality, we obtain

$$\varphi(t) \le 2\left(\int_0^t [\delta(s)\psi_1(\varphi(s))]^q \, ds\right)^{1/q}.$$

Let

$$c(t) = 2\left(\int_0^t [\delta(s)\psi_1(\varphi(s))]^q \, ds\right)^{1/q}.$$

It is clear that c is nondecreasing. Then $\varphi \equiv 0$ once we show $c(T) = 0$. Suppose the contrary, i.e., $c(T) > 0$. Then, since $c(0) = 0$, for each $\varepsilon \in (0, A)$, where $A = \min\{c(T), 2R\}$, there is a subinterval $[a,b] \subset [0,T]$ with

$$c(a) = \varepsilon, \;\; c(b) = A \;\text{ and }\; c(t) \in (\varepsilon,\; A) \qquad \text{for all } t \in (a,b).$$

Now $\varphi(t) \le c(t) \le 2R$ on $[a,b]$ and ψ_1 nondecreasing on $[0,2R]$ guarantee that

$$\left(c(s)^q\right)' = 2^q \delta(s)^q \psi_1(\varphi(s))^q \le 2^q \delta(s)^q \psi_1(c(s))^q$$

a.e. $s \in [a,b]$. Consequently

$$q\int_\varepsilon^A \frac{r^{q-1}}{\psi_1(r)^q}\, dr = \int_a^b \frac{(c(s)^q)'}{\psi_1(c(s))^q}\, ds \le 2^q |\delta|_q^q.$$

This, for $\varepsilon \searrow 0$, yields a contradiction to (4.23). Thus $c(T) = 0$ and so $\varphi \equiv 0$.

Finally (C) with (4.18) instead of (4.17) follows from (4.20), $g_1(\cdot, 0) = 0$, and (4.21). ∎

Remark. Condition (4.23) is less restrictive than (4.19). Indeed, if (4.19) holds, then

$$\begin{aligned}\int_0^{2R} (r^{q-1}/\psi_1(r)^q)\, dr &= \lim_{\varepsilon\to 0+} \int_\varepsilon^{2R} \left(r^{q-1}/\psi_1(r)^q\right) dr \\ &\ge \inf_{r\in(0,2R]} \left(r/\psi_1(r)\right)^q \lim_{\varepsilon\to 0+} \int_\varepsilon^{2R} r^{-1}\, dr = \infty;\end{aligned}$$

that is, (4.23) also holds.

We conclude with a general existence principle of lower semicontinuous type.

THEOREM 4.9. *Assume that* (4.3), (4.4) *hold. In addition suppose*

(h1) $N: \overline{U} \to P_{fc}(C([0,T];E))$ *is lower semicontinuous.*

(h2) $$\left.\begin{array}{l} M \subset \overline{U}, M \subset \text{conv}(\{0\} \cup N(M)) \\ \text{and } \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \end{array}\right\} \implies \overline{M} \text{ compact.}$$

(h3) $u \notin \lambda N(u)$ *for all* $u \in \overline{U}$ *with* $|u|_\infty = R$ *and* $\lambda \in (0,1)$.

Then (4.1) *has a solution in* $C([0,T];B)$.

Notice we can guarantee (h1) by using lower semicontinuity results for the Nemitsky set-valued operator from L^p into L^q. Such results can be found in Cardinali and Papageorgiou [4] and Hu and Papageorgiou [12].

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