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A MÖNCH TYPE GENERALIZATION OF THE EILENBERG-MONTGOMERY FIXED POINT THEOREM

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Abstract. We establish a Mönch type generalization of the Eilenberg-Montgomery fixed point theorem for multi-valued maps with acyclic values.

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In [1] Eilenberg and Montgomery established the following extension of the Bohnenblust-Karlin fixed point theorem for multi-valued maps.

Theorem 1 *Let Ξ be an acyclic, absolute neighborhood retract, Θ a compact metric space, $r : \Theta \to \Xi$ a continuous single valued map and $T : \Xi \to 2^{\Theta} \setminus \{\emptyset\}$ an upper semicontinuous map such that all the sets $T(x)$ are acyclic for $x \in \Xi$. Then the combined multi-valued map $rT : \Xi \to 2^{\Xi}$ has a fixed point.*

An extension of this theorem for condensing (noncompact) maps rT is due to Fitzpatrick and Petryshyn [2].

The aim of this note is to prove the following generalization of the Eilenberg-Montgomery theorem.

Theorem 2 *Let D be a closed, convex subset of a Banach space X, Y a metric space, $T : D \to 2^{Y} \setminus \{\emptyset\}$ a map with acyclic values, and $r : Y \to D$ continuous. Assume graph(T) is closed, T maps compact sets into relatively compact sets and that for some $x_0 \in D$ one has*

$$\left.\begin{array}{r} M \subset D,\ M = conv\left(\{x_0\} \cup rT(M)\right) \\ and\ \overline{M} = \overline{C}\ \ with\ C \subset M,\ C\ \ countable \end{array}\right\} \Longrightarrow \overline{M}\ \ compact. \tag{1}$$

Then there exists $x \in D$ with $x \in rT(x)$.

Proof. First note since r is continuous, the map $N := rT$ also has a closed graph and maps compact sets into relatively compact sets.

Following the steps (a) and (b) of the proof of Theorem 3.1 in [4], we find a convex set $M \subset D$ with $M = \text{conv}\,(\{x_0\} \cup rT\,(M))$ and $K := \overline{M}$ compact. Next, instead of steps (c)-(d) of the above mentioned proof, we follow:

(c*) Proof of inclusion $rT\,(K) \subset K$. Let $\varepsilon > 0$ be fixed. According to Theorem 1.2.23 in [3], rT is upper semicontinuous. Consequently, for each $x \in M$ there exists an open neighborhood V_x of x such that $rT\,(y) \subset rT\,(x) + B_\varepsilon\,(0)$ for all $y \in V_x$. Since for $x \in M$, one has $rT\,(x) \subset K$, it follows that $rT\,(y) \subset K_\varepsilon := K + B_\varepsilon\,(0)$ for every $y \in V_x$. M being dense in K, we have that $\{V_x : x \in M\}$ is a cover of K. Consequently, $rT\,(K) \subset K_\varepsilon$. Hence $rT\,(K) \subset \bigcap_{\varepsilon > 0} K_\varepsilon = K$.

(d*) Application of the Eilenberg-Montgomery theorem. Since every compact and convex subset of a Banach space is an absolute neighborhood retract and is acyclic, we may apply Theorem 1 to: $\Xi := K$ and $\Theta := T\,(K)$. ■

Remarks. (a) Under the assumptions of Theorem 2 the values of T are nonempty, compact and acyclic.

(b) Theorem 2 is in particular true if $T : D \to 2^Y$ has nonempty, compact and acyclic values and T is upper semicontinuous.

The next result is the continuation type version of Theorem 2.

Theorem 3 *Let K be a closed, convex subset of a Banach space X, U a convex, relatively open subset of K, Y a metric space, $T : \overline{U} \to 2^Y \setminus \{\emptyset\}$ with acyclic values and $r : Y \to K$ continuous. Assume graph$\,(T)$ is closed, T maps compact sets into relatively compact sets and that for some $x_0 \in U$, the following two conditions are satisfied:*

$$\left.\begin{array}{c} M \subset \overline{U},\ M \subset \ \textit{conv}\ (\{x_0\} \cup rT\,(M)) \\ \textit{and}\ \overline{M} = \overline{C}\ \ \textit{with}\ C \subset M,\ C\ \ \textit{countable} \end{array}\right\} \Longrightarrow \overline{M}\ \ \textit{compact;} \tag{2}$$

$$x \notin (1 - \lambda)\,x_0 + \lambda rT\,(x) \tag{3}$$

for all $x \in \overline{U} \setminus U$, $\lambda \in (0, 1)$. Then there exists $x \in \overline{U}$ with $x \in rT\,(x)$.

Proof. Let $D = \overline{\text{conv}}\,\left(\{x_0\} \cup rT\left(\overline{U}\right)\right)$. Clearly, $x_0 \in D \subset K$. Let $P : K \to \overline{U}$ be given by

$$P\,(x) = \begin{cases} x \text{ for } x \in \overline{U} \\ \overline{x} \text{ for } x \in K \setminus \overline{U} \end{cases}$$

Here $\overline{x} = (1 - \lambda)\,x_0 + \lambda x \in \overline{U} \setminus U$, $\lambda \in (0, 1)$. It is easy to see that P is continuous.

Let $\widetilde{T} : D \to 2^Y$, be given by $\widetilde{T}\,(x) = T\,(P\,(x))$. Clearly its values are nonempty and acyclic. Also, it is easily seen that graph$\left(\widetilde{T}\right)$ is closed and $\widetilde{T}$ maps compact sets into relatively compact sets. Next we check (1) for $r\widetilde{T}$. Let $M \subset D$ be such that $M = \text{conv}\,(\{x_0\} \cup r\widetilde{T}\,(M))$ and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Since

$$\begin{aligned} P\,(M) &\subset \text{conv}\ (\{x_0\} \cup M) \subset \text{conv}\ \left(\{x_0\} \cup r\widetilde{T}\,(M)\right) \\ &= \text{conv}\ (\{x_0\} \cup rTP\,(M)), \end{aligned} \tag{4}$$

$$\overline{P(M)} = \overline{P(C)}\ ,\quad P(C) \subset P(M) \quad \text{and} \quad P(C) \quad \text{is countable,}$$

from (2) we deduce that $P(M)$ is relatively compact. Then $r\widetilde{T}(M) = rTP(M)$ is relatively compact and Mazur's lemma implies that $\overline{M}$ is compact. Thus (1) holds for $r\widetilde{T}$.

Now we apply Theorem 2 to deduce that there exists an $x \in D \subset \overline{U}$ with $x \in r\widetilde{T}(x)$. We claim that $x \in D \cap \overline{U}$. Assume the contrary, that is $x \in D \setminus \overline{U}$. Then $x \in rT(\overline{x})$, where $\overline{x} = (1-\lambda)x_0 + \lambda x \in \overline{U} \setminus U$, $\lambda \in (0,1)$. Then $x = (1/\lambda)\overline{x} + (1-1/\lambda)x_0 \in rT(\overline{x})$. Hence $\overline{x} \in (1-\lambda)x_0 + \lambda rT(\overline{x})$, which contradicts (3). Thus $x \in D \cap \overline{U}$ and so $x \in rT(x)$. ∎

References

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