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Discrete continuation method for boundary value problems on bounded sets in Banach spaces

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Abstract

The paper deals with existence, uniqueness and iterative approximation of solutions to boundary value problems for second-order differential equations on bounded sets in a Banach space. The tools are an extension of Granas' continuation principle for contraction mappings to spaces endowed with two metrics and a computational procedure accompanying the continuation principle. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper deals with existence, uniqueness and iterative approximation of solutions to problems of the form

$$u'' = f(t,u,u'), \quad t \in I = [0,1], \tag{1.1}$$

$$V_1(u) = b_1, \quad V_2(u) = b_2 \tag{1.2}$$

in a Banach space E, where $b_1, b_2 \in E$, V_1, V_2 are linear continuous mappings from $C^1(I;E)$ into E and f is defined on a bounded subset of $I \times E^2$.

Boundary value problems of this form with particular boundary conditions occur frequently when modelling real processes and have been studied, with varying degrees of generality, by many authors.

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For recent results and references see the paper by Lee and O'Regan [5] which was the first motivation of the present article.

The approach to (1.1)–(1.2) is a fixed-point one. We express (1.1)–(1.2) as a fixed point problem for a certain mapping A from a subset of $C^1(I;E)$ into $C^1(I;E)$. As usually, if E is finite dimensional and f is a Carathéodory function, then, by the Ascoli–Arzela theorem, A is completely continuous and under some additional assumptions guaranteeing the a priori boundedness of solutions, the Leray–Schauder continuation principle for compact mappings applies. When E is infinite dimensional, the complete continuity of A fails. Then, assuming that f satisfies a Lipschitz condition we can arrange that A become a contraction mapping and so that the continuation principle for contractions applies. Also, since any contraction mapping on a subset of a Banach space is a set contraction in Darbo's sense, the continuation principle for set contractions equally applies. However, we can take advantage from the application of the first one in the same way that we obtain more information from Banach contraction principle than from Darbo fixed point theorem, namely the iterative procedure for approximating the unique fixed point. In this respect, in Section 2, we give a discrete version of the Granas continuation principle for contractions on metric spaces [2]. Notice that an elementary proof of a continuation principle for contractions on closed subsets of a Banach space is due to Gatica and Kirk [1]. Discrete continuation methods for solving nonlinear operator equations on finite or infinite dimensional spaces have been also described in connection to particular numerical procedures. For example, in [9], a discrete continuation method is presented in combination with Newton's method.

In addition, in our version of the continuation principle for contraction mappings, the Lipschitz condition is asked with respect to a noncomplete metric provided that suitable topological compatibilities between homotopy, the noncomplete metric and a complete metric hold. Here, the unusual term of a *complete* (resp., *noncomplete*) *metric* d on a set X is used to show that the metric space (X,d) is complete (resp., incomplete). The idea of using two metrics, one complete and other noncomplete, is patterned from Maia [6], where a version of Banach contraction principle is given for spaces endowed with two metrics. In studying (1.1)–(1.2), our continuation principle makes possible, even if f is defined only on a bounded subset of $I \times E^2$, to use together a (complete) sup-norm and an (noncomplete) L^p-norm on $C^1(I;E)$ in order that the contraction condition be relaxed. The technique has already been used to integral and differential equations (see Rus [12] and Petracovici [10]) but only together with a fixed point theorem for self-mappings of a metric space whose application to (1.1)–(1.2) requires that f be defined on the entire set $I \times E^2$.

2. The iterative discrete continuation principle in a space with two metrics

Given a space X endowed with two metrics d and δ, in order to precise the metric with respect to which a topological notion is considered, we shall indicate the corresponding metric in front of that notion. So, we shall speak about d-Cauchy and δ-Cauchy sequences, d-open, δ-open, d-closed, δ-closed sets, d-closure, δ-closure, d-interior and d-neighborhood. Also, we shall say that $A : D \to X$ is (d,δ)-continuous (resp., (δ,δ)-continuous), where $D \subset X$ or $D \subset X \times [0,1]$, if A is continuous from D into (X,δ), with respect to the topology induced by d (resp., δ) on D. The meaning of the notion of an uniformly (d,δ)-continuous mapping will be similar.

First we state a slight extension of a result by Maia [6] (see also [11]).

Lemma 2.1. *Let (X,d) be a metric space and δ a complete metric on X. Assume that for $A : X \to X$ the following conditions are satisfied*:
(a) *there is $l \in [0,1)$ such that*

$$d(A(x),A(y)) \leqslant l d(x,y) \quad \text{for all } x,y \in X, \tag{2.1}$$

(b) *A is uniformly (d,δ)-continuous*;
(c) *A is (δ,δ)-continuous.*
Then A has a unique fixed point x^*. Moreover, for any $x \in X$, we have

$$d(A^k(x),x^*) \leqslant \frac{l^k}{1-l} d(x,A(x)) \quad (k \in \mathbb{N}) \tag{2.2}$$

and

$$\delta(A^k(x),x^*) \to 0 \quad \text{as } k \to \infty. \tag{2.3}$$

Proof. Let $x \in X$. Denote $x_k = A^k(x)$, $k \in \mathbb{N}$. By (2.1), (x_k) is d-Cauchy. Next, from $\delta(x_k,x_j) = \delta(A(x_{k-1}),A(x_{j-1}))$ and (b), we deduce that (x_k) is δ-Cauchy too. Since δ is a complete metric on X, it follows that there exists $x^* \in X$ with $\delta(x_k,x^*) \to 0$ as $k \to \infty$. Then, by (c), $\delta(A(x_{k-1}),A(x^*)) \to 0$ as $k \to \infty$. But $\delta(A(x_{k-1}),A(x^*)) = \delta(x_k,A(x^*))$. Hence $A(x^*) = x^*$. By (2.1), x^* is the unique fixed point of A and so (2.3) is true for any $x \in X$. Again by (2.1),

$$d(x_k,x^*) = d(A^k(x),A^k(x^*)) \leqslant l^k d(x,x^*) \to 0 \quad \text{as } k \to \infty. \tag{2.4}$$

Finally, (2.2) follows by a standard argument.

Second proof. Let $(\tilde{X},\tilde{d})$ be the *completion* of (X,d) (see [4] for example). The elements of $\tilde{X}$ are classes of d-Cauchy sequences in X which are equivalent in the following sense: $(x_k) \sim (y_k)$ if $d(x_k,y_k) \to 0$ as $k \to \infty$. Denote by $\widehat{(x_k)}$ the class of the sequence (x_k). If $\xi,\eta \in \tilde{X}$, $\xi = \widehat{(x_k)}$ and $\eta = \widehat{(y_k)}$, then one sets $\tilde{d}(\xi,\eta) = \lim_{k\to\infty} d(x_k,y_k)$. Now we define the extension $\tilde{A}$ of A to $\tilde{X}$ by $\tilde{A}(\widehat{(x_k)}) = \widehat{(A(x_k))}$. The definition is correct because, by (2.1), $(A(x_k))$ is d-Cauchy whenever (x_k) is. Clearly $\tilde{A}$ is a contraction mapping on $\tilde{X}$ and so, by Banach fixed point theorem, there exists $\xi \in \tilde{X}$ with $\tilde{A}(\xi) = \xi$. Let $\xi = \widehat{(z_k)}$. Then $(z_k) \sim (A(z_k))$. Since (z_k) is d-Cauchy, by (b), it follows that $(A(z_k))$ is δ-Cauchy and so δ-convergent to some $x^* \in X$. Then, by (c), $(A^2(z_k))$ is δ-convergent to $A(x^*)$. By $(z_k) \sim (A(z_k))$, that is $d(z_k,A(z_k)) \to 0$, and (b), we obtain that $\delta(A(z_k),A^2(z_k)) \to 0$. Consequently, $\delta(x^*,A(x^*)) = 0$ and so $A(x^*) = x^*$. Finally, for any $x \in X$, we have (2.4) and, by (b), $\delta(A^{k+1}(x),A(x^*)) = \delta(A^{k+1}(x),x^*) \to 0$ too. □

The result in [6] corresponds to the case where $\delta \leqslant d$, when (b) is a consequence of (a).

The second proof shows that Maia's theorem in X is Banach's theorem in the completion $\tilde{X}$ but with the fixed point in X.

Before going to state the main result of this section, we introduce the following notation. For a mapping $H : D \times [0,1] \to X$, where $D \subset X$, and any $\lambda \in [0,1]$, we denote by H_λ the mapping $H(.,\lambda)$ from D into X.

Theorem 2.2. *Let (X,d) be a metric space and δ a complete metric on X. Let $D\subset X$ be δ-closed and U a d-open set of X with $U\subset D$. Let $H : D\times[0,1]\to X$ and assume that the following conditions are satisfied:*

(i) *there is $l\in[0,1)$ such that*

$$d(H(x,\lambda),H(y,\lambda))\leqslant ld(x,y)$$

for all $x,y\in D$ and $\lambda\in[0,1]$;

(ii) *$H(x,\lambda)\neq x$ for all $x\in D\setminus U$ and $\lambda\in[0,1]$;*

(iii) *H is uniformly (d,δ)-continuous;*

(iv) *H is (δ,δ)-continuous;*

(v) *$H(x,\lambda)$ is d-continuous in λ, uniformly for $x\in U$, i.e. for each $\varepsilon>0$ and $\lambda\in[0,1]$, there is $\rho>0$ such that $d(H(x,\lambda),H(x,\mu))<\varepsilon$ whenever $x\in U$ and $|\lambda-\mu|<\rho$.*

In addition suppose that H_0 has a fixed point. Then, for each $\lambda\in[0,1]$, there exists a unique fixed point $x(\lambda)$ of H_λ. Moreover, $x(\lambda)$ depends d-continuously on λ and there exists $0<r\leqslant\infty$, integers $m,n_1,n_2,\ldots,n_{m-1}$ and numbers $0<\lambda_1<\lambda_2<\cdots<\lambda_{m-1}<\lambda_m=1$ such that for any $x_0\in X$ satisfying $d(x_0,x(0))\leqslant r$, the sequences $(x_{j,k})_{k\geqslant 0}$, $j=1,2,\ldots,m$,

$$\begin{aligned}&x_{1,0}=x_0,\\ &x_{j,k+1}=H_{\lambda_j}(x_{j,k}),\quad k=0,1,\ldots,\\ &x_{j+1,0}=x_{j,n_j},\quad j=1,2,\ldots,m-1,\end{aligned}$$

are well defined and satisfy

$$d(x_{j,k},x(\lambda_j))\leqslant\frac{l^k}{1-l}d(x_{j,0},H_{\lambda_j}(x_{j,0}))\quad(k\in\mathbb{N})\tag{2.5}$$

and

$$\delta(x_{j,k},x(\lambda_j))\to 0\quad\text{as } k\to\infty.\tag{2.6}$$

Remark 2.3. Obviously, we have

$$x_{j,k}=H^{k}_{\lambda_j}(H^{n_{j-1}}_{\lambda_{j-1}}(\ldots(H^{n_1}_{\lambda_1}(x_0))\ldots))\quad(k\in\mathbb{N}),$$

$d(x_{j,k},x(\lambda_j))\to 0$ and $\delta(x_{j,k},x(\lambda_j))\to 0$ as $k\to\infty$ $(j=1,2,\ldots,m)$. In particular, for $j=m$, $(x_{m,k})_{k\geqslant 0}$ is a sequence of successive approximations of $x(1)$, with respect to both metrics d and δ.

Proof. (1) First we prove that for each $\lambda\in[0,1]$, H_λ has a fixed point. Let

$$\Lambda=\{\lambda\in[0,1]; H(x,\lambda)=x \text{ for some } x\in U\}.$$

We have $0\in\Lambda$ by the assumption that H_0 has a fixed point. Hence Λ is nonempty. We will show that Λ is both closed and open in $[0,1]$ and so, by the connectedness of $[0,1]$, $\Lambda=[0,1]$.

To prove that Λ is closed, let $\lambda_k\in\Lambda$ with $\lambda_k\to\lambda$ as $k\to\infty$. Since $\lambda_k\in\Lambda$, there is $x_k\in U$ so that $H(x_k,\lambda_k)=x_k$. Then, by (i), we obtain

$$\begin{aligned}d(x_k,x_j)&=d(H(x_k,\lambda_k),H(x_j,\lambda_j))\leqslant d(H(x_k,\lambda_k),H(x_k,\lambda))\\ &\quad+d(H(x_k,\lambda),H(x_j,\lambda))+d(H(x_j,\lambda),H(x_j,\lambda_j))\\ &\leqslant d(H(x_k,\lambda_k),H(x_k,\lambda))+ld(x_k,x_j)+d(H(x_j,\lambda),H(x_j,\lambda_j)).\end{aligned}$$

It follows that

$$d(x_k,x_j)\leqslant\frac{1}{1-l}[d(H(x_k,\lambda_k),H(x_k,\lambda))+d(H(x_j,\lambda),H(x_j,\lambda_j))].$$

This, by (v), it shows that (x_k) is d-Cauchy. Further, from $\delta(x_k,x_j)=\delta(H(x_k,\lambda_k),\ H(x_j,\lambda_j))$ and (iii), we see that (x_k) is also δ-Cauchy. Thus, by the completeness of δ, there is $x\in X$ with $\delta(x_k,x)\to 0$ as $k\to\infty$. Since $x_k\in D$ and D is δ-closed, we have $x\in D$ too. Then $\delta(x_k,H(x,\lambda))\to\delta(x,H(x,\lambda))$ and, by (iv), $\delta(x_k,H(x,\lambda))=\delta(H(x_k,\lambda_k),H(x,\lambda))\to 0$. Hence $\delta(x,H(x,\lambda))=0$, that is $H(x,\lambda)=x$. By (ii), $x\in U$ and so $\lambda\in\Lambda$.

To prove that Λ is open in $[0,1]$, let $\mu\in\Lambda$ and $z\in U$ such that $H(z,\mu)=z$. Since U is d-open, there exists $\rho>0$ such that

$$d(x,z)\leqslant\rho\quad\text{implies } x\in U.$$

Also, by (v), there is $\eta=\eta(\rho)>0$ such that

$$d(z,H(z,\lambda))=d(H(z,\mu),H(z,\lambda))\leqslant(1-l)\rho \tag{2.7}$$

for $|\lambda-\mu|\leqslant\eta$. Consequently,

$$\begin{aligned}d(z,H(x,\lambda)) &\leqslant d(z,H(z,\lambda))+d(H(z,\lambda),H(x,\lambda))\\ &\leqslant (1-l)\rho+ld(z,x)\leqslant\rho,\end{aligned}$$

whenever $d(z,x)\leqslant\rho$ and $|\lambda-\mu|\leqslant\eta$. This shows that for $|\lambda-\mu|\leqslant\eta$, H_λ sends B into itself, where $B=\{x\in X;\ d(z,x)\leqslant\rho\}$. Let $\bar{B}$ be the δ-closure of B. Since $B\subset U\subset D$ and D is δ-closed, we also have $\bar{B}\subset D$. Using (iv), it is easily seen that $H_\lambda(\bar{B})\subset\bar{B}$ for $|\lambda-\mu|\leqslant\eta$. Now we may apply Lemma 2.1 to $A=H_\lambda$. Consequently, there is $x(\lambda)\in\bar{B}\subset D$ a fixed point of H_λ for $|\lambda-\mu|\leqslant\eta$. This shows that μ is an interior point of Λ and hence Λ is open in $[0,1]$. Notice that for every $x\in B$ and $|\lambda-\mu|\leqslant\eta$, we also have by Lemma 2.1, that the sequence $(H_\lambda^k(x))_{k\geqslant 0}$ is well defined,

$$d(H_\lambda^k(x),x(\lambda))\leqslant\frac{l^k}{1-l}d(x,H_\lambda(x))\quad(k\in\mathbb{N})$$

and $\delta(H_\lambda^k(x),x(\lambda))\to 0$ as $k\to\infty$.

(2) The uniqueness of $x(\lambda)$ is a simple consequence of (i).

(3) $x(\lambda)$ is d-continuous on $[0,1]$. Indeed,

$$\begin{aligned}d(x(\lambda),x(\mu)) &= d(H(x(\lambda),\lambda),H(x(\mu),\mu))\\ &\leqslant d(H(x(\lambda),\lambda),H(x(\mu),\lambda))+d(H(x(\mu),\lambda),H(x(\mu),\mu))\\ &\leqslant ld(x(\lambda),x(\mu))+d(H(x(\mu),\lambda),H(x(\mu),\mu)).\end{aligned}$$

This, by (v), implies

$$d(x(\lambda),x(\mu))\leqslant\frac{1}{1-l}d(H(x(\mu),\lambda),H(x(\mu),\mu))\to 0\quad\text{as }\lambda\to\mu.$$

(4) Obtention of r. For any $\mu\in[0,1]$, denote

$$r(\mu)=\inf\{d(x,x(\mu));x\in X\setminus U\}.$$

Since $x(\mu) \in U$ and U is d-open, $r(\mu) > 0$. We claim that

$$\inf\{r(\mu); \mu \in [0,1]\} > 0. \tag{2.8}$$

To prove this, assume the contrary. Then, there are $\mu_k \in [0,1]$ such that $r(\mu_k) \to 0$ as $k \to \infty$. Clearly, we may assume that $\mu_k \to \mu$ for some $\mu \in [0,1]$. Then, from the d-continuity of $x(\lambda)$, we have

$$d(x(\mu_k), x(\mu)) < r(\mu)/2 \quad \text{for } k \geqslant k_1. \tag{2.9}$$

On the other hand, since $r(\mu_k) \to 0$,

$$r(\mu_k) < r(\mu)/2 \quad \text{for } k \geqslant k_2. \tag{2.10}$$

Let $k_0 = \max\{k_1, k_2\}$. By (2.10) and the definition of $r(\mu_{k_0})$ as infimum, there is $x \in X \setminus U$ with

$$d(x, x(\mu_{k_0})) < r(\mu)/2. \tag{2.11}$$

Then, by (2.9) and (2.11), we obtain

$$d(x, x(\mu)) \leqslant d(x, x(\mu_{k_0})) + d(x(\mu_{k_0}), x(\mu)) < 2r(\mu)/2 = r(\mu),$$

a contradiction. Thus (2.8) holds as claimed. Now we choose any $r > 0$ less than the infimum in (2.8), with the convention that $r = \infty$ if the infimum equals infinity.

(5) Obtention of m and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < 1$. Let $h = \eta(r)$, where r was fixed at the anterior step and $\eta(r)$ is chosen as in (2.7). Then, by what was shown at the end of step (1), for each $\mu \in [0,1]$,

$$d(x, x(\mu)) \leqslant r \quad \text{and } |\lambda - \mu| \leqslant h \quad \text{imply } (H_\lambda^k(x))_{k \geqslant 0} \text{ is well defined,} \tag{2.12}$$

$$d(H_\lambda^k(x), x(\lambda)) \leqslant \frac{l^k}{1-l} d(x, H_\lambda(x)) \quad (k \in \mathbb{N})$$

and

$$\delta(H_\lambda^k(x), x(\lambda)) \to 0 \quad \text{as } k \to \infty.$$

Now we choose any partition $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{m-1} < \lambda_m = 1$ of $[0,1]$ such that $\lambda_{j+1} - \lambda_j \leqslant h$, $j = 0, 1, \ldots, m-1$.

(6) Finding of integers $n_1, n_2, \ldots, n_{m-1}$. From $d(x_{1,0}, x(0)) = d(x_0, x(0)) \leqslant r$ and $\lambda_1 - \lambda_0 \leqslant h$, by (2.12), we have that $(x_{1,k})_{k \geqslant 0}$ is well defined and satisfies (2.5)–(2.6). By (2.5), we may choose $n_1 \in \mathbb{N}$ such that $d(x_{1,n_1}, x(\lambda_1)) \leqslant r$. Now $d(x_{2,0}, x(\lambda_1)) = d(x_{1,n_1}, x(\lambda_1)) \leqslant r$ and $\lambda_2 - \lambda_1 \leqslant h$ and we repeat the above argument in order to show that $(x_{2,k})_{k \geqslant 0}$ is well defined and satisfies (2.5)–(2.6). In general, at step j $(1 \leqslant j \leqslant m-1)$ we choose $n_j \in \mathbb{N}$ such that $d(x_{j,n_j}, x(\lambda_j)) \leqslant r$. Then $d(x_{j+1,0}, x(\lambda_j)) = d(x_{j,n_j}, x(\lambda_j)) \leqslant r$ and $\lambda_{j+1} - \lambda_j \leqslant h$, by (2.12), imply that sequence $(x_{j+1,k})_{k \geqslant 0}$ is well defined and satisfies (2.5)–(2.6). □

The above proof yields the following algorithm for the approximation of $x(1)$ under the assumptions of Theorem 2.2:

Suppose we know r and h and we wish to obtain an approximation $\bar{x}_1$ of $x(1)$ with $d(\bar{x}_1, x(1)) \leqslant \varepsilon$. Then we choose any partition $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < \lambda_m = 1$ of $[0,1]$ with $\lambda_{j+1} - \lambda_j \leqslant h$, $j = 0, 1, \ldots, m-1$, any element x_0 with $d(x_0, x(0)) \leqslant r$ and we follow the next

Iterative procedure:

Set $n_0 := 0$ and $x_{0,n_0} := x_0$;
For $j := 1$ to $m-1$ do
  $x_{j,0} := x_{j-1,n_{j-1}}$
  $k := 0$
  While $l^k(1-l)^{-1} d(x_{j,0}, H_{\lambda_j}(x_{j,0})) > r$
    $x_{j,k+1} := H_{\lambda_j}(x_{j,k})$
    $k := k+1$
  $n_j := k$
Set $k := 0$
While $l^k(1-l)^{-1} d(x_{m,0}, H_1(x_{m,0})) > \varepsilon$
  $x_{m,k+1} := H_1(x_{m,k})$
  $k := k+1$
Finally take $\bar{x}_1 = x_{m,k}$.

Remark 2.4. Clearly, if $d \leqslant \delta$ on X, then it suffices that the estimates in the above algorithm be made with respect to δ.

Notice that when $D = U = X$ and $H_\lambda = A$ for all $\lambda \in [0,1]$, Theorem 2.2 reduces to Lemma 2.1. In this case, $r = \infty$ and $m = 1$.

In case that $d = \delta$, Theorem 2.2 yields the following computational version of Granas continuation principle for contraction mappings on complete metric spaces.

Corollary 2.5. *Let (X,d) be a complete metric space and U be an open set of X. Let $H : \bar{U} \times [0,1] \to X$ and assume that the following conditions are satisfied*:

(a1) *there is $l \in [0,1)$ such that*

$$d(H(x,\lambda), H(y,\lambda)) \leqslant l d(x,y)$$

for all $x, y \in \bar{U}$ and $\lambda \in [0,1]$;

(a2) *$H(x,\lambda) \neq x$ for all $x \in \partial U$ and $\lambda \in [0,1]$*;

(a3) *H is continuous in λ, uniformly for $x \in \bar{U}$, i.e. for each $\varepsilon > 0$ and $\lambda \in [0,1]$, there is $\rho > 0$ such that $d(H(x,\lambda), H(x,\mu)) < \varepsilon$ whenever $x \in \bar{U}$ and $|\lambda - \mu| < \rho$.*

In addition suppose that H_0 has a fixed point. Then, for each $\lambda \in [0,1]$, there exists a unique fixed point $x(\lambda)$ of H_λ. Moreover, $x(\lambda)$ depends continuously on λ and there exists $0 < r \leqslant \infty$, integers $m, n_1, n_2, \ldots, n_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < \lambda_m = 1$ such that for any $x_0 \in X$ satisfying $d(x_0, x(0)) \leqslant r$, the sequences $(x_{j,k})_{k \geqslant 0}$, $j = 1, 2, \ldots, m$,

$$x_{1,0} = x_0,$$

$$x_{j,k+1} = H_{\lambda_j}(x_{j,k}), \quad k = 0, 1, \ldots,$$

$$x_{j+1,0} = x_{j,n_j}, \quad j = 1, 2, \ldots, m-1,$$

are well defined and satisfy

$$d(x_{j,k},x(\lambda_j))\leqslant \frac{l^k}{1-l}d(x_{j,0},H_{\lambda_j}(x_{j,0}))\quad (k\in\mathbb{N}).$$

Obviously, for $U=X$ and $H_\lambda=A$, $\lambda\in[0,1]$, Corollary 2.5 reduces to Banach contraction principle.

3. Boundary value problems on bounded sets in Banach spaces

We denote $C^k=C^k(I;E)$, $C=C^0$, $C^1_{\mathscr{B}_0}=\{u\in C^1; V_j(u)=0,\ j=1,2\}$ and $C^k_{\mathscr{B}_0}=C^k\cap C^1_{\mathscr{B}_0}$ $(k\geqslant 2)$. Similarly, $C^1_{\mathscr{B}}=\{u\in C^1;\ V_j(u)=b_j,\ j=1,2\}$ and $C^k_{\mathscr{B}}=C^k\cap C^1_{\mathscr{B}}$. Also, for an integer $m\geqslant 1$ and a real $1\leqslant p\leqslant\infty$, we shortly denote $L^p=L^p(I;E)$, $W^{m,p}=W^{m,p}(I;E)$, and $W^{m,p}_{\mathscr{B}_0}=W^{m,p}\cap C^1_{\mathscr{B}_0}$, $W^{m,p}_{\mathscr{B}}=W^{m,p}\cap C^1_{\mathscr{B}}$ $(m\geqslant 2)$. Recall that $W^{m,p}\subset C^{m-1}$.

In what follows we assume that the unique solution of $u''=0$ which satisfies $V_j(u)=0,\ j=1,2$, is the null function. Then, there is a unique solution to $u''=0$ such that $V_j(u)=b_j,\ j=1,2$, say $u_0(t)$, and there is a Green's function $g(t,s)$ corresponding to operator u'' and boundary conditions $V_j(u)=0,\ j=1,2$. Moreover, for each $p\in[1,\infty]$, the operator $L:W^{2,p}_{\mathscr{B}_0}\to L^p$, $Lu=u''$ is invertible and

$$L^{-1}v(t)=\int_0^1 g(t,s)v(s)\,\mathrm{d}s.$$

The same is true for the operator $L:C^2_{\mathscr{B}_0}\to C,\ Lu=u''$.

Also, we denote by $||\,.\,||_{1,p}$ the following complete norm on $W^{1,p}$ (noncomplete on C^1)

$$||u||_{1,p}=\max\{||u||_p,||u'||_p\},\qquad ||u||_p=\left(\int_0^1|u(t)|^p\,\mathrm{d}t\right)^{1/p}$$

$(1\leqslant p<\infty)$ and by $||\,.\,||_{1,\infty}$ the usual complete norm on C^1,

$$||u||_{1,\infty}=\max\{||u||_\infty,||u'||_\infty\},\qquad ||u||_\infty=\sup_{t\in I}|u(t)|.$$

Now we state a very general existence and uniqueness principle in a ball of $C^1_{\mathscr{B}}$.

Theorem 3.1. *Let $R>0$, $1<p\leqslant\infty$ and $A:D_R\to W^{2,p}_{\mathscr{B}}$ be any mapping, where $D_R=\{u\in C^1_{\mathscr{B}};\ ||u||_{1,\infty}\leqslant R\}$. Assume that $||u_0||_{1,\infty}<R$ and that the following conditions are satisfied:*

(H1) *$A(D_R)$ is bounded in $(C^1,||\,.\,||_{1,\infty})$ and there is $R'>0$ such that $|u''(t)|\leqslant R'$ for a.e. $t\in I$ and any $u\in A(D_R)$;*

(H2) *there exists a metric d on $C^1_{\mathscr{B}}$ equivalent to the metric induced by $||\,.\,||_{1,p}$ satisfying*

$$d(u,v)\leqslant c_0||u-v||_{1,p}\tag{3.1}$$

for all $u,v\in C^1_{\mathscr{B}}$ and some $c_0>0$, such that

$$||A(u)-A(v)||_{1,\infty}\leqslant cd(u,v)\tag{3.2}$$

and

$$d(A(u),A(v))\leqslant ld(u,v)\tag{3.3}$$

for all $u,v\in D_R$ and some $c>0$, $l\in[0,1)$;

(H3) *if* $u \in D_R$ *solves* $u = (1-\lambda)u_0 + \lambda A(u)$ *for some* $\lambda \in [0,1]$, *then* $||u||_{1,\infty} < R$.

Then A *has a unique fixed point in* D_R.

Proof. We shall apply Theorem 2.2. Denote by δ the metric induced by $||\,.\,||_{1,\infty}$ on $C^1_{\mathscr{B}}$. Recall that V_j, $j = 1,2$, were supposed continuous; consequently, $(C^1_{\mathscr{B}}, \delta)$ is a complete metric space. Let $X_0 = \mathrm{co}\{\{u_0\} \cup A(D_R)\}$, where "co" stands for the convex hull. Since $u_0 \in C^1_{\mathscr{B}}$, $A(D_R) \subset C^1_{\mathscr{B}}$ and $C^1_{\mathscr{B}}$ is convex, we also have $X_0 \subset C^1_{\mathscr{B}}$. Denote by X the δ-closure of X_0 in $C^1_{\mathscr{B}}$ and let $D = X \cap D_R$. Obviously, D is δ-closed in X.

From (H1), we see that any function u in X_0 satisfies $|u''(t)| \leqslant R'$ for a.e. $t \in I$. This property is the reason of the choice of X.

Define $H : D \times [0,1] \to X$, $H(u,\lambda) = (1-\lambda)u_0 + \lambda A(u)$. We now check that all the assumptions of Theorem 2.2 are satisfied, where U is the d-interior of D in X.

Condition (i) follows from (3.3) since $D \subset D_R$. By (3.2), since $A(D_R)$ is bounded in C^1, we have

$$\begin{aligned} ||H(u,\lambda) - H(v,\mu)||_{1,\infty} &\leqslant ||H(u,\lambda) - H(v,\lambda)||_{1,\infty} + ||H(v,\lambda) - H(v,\mu)||_{1,\infty} \\ &\leqslant ||A(u) - A(v)||_{1,\infty} + c'|\lambda - \mu| \leqslant c d(u,v) + c'|\lambda - \mu|, \end{aligned} \tag{3.4}$$

where c' is a constant depending only on R. It follows that H is uniformly (d,δ)-continuous, that is (iii). By (3.1) and $||\,.\,||_{1,p} \leqslant ||\,.\,||_{1,\infty}$, from (3.4) also follows (iv). Now, if in (3.4) we put $u = v$, then we obtain

$$\begin{aligned} d(H(u,\lambda), H(u,\mu)) &\leqslant c_0 ||H(u,\lambda) - H(u,\mu)||_{1,p} \\ &\leqslant c_0 ||H(u,\lambda) - H(u,\mu)||_{1,\infty} \leqslant c_0 c' |\lambda - \mu|. \end{aligned}$$

This proves (v).

It is clear that (ii) follows from (H3) if we prove that

$$u \in D \quad \text{and} \quad ||u||_{1,\infty} < R \quad \text{implies } u \in U. \tag{3.5}$$

So let $u \in D$ with $||u||_{1,\infty} < R$. We have to show that there exists $r > 0$ such that $v \in X$ and $||v - u||_{1,p} < r$ imply $v \in D_R$. Suppose the contrary. Then, there is a sequence $(u_k) \subset X$ with $||u_k - u||_{1,p} < 1/k$ and $u_k \notin D_R$. Then, $|u_k(t)| > R$ or $|u_k'(t)| > R$ for some $t \in I$. On the other hand, if we denote $R_0 = ||u||_{1,\infty}$, then $R_0 < R$ and $|u(t)| \leqslant R_0, |u'(t)| \leqslant R_0$ for all $t \in I$. Consequently, for each k there is at least one t such that:

(1) $|u_k(t) - u(t)| \geqslant |u_k(t)| - |u(t)| \geqslant |u_k(t)| - R_0 > R - R_0$

or

(2) $|u_k'(t) - u'(t)| \geqslant |u_k'(t)| - |u'(t)| \geqslant |u_k'(t)| - R_0 > R - R_0$.

We shall derive a contradiction by using the following result.

Lemma 3.2. *Let* $\chi \in C^1(I;E)$. *If* $|\chi(t)| \geqslant a > 0$ for some $t \in I$ and $|\chi'(t)| \leqslant M$ *for all* $t \in I$, *then*

$$\int_0^1 |\chi(s)|\,\mathrm{d}s \geqslant \min\{a/2, 3a^2/(8M)\}.$$

I. First suppose that $|u_k'(t)-u'(t)|\leqslant R-R_0$ for all $t\in I$ and for infinitely many values of k. Then, passing if necessary to a subsequence, we may assume that for any k, we have

$$|u_k'(t)-u'(t)|\leqslant R-R_0 \quad \text{for all } t\in I \text{ and}$$

$$|u_k(t)-u(t)|>R-R_0 \quad \text{for at least one } t.$$

Then, by Lemma 3.2, it follows that

$$\int_0^1 |u_k(s)-u(s)|\,\mathrm{d}s \geqslant 3(R-R_0)/8>0$$

for all k. This yields $||u_k-u||_{1,p}\nrightarrow 0$ as $k\to\infty$, a contradiction.

II. In the opposite case to I, we may suppose that for any k, we have

$$|u_k'(t)-u'(t)|>R-R_0 \quad \text{for at least one } t.$$

Let $\varepsilon>0$. Since $u,u_k\in X$, there are $\tilde{u},\tilde{u}_k\in X_0$ such that

$$|\tilde{u}_k'(t)-\tilde{u}'(t)|>R-R_0 \quad \text{for at least one } t,$$

$$\int_0^1 |u_k'(s)-\tilde{u}_k'(s)|\,\mathrm{d}s\leqslant\varepsilon/2 \quad \text{and} \quad \int_0^1 |u'(s)-\tilde{u}'(s)|\,\mathrm{d}s\leqslant\varepsilon/2.$$

From $\tilde{u},\tilde{u}_k\in X_0$, we also have

$$|\tilde{u}_k''(t)-\tilde{u}''(t)|\leqslant|\tilde{u}_k''(t)|+|\tilde{u}''(t)|\leqslant 2R' \quad \text{for all } t\in I.$$

Then, by Lemma 3.2,

$$\int_0^1 |\tilde{u}_k'(s)-\tilde{u}'(s)|\,\mathrm{d}s\geqslant C>0$$

for all k, where C depends only on $R-R_0$ and R'. Thus, we have

$$C\leqslant\int_0^1 |\tilde{u}_k'(s)-\tilde{u}'(s)|\,\mathrm{d}s\leqslant\varepsilon+\int_0^1 |u_k'(s)-u'(s)|\,\mathrm{d}s$$
$$\leqslant\varepsilon+||u_k-u||_{1,p}.$$

Hence $||u_k-u||_{1,p}\geqslant C-\varepsilon$ for all k. Choosing $\varepsilon<C$ this yields $||u_k-u||_{1,p}\nrightarrow 0$ as $k\to\infty$, a contradiction.

Thus (3.5) holds and Theorem 2.2 can be applied. □

Proof of Lemma 3.2. We have

$$||\chi(t)|-|\chi(s)||\leqslant|\chi(t)-\chi(s)|\leqslant M|t-s| \quad \text{for all } t,s\in I. \tag{3.6}$$

Two cases are possible:

(1) For all $t\in I$, $|\chi(t)|\geqslant a/2$. Then, clearly,

$$\int_0^1 |\chi(s)|\,\mathrm{d}s\geqslant a/2.$$

(2) There are $t_1,t_2\in I$ with $|\chi(t_1)|=a/2$, $|\chi(t_2)|=a$ and $|\chi(t)|\in[a/2,a]$ for all t between t_1 and t_2. Suppose $t_1<t_2$. Then, if we choose $t=t_1$ and $s=t_2$ in (3.6), we get $t_2-t_1\geqslant a/(2M)$. Also, again by (3.6),

$$|\chi(t)|\geqslant|\chi(t_2)|-M(t_2-t)=a-M(t_2-t) \quad \text{for all } t\in[t_1,t_2].$$

Integration from $t_2 - a/(2M)$ to t_2 yields

$$\int_0^1 |\chi(s)|\,\mathrm{d}s \geqslant \int_{t_2-a/(2M)}^{t_2} |\chi(s)|\,\mathrm{d}s \geqslant 3a^2/(8M). \qquad \square$$

Remark 3.3. In particular, if d is the metric on $C^1_{\mathscr{B}}$ induced by $||\,.\,||_{1,p}$ and in addition there is $r \in (0,R)$ such that in (H3), $||u||_{1,\infty} < R - r$ for any solution of $u = (1-\lambda)u_0 + \lambda A(u)$, $\lambda \in [0,1]$, then the unique fixed point of A can be approximated by means of the iterative procedure described in Section 2, where we may use this r and the first approximation $x_0 = u_0$.

Remark 3.4. For $p = \infty$, d and δ are equivalent metrics on $C^1_{\mathscr{B}}$ and Theorem 3.1 is a direct consequence of Corollary 2.5.

Denote $\bar{B}_R = \{u \in E;\ |u| \leqslant R\}$. Let $f : I \times \bar{B}_R^2 \to E$. Recall that f is said to be L^p-Carathéodory if $f(t,.)$ is continuous for a.e. $t \in I$; $f(.,u,v)$ is measurable for all $(u,v) \in \bar{B}_R^2$ and there exists $h \in L^p(I)$ such that $|f(t,u,v)| \leqslant h(t)$ a.e. $t \in I$, whenever $u, v \in \bar{B}_R$. If f is continuous (resp., L^p-Carathéodory), then the operator

$$F(u)(t) = f(t,u(t),u'(t)), \quad t \in I$$

is well defined from D_R into C (resp., L^p) and a function $u \in D_R$ is a classical (resp., Carathéodory) solution of (1.1)–(1.2) if and only if $u = A(u)$, where

$$A(u) = u_0 + L^{-1}F(u).$$

In order to state an existence and uniqueness principle for (1.1)–(1.2), we embed this problem into an one-parameter family of problems

$$u'' = \lambda f(t,u,u'), \quad t \in I, \tag{3.7}$$

$$V_1(u) = b_1, \quad V_2(u) = b_2, \tag{3.8}$$

where $\lambda \in [0,1]$.

Theorem 3.5. *Let $f : I \times \bar{B}_R^2 \to E$. Assume that $||u_0||_{1,\infty} < R$ and the following conditions are satisfied*:

(h1) *f is continuous* (*resp.*, *$f(.,u,v)$ is measurable for all $(u,v) \in \bar{B}_R^2$ and $f(.,0,0) \in L^\infty(I;E)$*);

(h2) *there exist numbers $K_0, K_1 \geqslant 0$, function $\phi \in L^\infty(I;I)$ and $p \in (1,\infty]$ such that*

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leqslant \phi(t)[K_0|u-\bar{u}| + K_1|v-\bar{v}|] \tag{3.9}$$

for a.e. $t \in I$ and all $u, \bar{u}, v, \bar{v} \in \bar{B}_R$, and

$$l_p = K_0 \left[\int_0^1 \left(\int_0^1 |g(t,s)|^q \phi(s)^q\,\mathrm{d}s\right)^{p/q} \mathrm{d}t\right]^{1/p} + K_1 \left[\int_0^1 \left(\int_0^1 |g_t(t,s)|^q \phi(s)^q\,\mathrm{d}s\right)^{p/q} \mathrm{d}t\right]^{1/p} < 1, \tag{3.10}$$

where $1/p + 1/q = 1$ (*for* $p = \infty$,

$$l_\infty = K_0 \max_{t\in I} \int_0^1 |g(t,s)|\phi(s)\,\mathrm{d}s + K_1 \max_{t\in I} \int_0^1 |g_t(t,s)|\phi(s)\,\mathrm{d}s).$$

(h3) *if* $u \in D_R$ *solves* (3.7)–(3.8) *for some* $\lambda \in [0,1]$, then $||u||_{1,\infty} < R$.

Then (1.1)–(1.2) *has a unique classical* (*resp., Carathéodory*) *solution in* D_R.

Proof. We shall apply Theorem 3.1. We first note that from (3.9) and $f(\cdot,0,0) \in L^\infty(I;E)$, it follows that f is L^∞-Carathéodory. Now we immediately see that the operator $A(u) = u_0 + L^{-1}F(u)$ is well defined from D_R into $W^{2,p}_{\mathscr{B}}$, $A(D_R)$ is bounded with respect to $||\,.\,||_{1,\infty}$ and that there is $R' > 0$ such that $|u''(t)| \leqslant R'$ a.e. on I, for any $u \in A(D_R)$. Hence condition (H1) is satisfied.

Without loss of generality, we may assume that $K_0 > 0$ and $K_1 > 0$. Otherwise, we take $K_0 + \varepsilon$ and $K_1 + \varepsilon$ instead of K_0, K_1 with $\varepsilon > 0$ small enough that inequality (3.10) remain true. Then, we define a modified L^p-norm on C^1 by

$$||u|| = K_0||u||_p + K_1||u'||_p.$$

Clearly, norms $||\,.\,||$ and $||\,.\,||_{1,p}$ are equivalent. Let d be the metric induced by $||\cdot||$ on $C^1_{\mathscr{B}}$. We now check (3.2) and (3.3). Let $u, v \in D_R$. Then using (3.9), we obtain

$$\begin{aligned} |A(u)(t) - A(v)(t)| &\leqslant \int_0^1 |g(t,s)|\,|f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\,\mathrm{d}s \\ &\leqslant \int_0^1 |g(t,s)|\phi(s)(K_0|u(s)-v(s)| + K_1|u'(s)-v'(s)|)\,\mathrm{d}s \\ &\leqslant \left(\int_0^1 |g(t,s)|^q\phi(s)^q\,\mathrm{d}s\right)^{1/q} ||u-v||. \end{aligned}$$

Also

$$\begin{aligned} |A(u)'(t) - A(v)'(t)| &\leqslant \int_0^1 |g_t(t,s)|\,|f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\,\mathrm{d}s \\ &\leqslant \left(\int_0^1 |g_t(t,s)|^q\phi(s)^q\,\mathrm{d}s\right)^{1/q} ||u-v||. \end{aligned}$$

These clearly yield (3.2). By (3.10), they also imply (3.3), where $l = l_p$. Hence (H2) is satisfied too. Finally (H3) follows from (h3) since a function $u \in D_R$ solves (3.7)−(3.8) if and only if $u = (1-\lambda)u_0 + \lambda A(u)$. Thus, Theorem 3.1 can be applied. □

Remark 3.6. (1) For $p = \infty$ and $\phi = 1$, the result in Theorem 3.5 follows from [5, Theorem 3.6].

(2) We will compare the contraction condition $l_p < 1$ for $p = \infty$ and $p = 2$. Suppose $V_1(u) = u(0)$ and $V_2(u) = u(1)$ and $\phi = 1$. Then, direct computation yields $l_\infty = K_0/8 + K_1/2$ while $l_2 = K_0/(3\sqrt{10}) + K_1/\sqrt{6}$. Thus the contraction condition $l_2 < 1$ is less restrictive than $l_\infty < 1$.

(3) Other modified L^p-norms on C^1 are possible and are expected to relax the contraction condition (3.10). For example, we may take the norm

$$||u|| = K_0||\psi u||_p + K_1||\psi u'||_p,$$

where $\psi \in C(I;\mathbb{R}_+^*)$. In this case, the contraction condition becomes

$$K_0\left[\int_0^1 \psi(t)^p\left(\int_0^1 |g(t,s)|^q\phi(s)^q\psi(s)^{-q}\,\mathrm{d}s\right)^{p/q}\mathrm{d}t\right]^{1/p}$$
$$+K_1\left[\int_0^1 \psi(t)^p\left(\int_0^1 |g_t(t,s)|^q\phi(s)^q\psi(s)^{-q}\,\mathrm{d}s\right)^{p/q}\mathrm{d}t\right]^{1/p}<1$$

for $p<\infty$, and

$$K_0\max_{t\in I}\int_0^1 \psi(t)|g(t,s)|\phi(s)\psi(s)^{-1}\,\mathrm{d}s$$
$$+K_1\max_{t\in I}\int_0^1 \psi(t)|g_t(t,s)|\phi(s)\psi(s)^{-1}\,\mathrm{d}s<1$$

for $p=\infty$.

For such tricks of contraction, we refer the interested reader to [3].

(4) Another interested choice of the norm $||\,.\,||$, based on Wirtinger's inequality, is possible in the case of the homogeneous Dirichlet conditions $u(0)=u(1)=0$, when $C^1_{\mathscr{B}}$ is simply denoted by C^1_0. There are well known: Wirtinger's inequality

$$||u||_2\leqslant\frac{1}{\pi}||u'||_2,\quad u\in C^1_0, \tag{3.11}$$

and Opial's inequality (see [8] for example)

$$\int_0^1 |u(t)|\,|u'(t)|\,\mathrm{d}t\leqslant\frac{1}{4}\int_0^1 |u'(t)|^2\,\mathrm{d}t,\quad u\in C^1_0.$$

Also,

$$||L^{-1}v||_2\leqslant\frac{1}{\pi^2}||v||_2,\quad v\in L^2. \tag{3.12}$$

Recall that π^2 is here the first eigenvalue corresponding to the differential operator $-u''$ and to the Dirichlet boundary conditions. Now, if $f: I\times\bar{B}^2_R\to E$ satisfies (h1), (h3) and the Lipschitz inequality (3.9) with $\phi=1$, then the contraction condition (3.10) can be replaced by

$$\frac{K_0^2}{\pi^4}+\frac{K_1^2}{\pi^2}+\frac{K_0K_1}{2\pi^2}<1. \tag{3.13}$$

Indeed, if we choose as d the metric on C^1_0 induced by the norm $||u||=||u'||_2$, then all the assumptions of Theorem 3.1 are satisfied for $p=2$. For example, (3.3) follows by (3.11)–(3.12):

$$\begin{aligned}d(A(u),A(v))&=||(L^{-1}(F(u)-F(v)))'||_2\\&=\{(F(v)-F(u),L^{-1}(F(u-F(v)))_2\}^{1/2}\leqslant\frac{1}{\pi}||F(u)-F(v)||_2\\&\leqslant\frac{1}{\pi}\left[\int_0^1(K_0|u-v|+K_1|u'-v'|)^2\,\mathrm{d}t\right]^{1/2}\end{aligned}$$

$$\leqslant \frac{1}{\pi}\left[K_0^2||u-v||_2^2 + K_1^2||u'-v'||_2^2 + \frac{K_0K_1}{2}||u'-v'||_2^2\right]^{1/2}$$
$$\leqslant \left(\frac{K_0^2}{\pi^4} + \frac{K_1^2}{\pi^2} + \frac{K_0K_1}{2\pi^2}\right)^{1/2} ||u'-v'||_2 = \left(\frac{K_0^2}{\pi^4} + \frac{K_1^2}{\pi^2} + \frac{K_0K_1}{2\pi^2}\right)^{1/2} \mathrm{d}(u,v).$$

We mention that (3.13) was obtained by Hai and Schmitt [3] and used in case that f is defined on the entire set $I \times E^2$ (see also the paper of Mawhin [7]). Therefore, our technique based on the use of two metrics makes possible that certain results involving conditions derived when working with energy L^p-norms can be extended to the case where f is defined, or has the required properties, only on a bounded region.

(5) As we have already remarked, for $p=\infty$, Theorem 3.1 is a consequence of Corollary 2.5. For an arbitrary $p<\infty$, according to the second proof of Lemma 2.1, we could think to use also Corollary 2.5, working in the completion of $C^1_{\mathscr{B}}$ with respect to d. For example, when $\mathscr{B}$ means $u(0)=u(1)=0$, the completion of C^1_0 is the Sobolev space $W_0^{1,p}(I;E)$. It is easily seen that such an approach has a major impediment, namely the bounded domain of A.

(6) In case that f is independent of u' and V_j, $j=1,2$, are linear continuous from C into E, we can regard A as a mapping from $D^0_R=\{u\in C_{\mathscr{B}};\ ||u||_\infty \leqslant R\} \subset C_{\mathscr{B}}$ into $C_{\mathscr{B}}$, where $C_{\mathscr{B}}=\{u\in C;\ V_j(u)=b_j,\ j=1,2\}$. This leads variants of Theorems 3.1 and 3.5 in which all reference to u' is dropped and the norms $||\,.\,||_\infty$, $||\,.\,||_p$ are used instead of $||\,.\,||_{1,\infty}$ and $||\,.\,||_{1,p}$ respectively.

Example. Consider the boundary value problem

$$\begin{aligned} &u''=f(u), \quad t\in I,\\ &u(0)=u(1)=0. \end{aligned} \tag{3.14}$$

Assume that for some $R>0$, $f\in C(\bar{B}_R;E)$,

$$\sup\{|f(u)|;\ |u|\leqslant R\}\leqslant 8R$$

and there exists $K_0 < 3\sqrt{10}$ such that

$$|f(u)-f(v)|\leqslant K_0|u-v| \quad \text{for all } u,v\in \bar{B}_R.$$

Then (3.14) has a unique solution (with sup-norm at most R). If in addition,

$$\sup\{|f(u)|;\ |u|\leqslant R\} < 8(R-r)$$

for some $0<r<R$, then the unique solution can be approximated by the iterative procedure described in Section 2, where: $l=K_0/(3\sqrt{10})$, $x_0\equiv 0$, $H(.,\lambda)=\lambda A$ and d is the metric on C_0 induced by $||\,.\,||_2$. According to Remark 2.4, since $||\,.\,||_2\leqslant ||\,.\,||_\infty$, it suffices that the estimates in the iterative procedure be made with respect to metric δ induced by $||\,.\,||_\infty$.

The above example shows in what way the continuation principle for contractions applies to problems with superlinear nonlinearity provided that a Lipschitz condition holds in some bounded set. In particular, problem (3.14) for $E=\mathbb{R}$ and $f(u)=-\mathrm{e}^u$ which comes from thermodynamics, was discussed in [5].

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