Discrete continuation method for nonlinear integral equations in Banach spaces*

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Abstract: The continuation theorem for contractive mappings on spaces endowed with two metrics is used to obtain existence, uniqueness and iterative approximation results for nonlinear integral equations in Banach spaces.

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1 Introduction

In our recent paper [9], Granas' continuation principle [2] (see also [8]) for contractive mappings on complete metric spaces was extended to spaces endowed with two metrics, and completed by an iterative approximation procedure in the spirit of [5, Ch. 19]. The result was used to discuss existence, uniqueness and iterative approximation of solutions to some boundary value problems for second order ordinary differential equations in a ball of a Banach space. In this specific case the two metrics were induced by a complete max-norm and an incomplete L^p-norm, respectively. The contraction condition was required with respect to the L^p-norm, which in general is less restrictive than with respect to the max-norm.

The purpose of this paper is to extend the use of the above technique to Urysohn integral equations of the form

$$u(t) = \int_0^T f(t, s, u(s))\, ds, \quad t \in [0, T], \tag{1.1}$$

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in a Banach space $(E, |\,.\,|)$; here the integral is understood in the sense of Bochner (see [10]). We look for solutions in $C([0,T]; B)$, where B is the closed ball $\{x \in E;\ |x| \le R\}$.

For various results and methods on such equations in finite and infinite dimensions we refer the reader to [1], [4], [6] and [7].

In the present paper, the approach to (1.1) is based on the following continuation principle accompanying Banach contraction principle.

([9]) Let (X, δ) be a complete metric space and d another metric on X. Let $D \subset X$ be δ-closed and U a d-open set of X with $U \subset D$. Let $H : D \times [0,1] \to X$ and assume that the following conditions are satisfied:

(i) there is $l \in [0,1)$ such that

$$d(H(x,\lambda), H(y,\lambda)) \le l\, d(x, y)$$

for all $x, y \in D$ and $\lambda \in [0,1]$;

(ii) $H(x,\lambda) \ne x$ for all $x \in D \setminus U$ and $\lambda \in [0,1]$;

(iii) H is uniformly (d, δ)-continuous;

(iv) H is (δ, δ)-continuous;

(v) $H(x,\lambda)$ is d-continuous in λ, uniformly for $x \in U$, i.e. for each $\varepsilon > 0$ and $\lambda \in [0,1]$, there is $\rho > 0$ such that $d(H(x,\lambda), H(x,\mu)) < \varepsilon$ whenever $x \in U$ and $|\lambda - \mu| < \rho$.

In addition suppose that H_0 has a fixed point. Then, for each $\lambda \in [0,1]$, there exists a unique fixed point $x(\lambda)$ of $H_\lambda := H(\,.\,,\lambda)$. Moreover, $x(\lambda)$ depends d-continuously on λ and there exists $0 < r \le \infty$, integers $m, n_1, n_2, ..., n_{m-1}$ and numbers $0 < \lambda_1 < \lambda_2 < ... < \lambda_{m-1} < \lambda_m = 1$ such that for any $x_0 \in X$ satisfying $d(x_0, x(0)) \le r$, the sequences $(x_{j,k})_{k\ge 0}$, $j = 1, 2, ..., m$,

$$\begin{aligned}&x_{1,0} = x_0\\ &x_{j,k+1} = H_{\lambda_j}(x_{j,k}), \qquad k = 0, 1, ...\\ &x_{j+1,0} = x_{j,n_j}, \qquad j = 1, 2, ..., m-1\end{aligned}$$

are well defined and satisfy

$$d(x_{j,k}, x(\lambda_j)) \le \frac{l^k}{1-l}\, d(x_{j,0}, H_{\lambda_j}(x_{j,0})) \quad (k \in \mathbf{N})$$

and

$$\delta(x_{j,k},\, x(\lambda_j)) \to 0 \quad as\, k \to \infty.$$

Throughout this paper we use the following notation:

$$\|u\|_p = \left(\int_0^T |u\,(s)|^p\, ds \right)^{1/p} \quad (u \in L^p\,([0,T]\,;E)\,,\; 1 \le p < \infty)$$

and

$$\|u\|_\infty = \inf\{M \ge 0;\; |u\,(t)| \le M\;\, fora.e.t \in [0,T]\} \quad (u \in L^\infty\,([0,T]\,;E))\,.$$

It is clear that if $u \in C\,([0,T]\,;E)\,,$ then $\|u\|_\infty = \max_{t\in[0,T]} |u\,(t)|\,.$ When it will be important, we shall denote $\|\,.\,\|_p$ also by $\|\,.\,\|_{L^p([0,T];\,E)}\,.$

We shall denote $C\,([0,T]\,;E)$ shortly by $C.$ Let $w : [0,T] \to \mathbf{R}_+$ be continuous at 0 and $w\,(0) = 0.$ By C_w we mean the set

$$C_w = \{u \in C;\; |u\,(t) - u\,(s)| \le w\,(|t-s|) \quad forall\, t,\, s \in [0,T]\}\,.$$

2 Results

First we prove a general existence and uniqueness fixed point result in a bounded region of the space $C.$ It is of the same type as Theorem 3.1 [9].

Let $K \subset C$ be closed and convex, $R > 0$ and $A : K_R \to K,$ where $K_R = \{u \in K;\; \|u\|_\infty \le R\}\,.$ Let $w : [0,T] \to \mathbf{R}_+$ be continuous at 0 with $w\,(0) = 0$ and let $u_0 \in K \cap C_w$ with $\|u_0\|_\infty < R.$ Suppose

(H1) $A\,(K_R)$ is bounded in $(C,\; \|\,.\,\|_\infty)$ and $A\,(K_R) \subset C_w\,;$

(H2) there exists a metric d on K equivalent to the metric induced by $\|\,.\,\|_p$ satisfying

$$d\,(u,\, v) \le c_0\, \|u - v\|_p \tag{2.1}$$

for all $u,\, v \in K$ and some $c_0 > 0,$ such that

$$\|A\,(u) - A\,(v)\|_\infty \le c\, d\,(u,\, v) \tag{2.2}$$

and

$$d\,(A\,(u)\,,\, A\,(v)) \le l\, d\,(u, v) \tag{2.3}$$

for all $u,\, v \in K_R$ and some $c > 0,\; l \in [0,1);$

(H3) for each $\lambda \in (0,1)$, each possible solution $u \in K_R$ of equation $u = (1-\lambda) u_0 + \lambda A(u)$ is such that $\|u\|_\infty < R$.

Then A has a unique fixed point in K_R.

ProofWe shall apply Theorem 1.1. Let δ be the metric induced by $\|\,.\,\|_\infty$ on K. Since K is closed, (K, δ) is a complete metric space. Let

$$X_0 = co\,\{\{u_0\} \cup A(K_R)\}.$$

Since $u_0 \in K$, $A(K_R) \subset K$ and K is convex, we have $X_0 \subset K$. Let X be the δ-closure of X_0 and let $D = X \cap K_R$. Obviously, D is δ-closed in X.

Notice $u_0 \in C_w$ and $A(K_R) \subset C_w$ imply $X \subset C_w$.

Let $H : D \times [0,1] \to X$ be given by $H(u,\lambda) = (1-\lambda) u_0 + \lambda A(u)$ and let U be the d-interior of D in X.

Check of (i) is immediate from (2.3).

Check of (ii): According to (H3) it is sufficient to show that

$$u \in D \textit{ and } \|u\|_\infty < R \textit{ imply } u \in U. \tag{2.4}$$

To see this, let $u \in D$ with $\|u\|_\infty < R$. Then (2.4) will be proved once we show that there exists $r > 0$ such that $v \in X$ and $\|u - v\|_p < r$ imply $v \in K_R$. Assume the contrary. Then there exists a sequence $(u_k) \subset X$ with $\|u_k - u\|_p < 1/k$ and $u_k \notin K_R$. Then, $|u_k(t)| > R$ for some $t \in [0,T]$ (depending on k). Let $R_0 = \|u\|_\infty$. We have $R_0 < R$ and $|u(t)| \leq R_0$ for all $t \in [0,T]$. Hence, for each k there is a $t \in [0,T]$ with

$$|u_k(t) - u(t)| \geq |u_k(t)| - |u(t)| \geq |u_k(t)| - R_0 > R - R_0. \tag{2.5}$$

In order to derive a contradiction, we use the following lemma:

Let $W : [0,T] \to \mathbf{R}_+$ be continuous at $t = 0$ with $W(0) = 0$ and let $a > 0$. Then, there exists $b > 0$ only depending on W and a, such that

$$\int_0^T |\chi(s)|\, ds \geq b \tag{2.6}$$

for any $\chi \in C_W$ with $|\chi(t)| \geq a$ for at least one $t \in [0,T]$.

Now we apply Lemma 2.2 to $\chi = u_k - u$, $a = R - R_0$ and $W = 2w$ (notice u_0, $u \in C_w$ implies $u_k - u \in C_{2w}$). It follows that

$$\int_0^T |u_k(s) - u(s)|\, ds \geq b > 0,$$

where b does not depend on k. Consequently, $\|u_k - u\|_p \not\to 0$ as $k \to \infty$, which is a contradiction. Therefore (ii) also holds.

Check of (iii): Using (2.2) and the boundedness of $A(K_R)$ we obtain

$$\|H(u,\lambda) - H(v,\mu)\|_\infty \le \|H(u,\lambda) - H(v,\lambda)\|_\infty + \|H(v,\lambda) - H(v,\mu)\|_\infty$$

$$\le \|A(u) - A(v)\|_\infty + c' |\lambda - \mu| \le c\, d(u, v) + c' |\lambda - \mu| \; ; \qquad (2.7)$$

here c' is a constant depending only on R. This shows that H is uniformly (d, δ)-continuous.

Check of (iv): Use (2.7) and take into account (2.1) and $\| \, . \, \|_p \le \| \, . \, \|_\infty$.

Check of (v): Put $v = u$ in (2.7) and use again (2.1) to obtain

$$d(H(u,\lambda), H(u,\mu)) \le c_0 \|H(u,\lambda) - H(u,\mu)\|_p$$

$$\le c_0 \|H(u,\lambda) - H(u,\mu)\|_\infty \le c_0 c' |\lambda - \mu| .$$

Thus all the assumptions of Theorem 1.1 are satisfied. End Proof

Proof of Lemma 2.2. Since $\chi \in C_W$, we have

$$||\chi(t)| - |\chi(s)|| \le |\chi(t) - \chi(s)| \le W(|t-s|) . \qquad (2.8)$$

If $|\chi(t)| \ge a/2$ for all $t \in [0,T]$, then

$$\int_0^T |\chi(s)|\, ds \ge aT/2.$$

If not, we may find $t_1, t_2 \in [0,T]$ with $|\chi(t_1)| = a/2$, $|\chi(t_2)| = a$ and $|\chi(t)| \in [a/2, a]$ for all t between t_1 and t_2. Suppose $t_1 < t_2$. Then, from (2.8) with $t = t_1$ and $s = t_2$, we see that $W(t_2 - t_1) \ge a/2$. Now W being continuous at $t = 0$, there exists $\delta > 0$ with $W(t) < a/2$ on $[0, \delta)$. Then $t_2 - t_1 \ge \delta$ and so

$$\int_0^T |\chi(s)|\, ds \ge \int_{t_1}^{t_2} |\chi(s)|\, ds \ge (t_2 - t_1)\, a/2 \ge a\delta/2.$$

Thus (2.6) holds with $b = \min\{aT/2, a\delta/2\}$. End Proof

We have the following existence and uniqueness principle for (1.1).

Let $f : [0,T]^2 \times B \to E$. Suppose

(h1) for any $t \in [0,T]$ and $x \in B$, the mapping $f(t, \, . \, , x)$ is strongly measurable and $f(t, \, . \, , 0) \in L^1([0,T]; E)$;

(h2) there exists $\phi : [0,T]^2 \to \mathbf{R}_+$ and $q \in [1,\infty]$ such that

$$\begin{cases} themapping\, t \longmapsto \phi\,(t,\,.\,)\ (alsodenotedby\, \phi) belongsto \\ L^{\infty}\left([0,T]\,;\,L^{q}\,[0,T]\right)\ and \\ \|\phi\|_{L^p([0,T];\ L^q[0,T])} < 1\ \ (1/p + 1/q = 1) \end{cases} \tag{2.9}$$

and

$$|f\,(t,s,x) - f\,(t,s,y)| \le \phi\,(t,s)\,|x-y| \tag{2.10}$$

for a.e. $s \in [0,T]$, all $x,\ y \in B$ and each $t \in [0,T]$;

(h3) there exists $w : [0,T] \to \mathbf{R}_+$ bounded, continuous at 0 and with $w\,(0) = 0$, such that

$$\int_0^T \sup_{|x|\le R} |f\,(t,s,x) - f\,(t',s,x)|\,ds \le w\,(|t-t'|) \tag{2.11}$$

for all $t,\ t' \in [0,T]$;

(h4) for each $\lambda \in (0,1)$, each possible solution $u \in K_R$ of equation

$$u\,(t) = \lambda \int_0^T f\,(t,s,u\,(s))\,ds, \quad t \in [0,T] \tag{2.12}$$

is such that $\|u\|_\infty < R$.

Then (1.1) has a unique solution in K_R.

ProofThe result follows from Theorem 2.1; here $K = C,\ u_0 = 0$ and $A : C_R \to C$ is given by

$$A\,(u)\,(t) = \int_0^T f\,(t,s,u\,(s))\,ds. \tag{2.13}$$

First we prove that A is well-defined (i.e. $A\,(u) \in C$) and $A\,(C_R) \subset C_w$. For this, we use the following estimate

$$\begin{aligned} |f\,(t,s,x)| &\le |f\,(t,s,x) - f\,(t,s,0)| + |f\,(t,s,0)| \\ &\le \phi\,(t,s)\,|x| + |f\,(t,s,0)| \le R\phi\,(t,s) + |f\,(t,s,0)|\,. \end{aligned} \tag{2.14}$$

Since $\phi\,(t,\,.\,) \in L^q\,[0,T] \subset L^1\,[0,T]$ and $f\,(t,\,.\,,0) \in L^1\,([0,T]\,;E)$, from (2.14) we see that $f\,(t,\,.\,,u\,(\,.\,))$ is Bochner integrable on $[0,T]$ and so the integral in (2.13) makes sense. Furthermore, from (2.11),

$$|A\,(u)\,(t) - A\,(u)\,(t')| \le \int_0^T |f\,(t,s,u\,(s)) - f\,(t',s,u\,(s))|\,ds$$

$$\leq w\left(\left|t-t'\right|\right) \to 0 \ \ as \ \ t' \to t.$$

Hence $A\left(u\right) \in C_w$. Now

$$\left|A\left(u\right)\left(t\right)\right| \leq \int_0^T \left|f\left(t,s,u\left(s\right)\right)\right| ds \leq \int_0^T \left|f\left(t,s,u\left(s\right)\right) - f\left(0,s,u\left(s\right)\right)\right| ds$$

$$+ \int_0^T \left|f\left(0,s,u\left(s\right)\right) - f\left(0,s,0\right)\right| ds + \int_0^T \left|f\left(0,s,0\right)\right| ds$$

$$\leq \int_0^T \sup_{|x|\leq R} \left|f\left(t,s,x\right) - f\left(0,s,x\right)\right| ds + \int_0^T \left[R\phi\left(0,s\right) + \left|f\left(0,s,0\right)\right|\right] ds$$

$$\leq \sup_{\tau\in[0,T]} w\left(\tau\right) + \int_0^T \left[R\phi\left(0,s\right) + \left|f\left(0,s,0\right)\right|\right] ds < \infty.$$

This clearly shows that $A\left(C_R\right)$ is bounded. Therefore (H1) holds.

Next we prove that (H2) is satisfied if d is the metric induced by $\left\|\,.\,\right\|_p$ on C. Let $u,\ v \in C_R$. Then, using (2.10) and Hölder's inequality, we obtain

$$\left|A\left(u\right)\left(t\right) - A\left(v\right)\left(t\right)\right| \leq \int_0^T \left|f\left(t,s,u\left(s\right)\right) - f\left(t,s,v\left(s\right)\right)\right| ds$$

$$\leq \int_0^T \phi\left(t,s\right)\left|u\left(s\right) - v\left(s\right)\right| ds \leq \left\|\phi\left(t,\,.\,\right)\right\|_q \left\|u-v\right\|_p.$$

Consequently

$$\left\|A\left(u\right) - A\left(v\right)\right\|_\infty \leq \left\|\phi\right\|_{L^\infty\left([0,T];\,L^q[0,T]\right)} \left\|u-v\right\|_p$$

which proves (2.2). Also

$$\left\|A\left(u\right) - A\left(v\right)\right\|_p \leq \left\|\phi\right\|_{L^p\left([0,T];\,L^q[0,T]\right)} \left\|u-v\right\|_p$$

which proves (2.3).

Finally (H3) is exactly (h4). End Proof

Now we shall specialize Theorem 2.3 for the case of the Hammerstein equation

$$u\left(t\right) = \int_0^T k\left(t,s\right) g\left(s,u\left(s\right)\right) ds,\ \ t\in\left[0,T\right]. \tag{2.15}$$

Let $k : \left[0,T\right]^2 \to \mathbf{R}$ and $g : \left[0,T\right] \times B \to E$. Suppose

(a) for any $x \in B$, $g\left(\,.\,,x\right)$ is strongly measurable and $g\left(\,.\,,0\right) \in L^\alpha\left(\left[0,T\right];E\right)$ for some $\alpha \in \left[1,\infty\right]$;

(b) there exists $\psi \in L^{\alpha}[0,T]$ such that

$$|g(s,x) - g(s,y)| \le \psi(s)\,|x-y|$$

for a.e. $s \in [0,T]$ and all $x,\ y \in B$;

(c) the mapping $t \longmapsto k(t,\,.)$ belongs to $C\left([0,T]\,;L^{\beta}[0,T]\right)$ $(1/\alpha + 1/\beta = 1/q,\ 1 \le q \le \infty)$ and

$$\|\phi\|_{L^{p}([0,T];\,L^{q}[0,T])} < 1,$$

where $\phi(t,s) = k(t,s)\,\psi(s)$ and $1/p + 1/q = 1$;

(d) for each $\lambda \in (0,1)$, each possible solution $u \in C_R$ of equation

$$u(t) = \lambda \int_0^T k(t,s)\,g(s,u(s))\,ds, \quad t \in [0,T] \tag{2.16}$$

is such that $\|u\|_{\infty} < R$.

Then (2.15) has a unique solution in C_R.

ProofIt is easy to check that all the assumptions of Theorem 2.3 are satisfied with

$$f(t,s,x) = k(t,s)\,g(s,x) \;\; \textit{and} \;\; \phi(t,s) = |k(t,s)|\,\psi(s)\,;$$

here $w(\delta)$ is the continuity modulus of the mapping $t \longmapsto k(t,\,.)$ from $[0,T]$ to $L^{\beta}[0,T]$, multiplied by a suitable positive constant. End Proof

We note that Theorem 3.3 in [9] is a direct consequence of Corollary 2.4.

Finally we provide an application of Corollary 2.4 to the Dirichlet boundary value problem

$$\begin{cases} -u'' = g(t,u)\,, & 0 < t < T \\ u(0) = u(T) = 0. \end{cases} \tag{2.17}$$

Let G be the Green's function

$$G(t,s) = \begin{cases} t(T-s)/T, & 0 \le t \le s \le T \\ s(T-t)/T, & 0 \le s \le t \le T. \end{cases} \tag{2.18}$$

Let $R > 0$ and $g : [0,T] \times [-R,R] \to \mathbf{R}$ continuous. Suppose

(j) there exists $\psi \in C[0,T]$ such that

$$|g(s,x) - g(s,y)| \leq \psi(s)|x-y|$$

for all $s \in [0,T]$ and $x, y \in [-R,R]$;

(jj) there exists $q \in [1,\infty]$ such that

$$\|\phi\|_{L^p([0,T];\,L^q[0,T])} < 1,$$

where $\phi(t,s) = G(t,s)\psi(s)$ and $1/p + 1/q = 1$;

(jjj) $g(s,-R) > 0$ and $g(s,R) < 0$ for all $s \in [0,T]$.

Then (2.17) has a unique solution satisfying $|u(t)| \leq R$ for all $t \in [0,T]$.

Indeed, it is well known that (2.17) is equivalent to the integral equation (2.15), where $k = G$. It is clear that the assumptions (j)-(jj) guarantee (a)-(c). To show (d), let u be a solution of (2.16). We claim that $\|u\|_\infty < R$. Indeed, if $\|u\|_\infty = R$, then since $u(0) = u(T) = 0$, there exists $t_0 \in (0,T)$ with $|u(t_0)| = R$. If $u(t_0) = R$, then $u'(t_0) = 0$ and $u''(t_0) \leq 0$. Since $-u'' = \lambda g(t,u)$, we deduce that $g(t_0,R) \geq 0$, a contradiction to (jjj). Similarly, if $u(t_0) = -R$, then $u'(t_0) = 0$ and $u''(t_0) \geq 0$. Hence $g(t_0,-R) \leq 0$, again a contradiction to (jjj). Thus our claim is proved and Corollary 2.4 applies.

Now choose for the above example: $q = \alpha = 1$ and $p = \beta = \infty$. Routine calculation gives

$$\max_{t\in[0,T]} \int_0^T G(t,s)\,ds = \max_{t\in[0,T]} t(T-t)/2 = T^2/8.$$

Then (jj) is satisfied if

$$T^2\|\psi\|_\infty/8 < 1.$$

We may think that the existence and uniqueness of the solution could be directly obtained from Banach's contraction theorem. This is true if the map $A : C_R \to C$ given by

$$A(u)(t) = \int_0^T G(t,s)\,g(s,u(s))\,ds$$

satisfies

$$A(C_R) \subset C_R. \tag{2.19}$$

For any $u \in C_R$, by (j), we have

$$|A(u)(t)| \leq \int_0^T G(t,s)\,|g(s,u(s))|\,ds$$

$$\leq \int_0^T G(t,s)\left[\psi(s)\,|u(s)| + |g(s,0)|\right] ds$$

$$\leq \left[\|\psi\|_\infty R + \|g(\,.\,,0)\|_\infty\right] T^2/8.$$

Thus, if

$$\left[\|\psi\|_\infty R + \|g(\,.\,,0)\|_\infty\right] T^2/8 \leq R, \tag{2.20}$$

then (2.19) holds and the conclusion follows from Banach's contraction principle. This is not the case when (2.20) does not hold.

For much more about the "a priori" bounds technique guarantying conditions like (d) we refer the reader to [3].

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